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Metacompactness and subparacompactness of product spaces

By Takemi MIZOKAMI

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§1. Introduction.

The aim of this paper is to generalize Kramer [3, Theorems 4.3, 4.4, 4.9] which gives some sufficient conditions for the product space to be metacompact or subparacompact.

Alster and Engelking [1] constructed a subparacompact space X such that X^2 is not subparacompact. The product space S^2 of Sorgenfrey lines S is not metacompact, though S is metacompact. Thus subparacompactness and metacompactness do not have productive property. This is the case even if one factor is metrizable and the other factor is paracompact. Przymusiński [4] constructed a separable metric space M and a separable first countable Lindelöf regular (and hence paracompact) space Y such that $M\times Y$ is neither subparacompact nor metacompact.

We consider the subparacompactness and (countable) metacompactness of the product space $X\times Y$, where X is a P-space due to Morita [2, Definition 56.1] and Y is a Σ-space due to Nagami [2, Definition 57.1]. It is seen in [2, Theorem 57.14] that these notions are very effective to our consideration. This is why we restrict Y to the class of Σ-spaces.

In the sequel, all spaces are assumed to be T_1 and N to be the positive integers.

§2. Theorems.

DEFINITION 1. A space X is said to be a *Σ-space* if there exists a sequence $\{\mathcal{F}_n : n\in N\}$ of locally finite closed covers of X satisfying the following (Σ):

(Σ): If $p_n\in C(p, \mathcal{F}_n)=\cap\{F: p\in F\in\mathcal{F}_n\}$ for every $n\in N$, then $\{p_n : n\in N\}$ clusters in X.

Moreover, if for every point $p\in X$,

$$C(p)=\bigcap_n C(p, \mathcal{F}_n)$$

is compact, then X is called a *strong Σ-space.*

LEMMA 1 (Nagami [2, Lemma 57.3]). *Let X be a Σ-space. Then X has a Σ-net $\{\mathcal{F}_i : i \in N\}$ which satisfies the following:*

(i) *Every $\mathcal{F}_i$ is (finitely) multiplicative.*

(ii) $\mathcal{F}_i = \{F(\alpha_1, \cdots, \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega\}$.

(iii) $F(\alpha_1, \cdots, \alpha_i) = \cup \{F(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}) : \alpha_{i+1} \in \Omega\}$.

(iv) *For every point $x \in X$, there exists a sequence $\{\alpha_i : i \in N\}$ such that $C(x) \subset F(\alpha_1, \cdots, \alpha_i)$ for every i and if $C(x) \subset U$ with U open, then*

$$C(x) \subset F(\alpha_1, \cdots, \alpha_i) \subset U$$

for some i.

DEFINITION 2. A space X is called *subparacompact* or *metacompact* if every open cover of X has respectively a σ-discrete closed refinement or a point-finite open refinement.

Recall that X is subparacompact if and only if every open cover of X has a σ-locally finite closed refinement, [2, Theorem 43.4].

THEOREM 1. *Let X be a regular strong Σ-space. Then X is subparacompact.*

PROOF. To prove this, it suffices to show the following:

LEMMA 2. *Let X be a Σ-space such that every open cover of $C(x)$ has a locally finite closed refinement. Then X is subparacompact.*

PROOF OF LEMMA. Let $\mathcal{U}$ be an arbitrary open cover of X. Let $\mathcal{F}_i = \{F_{i\alpha} : \alpha \in A_i\}$, $i \in N$ be a Σ-net of X, where each $\mathcal{F}_i$ is assumed to be multiplicative. Put for each $x \in X$,

$$\mathcal{U}(x) = \{U : U \cap C(x) = \emptyset, U \in \mathcal{U}\}.$$

Then by assumption, $\mathcal{U}(x)$ is refined by a locally finite cover $\mathcal{H}(x)$. Put

$$U(x) = \cup \{U : U \in \mathcal{U}(x)\}.$$

Then $U(x)$ is an open set containing $C(x)$. Therefore there exists an $F_{i(x)\alpha(x)} \in \mathcal{F}_{i(x)}$ with $C(x) \subset F_{i(x)\alpha(x)} \subset U(x)$. Set

$$\mathcal{H}_i = \{H \cap F_{i(x)\alpha(x)} : H \in \mathcal{H}(x), x \in X, i(x) = i\}.$$

Then $\mathcal{H} = \cup \mathcal{H}_i$ is a σ-locally finite closed refinement of $\mathcal{U}$.

