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A characterization of crossed products of factors by discrete outer automorphism groups

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§1. Introduction.

Let $\mathcal{A}$ be a factor, $\mathcal{G}$ a discrete countable group and α a faithful representation of $\mathcal{G}$ into $Out(\mathcal{A})$ (the group of all outer automorphisms of $\mathcal{A}$). Then it is known that there is a faithful normal expectation of the crossed product $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha)$ of $\mathcal{A}$ by $\mathcal{G}$ onto $\mathcal{A}$ and that the relative commutant $\mathcal{A}'\cap\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha)$ is the scalar multiples of the identity operator (cf, [**3**; Lemma 4]). In [**1**], we considered the crossed product $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ of a general von Neumann algebra $\mathcal{A}$ by a discrete countable group $\mathcal{G}$ with a factor set $\{v(g, h); g, h\in\mathcal{G}\}$ of unitaries in $\mathcal{A}$ and showed that there is a faithful normal expectation of $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ onto $\mathcal{A}$, where α is a semirepresentation of $\mathcal{G}$ into $Aut(\mathcal{A})$ (the group of all automorphisms of $\mathcal{A}$). Using this expectation, we can show that the relative commutant of $\mathcal{A}$ in $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ is contained in the center of $\mathcal{A}$ if α_g $(g\neq 1)$ is freely acting on $\mathcal{A}$ (Theorem 2), so that the relative commutant of a factor $\mathcal{A}$ in $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ is the scalar multiples of the identity operator if α is a semirepresentation of $\mathcal{G}$ into $Out(\mathcal{A})$ (Corollary 3). Is the converse of this result true? The purpose of this paper is to show that the converse of this result is true. That is, for a von Neumann algebra $\mathcal{M}$ generated by the normalizer of a subfactor $\mathcal{A}$ if there is a faithful normal expectation of $\mathcal{M}$ onto $\mathcal{A}$ and the relative commutant of $\mathcal{A}$ in $\mathcal{M}$ is the scalar multiples of the identity operator, then there exists a discrete countable group $\mathcal{G}$ such that $\mathcal{M}$ is isomorphic to $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ for some semirepresentation α of $\mathcal{G}$ into $Out(\mathcal{A})$ (Theorem 4).

§2. Crossed products with factor sets.

Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{G}$ a countable discrete group. A mapping α of $\mathcal{G}$ into $Aut(\mathcal{A})$ is called a *semirepresentation* if, for each g and h in $\mathcal{G}$, there exists an inner automorphism $\iota(g, h)$ of $\mathcal{A}$ such that

$$\alpha_g\alpha_h=\alpha_{gh}\iota(g, h) \qquad (g, h\in\mathcal{G}). \tag{1}$$

Let α be a semirepresentation of $\mathcal{G}$ into $Aut(\mathcal{A})$. A family $\{v(g, h); g, h\in\mathcal{G}\}$ of unitaries in $\mathcal{A}$ is called a *factor set associated with* $(\mathcal{G}, \alpha)$ if it satisfies the following conditions:

(2) $$Ad\, v(g, h)=\iota(g, h) \qquad (g, h\in\mathcal{G})$$

and

(3) $$v(g, hk)v(h, k)=v(gh, k)\alpha_k^{-1}(v(g, h)) \qquad (g, h, k\in\mathcal{G}).$$

Let $\mathcal{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, $\mathcal{G}$ a discrete countable group, α a semirepresentation of $\mathcal{G}$ into $Aut(\mathcal{A})$, and $\{v(g, h); g, h\in\mathcal{G}\}$ a factor set of unitaries in $\mathcal{A}$ associated with $(\mathcal{G}, \alpha)$. Let $l^2(\mathfrak{H}, \mathcal{G})$ be the Hilbert space of all $\mathfrak{H}$-valued functions ξ on $\mathcal{G}$ such that

$$\sum_{g\in\mathcal{G}}\|\xi(g)\|^2<+\infty .$$

For each x in $\mathcal{A}$ and g in $\mathcal{G}$, we shall define operators $\pi_\alpha(x)$ and $\rho(g)$ on $l^2(\mathfrak{H}, \mathcal{G})$ by

(4) $$(\pi_\alpha(x)\xi)(g)=\alpha_g^{-1}(x)\xi(g) \qquad (g\in\mathcal{G}, \xi\in l^2(\mathfrak{H}, \mathcal{G}))$$

and

(5) $$(\rho(g)\xi)(h)=v(g, g^{-1}h)\xi(g^{-1}h) \qquad (h\in\mathcal{G}, \xi\in l^2(\mathfrak{H}, \mathcal{G})).$$