COROLLARY. *If for each $i \in N$, X_i is a regular strong Σ-space, then $\prod_i X_i$ is subparacompact.*

PROOF. Use the fact that $\prod X_i$ is also a regular strong Σ-space by [2, Theorem 57.12].

LEMMA 3. *If X is a Σ-space, then X is countably metacompact.*

PROOF. Recall that a space X is countably metacompact if and only if for every increasing countable open cover $\{U_i : i \in N\}$, there exists a cover $\{F_i : i \in N\}$ such that each F_i is an F_σ-set and $F_i \subset U_i$ for each $i \in N$.

Suppose we are given such a cover $\{U_i : i \in N\}$. Put

$$F_i = \cup \{F : F \in \mathcal{F} = \cup \mathcal{F}_i, F \subset U_i\},$$

where $\{\mathcal{F}_i : i \in N\}$ is a Σ-net of X such that each $\mathcal{F}_i$ is multiplicative. Then each F_i is an F_σ-set contained in U_i. It is easy to see that $\{F_i\}$ covers X. Thus X is countably metacompact.

COROLLARY. *If for each $i \in N$ X_i is a strong Σ-space, then $\prod X_i$ is countably metacompact.*

DEFINITION 3. A space X is called *almost expandable* [**5**, Definition 1.5] if for every locally finite collection $\{F_\lambda : \lambda \in \Lambda\}$ of X there exists a point-finite open collection $\{G_\lambda : \lambda \in \Lambda\}$ such that $F_\lambda \subset G_\lambda$ for each λ.

It is well known that a space X is metacompact if and only if it is θ-refinable and almost expandable [**5**, Theorem 4.3 (ii)]. Recall that a θ-refinable countably compact space is compact [**6**, p. 824].

THEOREM 2. *If X is an almost expandable strong Σ-space, then X is metacompact.*

To prove this, it suffices to prove the following:

LEMMA 4. *Suppose X is an almost expandable Σ-space with the property that for every open cover $\mathcal{U}$ of $C(x)$, there exists a point-finite (in X) open cover of $C(x)$ refining $\mathcal{U}$. Then X is metacompact.*

PROOF. Let $\mathcal{F}_i = \{F_{i\alpha} : \alpha \in A_i\}$, $i \in N$ be a mutiplicative Σ-net of X. In the light of Lemma 3, it suffices to see that every open cover of X can be refined by a σ-point-finite open cover. To see this, let $\mathcal{U}$ be an open cover of X. Then by assumption,

$$\mathcal{U}(x) = \{U : U \cap C(x) \neq \emptyset, U \in \mathcal{U}\}$$

can be refined by a point-finite open (in X) cover $\mathcal{V}(x)$. Since X is almost expandable, there exists a point-finite open collection $\mathcal{H}_i = \{H_{i\alpha} : \alpha \in A_i\}$ such that $F_{i\alpha} \subset H_{i\alpha}$ for each $\alpha \in A_i$. Put

$$V(x) = \cup \{V : V \in \mathcal{V}(x)\},$$

$$\mathcal{V} = \{V(x) : x \in X\},$$

$$\mathcal{F}'_i = \{F_{i\beta} : \beta \in B_i\} = \{F \in \mathcal{F}_i, F < \mathcal{V}\}.$$

Observe that $\cup \mathcal{F}'_i$ covers X. Take $F_{i\beta} \in \mathcal{F}'_i$ and a point $x_{i\beta}$ with $F_{i\beta} \subset V(x_{i\beta})$. Put

$$\mathcal{W} = \{V \cap H_{i\beta} : V \in \mathcal{V}(x_{i\beta}), \beta \in B_i, i \in N\}.$$

Then $\mathcal{W}$ is a σ-point-finite open refinement of $\mathcal{U}$. Thus X is metacompact.

LEMMA 5. *If X is a Σ-space with a σ-point-finite base, then X is a metacompact strong Σ-space.*

PROOF. X is countably metacompact by Lemma 3. Since X has a σ-point-finite base, every open cover of X has a σ-point-finite open refinement. It follows that X is metacompact, and necessarily X is a strong Σ-space.

COROLLARY. *If for each $i \in N$ X_i is a Σ-space with a σ-point-finite base, then $\prod X_i$ is metacompact.*

PROOF. It suffices to prove that $X = \prod X_i$ has a σ-point-finite base. Let $\mathcal{V}_i = \bigcup_{j=1}^{\infty} \mathcal{V}_{ij}$ be a base of X_i such that each $\mathcal{V}_{ij}$ is a point-finite open cover and $\mathcal{V}_{ij} \subset \mathcal{V}_{ij+1}$ for each $j \in N$. Construct for each $n \in N$

$$\mathcal{W}_n = \{V_1 \times \cdots \times V_n \times \prod_{j>n} X_j : V_1 \in \mathcal{V}_{1n}, \cdots, V_n \in \mathcal{V}_{nn}\},$$

$$\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n .$$

Then $\mathcal{W}$ is a σ-point-finite base of X.

DEFINITION 4. A space X is called a *P-space* if for each collection $\{G(\alpha_1, \cdots, \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega, i \in N\}$ of open sets of X such that

$$G(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$$

for each sequence $\alpha_1, \alpha_2, \cdots \in \Omega$, there exists a collection $\{C(\alpha_1, \cdots, \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega, i \in N\}$ of F_σ-sets of X such that for each sequence $\alpha_1, \alpha_2, \cdots \in \Omega$,

(i) $$C(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i),$$

(ii) $$X = \bigcup_{i=1}^{\infty} C(\alpha_1, \cdots, \alpha_i) \quad \text{if} \quad X = \bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i).$$

If moreover the condition (ii) is strengthened to the following (iii), X is called to have the *property P^**.

(iii) $$X = \bigcup_{i=1}^{\infty} \operatorname{Int} C(\alpha_1, \cdots, \alpha_i) \quad \text{if} \quad X = \bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i).$$

Note that a normal P-space has the property P^* as seen in [2, Proposition 56.2]. If X is perfect i.e., every open set of X is an F_σ-set, then X has the property P^*.

LEMMA 6. *If X is a P-space and Y is a strong Σ-space, then $X \times Y$ is countably metacompact.*

PROOF. Let $\{U_j : j \in N\}$ be an arbitrary increasing countable open cover of $X \times Y$. As stated in the proof of Lemma 3, it suffices to prove that there exists a countable refinement by F_σ-sets. Let $\{F(\alpha_1, \cdots, \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega\}$, $i \in N$ be a strong Σ-net of Y described in Lemma 1. Put

$$G(\alpha_1, \cdots, \alpha_i) = \cup \{P : P \text{ is an open set of } X \text{ such that } P \times F(\alpha_1, \cdots, \alpha_i) \subset U_i\}.$$

Then

(1) $$G(\alpha_1, \cdots, \alpha_i)\times F(\alpha_1, \cdots, \alpha_i)\subset U_i$$

and

$$G(\alpha_1, \cdots \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}).$$

Since X is a P-space, there exists a collection $\{C(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i\in\Omega, i\in N\}$ of F_σ-sets of X such that

(2) $$C(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i),$$

(3) $$X=\bigcup_{i=1}^{\infty} C(\alpha_1, \cdots, \alpha_i) \quad \text{if} \quad X=\bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i).$$

Put for each $i\in N$

$$V_i=\cup\{C(\alpha_1, \cdots, \alpha_i)\times F(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i\in\Omega\}.$$

Then each V_i is an F_σ-set satisfying $V_i\subset U_i$ because of (1) and (2). It remains to prove that $\{V_i\}$ covers $X\times Y$. Let (p, q) be an arbitrary point of $X\times Y$. Let $\{\alpha_i: i\in N\}$ be a sequence such that $\{F(\alpha_1, \cdots, \alpha_i): i\in N\}$ satisfies (iv) in Lemma 1. To see $X=\bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i)$, let $x\in X$. Since $(x, C(q))$ is countably compact, $(x, C(q))\subset U_j$ for some j. Moreover the compactness of $(x, C(q))$ implies that there exist open sets U, V of X, Y, respectively, such that