Then we have that π_α is a faithful normal representation of $\mathcal{A}$ and ρ is a semirepresentation of $\mathcal{G}$ into the group of unitaries on $l^2(\mathfrak{H}, \mathcal{G})$:

(6) $$\rho(g)\rho(h)=\rho(gh)\pi_\alpha(v(g, h)) \qquad (g, h\in\mathcal{G}).$$

Also π_α and ρ satisfy the following condition:

(7) $$\pi_\alpha(\alpha_g(x))=\rho(g)\pi_\alpha(x)\rho(g)^* \qquad (g\in\mathcal{G}, x\in\mathcal{A}).$$

The von Neumann algebra on $l^2(\mathfrak{H}, \mathcal{G})$ generated by $\pi_\alpha(\mathcal{A})$ and $\rho(\mathcal{G})$ is called the *crossed product of* $\mathcal{A}$ *by* $\mathcal{G}$ *with the factor set* $\{v(g, h); g, h\in\mathcal{G}\}$ associated with $(\mathcal{G}, \alpha)$ and denoted by $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$. It is known that there exists a faithful normal expectation e of $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ onto $\pi_\alpha(\mathcal{A})$ such that $e(\rho(g))=0$ for all g $(\neq 1)$ in $\mathcal{G}$ ([**1**; Theorem 6]). Using this expectation, for each element x in $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$, we have so-called Fourier expansion of x.

LEMMA 1. *Let* $\mathcal{A}$ *be a von Neumann algebra and* $\mathcal{G}$ *a discrete group. Then each* x *in* $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ *is expressed by the following form;*

$$x=\sum_{g\in\mathcal{G}}x(g)\rho(g) \qquad (x(g)\in\pi_\alpha(\mathcal{A})) \qquad \textit{in the } \sigma\textit{-strong topology.}$$

PROOF. Let e be a faithful normal expectation of $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ onto $\pi_\alpha(\mathcal{A})$ such that $e(\rho(g))=0$ $(g\neq 1)$. A faithful normal state ϕ on $\pi_\alpha(\mathcal{A})$ is extended to $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ by $\sigma=\phi\cdot e$. Let $\mathfrak{K}$ be the Gelfand-Segal representation space of

$\pi_\alpha(\mathcal{A})$ by σ. Then the representation space of $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ by σ is $l^2(\mathfrak{K}, \mathcal{G})$. Since every element of $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ can be considered as an element of $l^2(\mathfrak{K}, \mathcal{G})$, it follows that every x in $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ has the Fourier expansion

$$x=\sum_{g\in\mathcal{G}} x(g)\rho(g) \qquad (x(g)\in\mathfrak{K}).$$

It is sufficient to show that for each g in $\mathcal{G}$, $x(g)$ belongs to $\pi_\alpha(\mathcal{A})$. By the property of the expectation e of $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ onto $\pi_\alpha(\mathcal{A})$, e induces the projection e of $l^2(\mathfrak{K}, \mathcal{G})$ onto $\mathfrak{K}$ such that

$$e(x)=e(\sum_{g\in\mathcal{G}} x(g)\rho(g))=x(1)\rho(1) \qquad (x(g)\in\mathfrak{K}).$$

Put $g=h=1$ in the equality (3), then we have that

$$v(1, k)v(1, k)=v(1, k)\alpha_k{}^{-1}(v(1, 1)) \qquad (k\in\mathcal{G}),$$

so that

$$v(1, k)=\alpha_k{}^{-1}(v(1, 1)) \qquad (k\in\mathcal{G}).$$

By the definition of π_α and ρ, it follows that

$$\begin{aligned}(\rho(1)\xi)(k)&=v(1, k)\xi(k)=\alpha_k{}^{-1}(v(1, 1))\xi(k)\\ &=(\pi_\alpha(v(1, 1))\xi)(k) \qquad (k\in\mathcal{G}, \xi\in l^2(\mathfrak{H}, \mathcal{G})).\end{aligned}$$

Hence we have that $\rho(1)=\pi_\alpha(v(1, 1))$.