$$(x, C(q))\subset U\times V\subset U_j.$$

Then there exists an $i\in N$ with $C(q)\subset F(\alpha_1, \cdots, \alpha_i)\subset V$. In either case of $i\geqq j$ and $i<j$, x belongs to some $G(\alpha_1, \cdots, \alpha_i)$. This implies that $X=\bigcup_{i=1}^{\infty} C(\alpha_1, \cdots, \alpha_i)$ by (3). Thus $x\in C(\alpha_1, \cdots, \alpha_i)$ for some i, proving

$$(p, q)\in C(\alpha_1, \cdots, \alpha_i)\times F(\alpha_1, \cdots, \alpha_i)\subset V_i.$$

COROLLARY. *If X is a first countable P-space and Y is a Σ-space, then $X\times Y$ is countably metacompact.*

PROOF. We modify the preceding proof slightly. Suppose $(p, C(q))\subset U_j$. Let $\{V_n(p): n\in N\}$ be a local base of p in X. Put for each $n\in N$

$$W_n=\cup\{P: P \text{ is an open set of } Y \text{ such that } V_n(p)\times P\subset U_j\}.$$

Then $\{W_n: n\in N\}$ covers $C(q)$. Since $C(q)$ is countably compact, there exists a finite subcover $\{W_{n_j}: j=1, \cdots, k\}$. Take U, V as follows:

$$U=\bigcap_{j=1}^{k} V_{n_j}(p), \qquad V=\bigcup_{j=1}^{k} W_{n_j}.$$

Then obviously we have

$$(p, C(q))\subset U\times V\subset U_j.$$

THEOREM 3. *If X is a metacompact space with the property P^* and Y is an almost expandable strong Σ-space, then $X\times Y$ is metacompact.*

PROOF. Let $\mathcal{F}_i=\{F(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i\in\Omega\}$, $i\in N$, be a Σ-net described in Lemma 1. Since Y is almost expandable, for each i there exists a point-finite open collection $\mathcal{H}_i=\{H(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i\in\Omega\}$ such that

$$F(\alpha_1, \cdots, \alpha_i)\subset H(\alpha_1, \cdots, \alpha_i)$$

for each sequence $\alpha_1, \alpha_2, \cdots \in\Omega$. Taking Lemma 6 into consideration, it suffices to prove that every open cover of $X\times Y$ can be refined by a σ-point-finite open cover. Let $\mathcal{G}$ be an arbitrary open cover of $X\times Y$, and $\Delta\mathcal{G}$ the collection of all finite unions of members of $\mathcal{G}$. For each $\alpha_1, \cdots, \alpha_i\in\Omega$, let $\mathcal{W}(\alpha_1, \cdots, \alpha_i)$ be the maximal collection of basic open sets $U_\lambda\times V_\lambda$ such that

$$\mathcal{W}(\alpha_1, \cdots, \alpha_i)=\{U_\lambda\times V_\lambda: \lambda\in\Lambda(\alpha_1, \cdots, \alpha_i)\},$$

$$F(\alpha_1, \cdots, \alpha_i)\subset V_\lambda\subset H(\alpha_1, \cdots, \alpha_i), \tag{1}$$

$$\mathcal{W}(\alpha_1, \cdots, \alpha_i)<\Delta\mathcal{G}. \tag{2}$$

Set

$$U(\alpha_1, \cdots, \alpha_i)=\cup\{U_\lambda: \lambda\in\Lambda(\alpha_1, \cdots, \alpha_i)\}.$$

Then by maximality of $\mathcal{W}(\alpha_1, \cdots, \alpha_i)$,

$$U(\alpha_1, \cdots, \alpha_i)\subset U(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}) \tag{3}$$

for each sequence $\alpha_1, \alpha_2, \cdots \in\Omega$. Since X has the property P^*, there exists an F_σ-set $C(\alpha_1, \cdots, \alpha_i)$ such that

$$C(\alpha_1, \cdots, \alpha_i)\subset U(\alpha_1, \cdots, \alpha_i) \tag{4}$$

for each sequence $\alpha_1, \alpha_2, \cdots \in\Omega$ and

$$X=\bigcup_{i=1}^{\infty}\operatorname{Int} C(\alpha_1, \cdots, \alpha_i) \qquad \text{if} \quad X=\bigcup_{i=1}^{\infty} U(\alpha_1, \cdots, \alpha_i). \tag{5}$$

Since $C(\alpha_1, \cdots, \alpha_i)$ is an F_σ-set of a metacompact space, $\{U_\lambda\cap C(\alpha_1, \cdots, \alpha_i): \lambda\in\Lambda(\alpha_1, \cdots, \alpha_i)\}$ can be refined by a point-finite open (in $C(\alpha_1, \cdots, \alpha_i)$) cover $\{E'_\lambda: \lambda\in\Lambda(\alpha_1, \cdots, \alpha_i)\}$ such that

$$E'_\lambda\subset U_\lambda\cap C(\alpha_1, \cdots, \alpha_i) \tag{6}$$

for each λ. Put

$$E_\lambda=\operatorname{Int} C(\alpha_1, \cdots \alpha_i)\cap E'_\lambda,$$

$$\mathcal{V}(\alpha_1, \cdots, \alpha_i)=\{E_\lambda\times V_\lambda: \lambda\in\Lambda(\alpha_1, \cdots, \alpha_i)\},$$

$$\mathcal{V}_i=\cup\{\mathcal{V}(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i\in\Omega\},$$

$$\mathcal{V}=\cup\{\mathcal{V}_i: i\in N\}.$$

We shall show that $\mathcal{V}$ has the following properties:

Claim 1: $\mathcal{V}$ is an open cover of $X\times Y$.

Let (x, y) be any point in $X\times Y$. Let $\{\alpha_i : i\in N\}$ be a sequence satisfying (iv) in Lemma 1. In this case, we firstly show that $X=\bigcup_{i=1}^{\infty} U(\alpha_1, \cdots, \alpha_i)$. Let $p\in X$. Since $(p, C(y))$ is compact, there exists $G'\in\Delta\mathcal{G}$ with $(p, C(y))\subset G'$. Because of compactness of $C(y)$ there exist open sets U, V of X, Y, respectively, such that

$$(p, C(y))\subset U\times V\subset G'.$$

Take an $i\in N$ such that $F(\alpha_1, \cdots, \alpha_i)\subset V$. Put $U=U_\lambda$ and $V\cap H(\alpha_1, \cdots, \alpha_i)=V_\lambda$. Then $U_\lambda\times V_\lambda\in\mathcal{W}(\alpha_1, \cdots, \alpha_i)$, proving $X=\bigcup_{i=1}^{\infty} U(\alpha_1, \cdots, \alpha_i)$. Therefore by (5) we have $X=\bigcup_{i=1}^{\infty}\operatorname{Int} C(\alpha_1, \cdots, \alpha_i)$. Thus $x\in\operatorname{Int} C(\alpha_1, \cdots, \alpha_i)$ for some i. From this $x\in E_\lambda$ for some $\lambda\in\Lambda(\alpha_1, \cdots, \alpha_i)$. Since $y\in F(\alpha_1, \cdots, \alpha_i)\subset V_\lambda\subset H(\alpha_1, \cdots, \alpha_i)$,

$$(x, y)\in E_\lambda\times V_\lambda$$

for $\lambda\in\Lambda(\alpha_1, \cdots, \alpha_i)$. Hence $\mathcal{V}$ is a cover of $X\times Y$.

Claim 2: $\mathcal{V}$ is a refinement of $\Delta\mathcal{G}$.

This follows from (2) and (4).

Claim 3: $\mathcal{V}$ is a σ-point-finite collection in $X\times Y$.

To see this, we shall show that each $\mathcal{V}_i$ is point-finite in $X\times Y$. Let $(x, y)\in X\times Y$. Since $\mathcal{H}_i$ is point-finite in Y, y belongs to at most finitely many members $H(\alpha_1, \cdots, \alpha_i)$. For each sequence $\alpha_1, \cdots, \alpha_i$, there exists a finite subset $\Lambda_0(\alpha_1, \cdots, \alpha_i)$ of $\Lambda(\alpha_1, \cdots, \alpha_i)$ such that $x\in E_\lambda$ implies $\lambda\in\Lambda_0(\alpha_1, \cdots, \alpha_i)$. Then $(x, y)\in E_\lambda\times V_\lambda$ implies $\lambda\in\bigcup\Lambda_0(\alpha_1, \cdots, \alpha_i)$, where the union is a finite union. Hence (x, y) belongs to at most finitely many members of $\mathcal{V}_i$.

Thus we have a σ-point-finite open refinement $\mathcal{V}$ of $\Delta\mathcal{G}$. For each $V\in\mathcal{V}$, take $G(V)\in\Delta\mathcal{G}$ with $V\subset G(V)$. Denote $G(V)$ by

$$G(V)=G_1(V)\cup\cdots\cup G_k(V), \qquad G_j(V)\in\mathcal{G}.$$

Put

$$\mathcal{W}=\{G_j(V)\cap V : j=1, \cdots, k,\ V\in\mathcal{V}\}.$$

Then $\mathcal{W}$ is a σ-point-finite open refinement of the original cover.

COROLLARY 1. *If X is a metacompact space with the property P^* and Y is a metacompact Σ-space, then $X\times Y$ is metacompact.*

PROOF. Recall that metacompactness is equivalent with θ-refinability plus almost expandability, and every θ-refinable countably compact space is compact.

COROLLARY 2. *If X is a metacompact and perfect space and Y is an almost expandable strong Σ-space, then $X\times Y$ is metacompact.*

PROOF. Perfectness implies that every closed set is a G_δ-set, and therefore

X has the property P^*.

COROLLARY 3 (Kramer [3, Theorem 4.4]). *If X is a metacompact perfect space and Y is a σ-space with a σ-point-finite base, then $X\times Y$ is metacompact.*

PROOF. This follows immediately from Lemma 5 and the above corollary.

THEOREM 4. *If X is a subparacompact P-space and Y is a regular strong Σ-space, then $X\times Y$ is subparacompact.*

PROOF. Let $\mathscr{F}_i=\{F(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i\in\Omega\}$, $i\in N$ be a strong Σ-net of Y described in Lemma 1. Suppose we are given an arbitrary open cover $\mathscr{G}$ of $X\times Y$. Since Y is regular, there exists an open cover $\{U_\lambda\times V_\lambda: \lambda\in\Lambda\}$ of $X\times Y$ such that each $U_\lambda\times\bar{V}_\lambda\subset G$ for some $G\in\mathscr{G}$.

Let Δ be the totality of finite subsets of Λ. For each $\delta\in\Delta$ we put

$$P_\delta=\cap\{U_\lambda: \lambda\in\delta\}\qquad Q_\delta=\cup\{V_\lambda: \lambda\in\delta\}$$

$$W_\delta=P_\delta\times Q_\delta.$$

Put for each sequence $\alpha_1, \cdots, \alpha_i\in\Omega$ and $\delta\in\Delta$,

(1) $$G(\alpha_1, \cdots, \alpha_i: \delta)=\cup\{P: P \text{ is an open set of } X \text{ such that } P\times F(\alpha_1, \cdots, \alpha_i)\subset W_\delta\}.$$

$$G(\alpha_1, \cdots, \alpha_i)=\cup\{G(\alpha_1, \cdots, \alpha_i: \delta): \delta\in\Delta\}.$$

Then

$$G(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$$

for each sequence $\alpha_1, \cdots, \alpha_i, \alpha_{i+1}\in\Omega$. By assumption, there exists a collection $\{H(\alpha_1, \cdots, \alpha_i)\}$ of closed sets satisfying

$$H(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i)$$

for each sequence $\alpha_1, \cdots, \alpha_i\in\Omega$ and

(2) $$X=\bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i) \quad \text{implies} \quad X=\bigcup_{i=1}^{\infty} H(\alpha_1, \cdots, \alpha_i).$$

Since every closed set is also subparacompact,

$$\{G(\alpha_1, \cdots, \alpha_i: \delta)\cap H(\alpha_1, \cdots, \alpha_i): \delta\in\Delta\}$$

can be refined by a σ-discrete closed (in X) refinement

$$\mathscr{K}(\alpha_1, \cdots, \alpha_i)=\cup\{\mathscr{K}_j(\alpha_1, \cdots, \alpha_i): j\in N\},$$

where

$$\mathscr{K}_j(\alpha_1, \cdots, \alpha_i)=\{K_j(\alpha_1, \cdots, \alpha_i: \delta): \delta\in\Delta\}$$

is a discrete closed collection of X such that

(3) $$K_j(\alpha_1, \cdots, \alpha_i: \delta)\subset G(\alpha_1, \cdots, \alpha_i: \delta)\cap H(\alpha_1, \cdots, \alpha_i)$$

for each sequence $\alpha_1, \cdots, \alpha_i \in \Omega$, $j \in N$ and $\delta \in \Delta$. Put

$$\mathcal{L}_j(\alpha_1, \cdots, \alpha_i) = \{K_j(\alpha_1, \cdots, \alpha_i : \delta) \times \overline{F(\alpha_1, \cdots, \alpha_i) \cap V_\lambda} : \lambda \in \delta, \delta \in \Delta\},$$

$$\mathcal{L}_{ji} = \cup \{\mathcal{L}_j(\alpha_1, \cdots, \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega\}$$

$$\mathcal{L} = \cup \{\mathcal{L}_{ji} : j, i \in N\}.$$

Claim 1: $\mathcal{L}$ is a cover of $X \times Y$.

Let (x, y) be any point of $X \times Y$. Take a sequence $\{\alpha_i : i \in N\}$ such that (iv) in Lemma 1 is satisfied. We can show that $X = \bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i)$. Suppose $p \in X$. Then $(p, C(y))$ is covered by a finite collection of $\mathcal{G}$, and therefore $(p, C(y)) \subset P_\delta \times Q_\delta = W_\delta$ for some $\delta \in \Delta$. Take an $i \in N$ with

$$C(y) \subset F(\alpha_1, \cdots, \alpha_i) \subset Q_\delta.$$

Thus we have

$$(p, C(y)) \subset P_\delta \times F(\alpha_1, \cdots, \alpha_i) \subset W_\delta,$$

which implies

$$p \in G(\alpha_1, \cdots, \alpha_i : \delta) \subset G(\alpha_1, \cdots, \alpha_i).$$

By (2),

$$X = \bigcup_{i=1}^{\infty} H(\alpha_1, \cdots, \alpha_i).$$

Thus there exists a $j \in N$ with $x \in H(\alpha_1, \cdots, \alpha_j)$. In this case we have

$$x \in K_m(\alpha_1, \cdots, \alpha_j : \delta).$$

Observe that for this

$$F(\alpha_1, \cdots, \alpha_j) = \cup \{\overline{F(\alpha_1, \cdots, \alpha_j) \cap V_\lambda} : \lambda \in \delta\}.$$

Thus for some $\lambda \in \delta$

$$y \in \overline{F(\alpha_1, \cdots, \alpha_j) \cap V_\lambda}.$$

These mean

$$(x, y) \in K_m(\alpha_1, \cdots, \alpha_j : \delta) \times \overline{F(\alpha_1, \cdots, \alpha_j) \cap V_\lambda},$$

proving that $\mathcal{L}$ is a cover of $X \times Y$.

Claim 2: $\mathcal{L}$ is a refinement of $\mathcal{G}$.

This follows from the fact that

$$K_j(\alpha_1, \cdots, \alpha_i : \delta) \times \overline{F(\alpha_1, \cdots, \alpha_i) \cap V_\lambda} \subset P_\delta \times \bar{V}_\lambda \subset U_\lambda \times \bar{V}_\lambda \subset G$$

for some $G \in \mathcal{G}$.

Claim 3: $\mathcal{L}$ is a σ-locally finite closed collection of $X \times Y$.

This follows from the local finiteness of each $\mathcal{F}_i$ and discreteness of $\mathcal{K}_j(\alpha_1,$

$\cdots, \alpha_\iota)$.

Thus we get a σ-locally finite closed refinement of $\mathcal{G}$.

COROLLARY. *If X is a subparacompact and perfect space and Y is a regular strong Σ-space, then $X \times Y$ is subparacompact.*

This is a refinement of Kramer [**3**, Theorem 4.3].

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Takemi MIZOKAMI
Department of Mathematics,
Sasebo Technical College,
Okishin-cho, Sasebo,
Nagasaki, Japan.