Therefore we have that for every element $\sum_{g\in\mathcal{G}} x(g)\rho(g)$ in $l^2(\mathfrak{K}, \mathcal{G})$,

$$e(\sum_{g\in\mathcal{G}} x(g)\rho(g))=x(1)\pi_\alpha(v(1, 1)).$$

Let $x=\sum_{k\in\mathcal{G}} x(k)\rho(k)$ be an element in $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$. By the property (6) and (7), we have that

$$\begin{aligned}x\rho(g^{-1})&=\sum_{k\in\mathcal{G}} x(k)\rho(kg^{-1})\pi_\alpha(v(k, g^{-1}))\\ &=\sum_{k\in\mathcal{G}} x(k)\pi_\alpha(\alpha_{kg}^{-1}(v(k, g^{-1})))\rho(kg^{-1}).\end{aligned}$$

Thus we have that for each g in $\mathcal{G}$

$$e(x\rho(g^{-1}))=x(g)\pi_\alpha(\alpha_1(v(g, g^{-1}))\rho(1)=x(g)\pi_\alpha(\alpha_1(v(g, g^{-1}))v(1, 1)),$$

which implies that

$$x(g)=e(x\rho(g^{-1}))\pi_\alpha(v(1, 1)^*\alpha_1(v(g, g^{-1})^*)) \qquad (g\in\mathcal{G}).$$

Hence $x(g)$ belongs to $\pi_\alpha(A)$.

THEOREM 2. *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{G}$ a discrete group. For a semirepresentation α of $\mathcal{G}$ into Aut($\mathcal{A}$), if every α_g $(g\neq 1)$ is freely acting on $\mathcal{A}$, then the relative commutant of $\pi_\alpha(\mathcal{A})$ in $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ is contained in the center of $\pi_\alpha(\mathcal{A})$.*

PROOF. Let x be an element in $\pi_\alpha(\mathcal{A})' \cap \mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$. By Lemma 1, x is expanded by the following form; $x = \sum_{g \in \mathcal{G}} x(g)\rho(g)$ $(x(g) \in \pi_\alpha(\mathcal{A}))$. Then we have that, for any a in $\pi_\alpha(\mathcal{A})$,

$$\sum_{g \in \mathcal{G}} a\,x(g)\rho(g) = \sum_{g \in \mathcal{G}} x(g)\rho(g)a = \sum_{g \in \mathcal{G}} x(g)\alpha_g(a)\rho(g).$$

Therefore, $a\,x(g) = x(g)\alpha_g(a)$ for every g in $\mathcal{G}$ and a in $\pi_\alpha(\mathcal{A})$. On the other hand, for g $(g \neq 1)$, α_g is freely acting. Hence we have that $x(g)=0$ for all g $(g \neq 1)$. Thus $x = x(1)$ is contained in the center of $\pi_\alpha(\mathcal{A})$.

For an automorphism of a factor, to be free is equivalent to be outer. So, we have the following corollary:

COROLLARY 3. *For a factor* $\mathcal{A}$ *and a discrete group* $\mathcal{G}$, *if* α *is a semirepresentation of* $\mathcal{G}$ *into* $Out(\mathcal{A})$, *then the relative commutant of* $\pi_\alpha(\mathcal{A})$ *in* $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ *is the scalar multiples of the identity.*

In the case where $\mathcal{A}$ is a II_1-factor and α the natural mapping of $Aut(\mathcal{A})$ onto $Out(\mathcal{A})$, Corollary 3 is showed in [**4**; Theorem 1].

We shall consider the converse version of Corollary 3.

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{A}$ a von Neumann subalgebra of $\mathcal{M}$. The set $\{\text{unitary } u \in \mathcal{M};\ u\mathcal{A}u^* = \mathcal{A}\}$ is called the normalizer of $\mathcal{A}$ in $\mathcal{M}$ and denoted by $N(\mathcal{A})$.

THEOREM 4. *Let* $\mathcal{M}$ *be a von Neumann algebra acting on a separable Hilbert space and* $\mathcal{A}$ *a subfactor of* $\mathcal{M}$. *If* $\mathcal{M}$ *and* $\mathcal{A}$ *satisfy the following conditions;*

(8) $\mathcal{M}$ *is generated by* $N(\mathcal{A})$,

(9) *the relative commutant of* $\mathcal{A}$ *in* $\mathcal{M}$ *is the scalar multiples of the identity and*

(10) *there is a faithful normal expectation of* $\mathcal{M}$ *onto* $\mathcal{A}$,

then $\mathcal{M}$ *is isomorphic to* $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ *for some discrete countable group* $\mathcal{G}$ *and some semirepresentation* α *of* $\mathcal{G}$ *into* $Out(\mathcal{A})$.

PROOF. The von Neumann algebra $\mathcal{M}$ is acting on a separable Hilbert space, so the unit ball of $\mathcal{M}$ is a σ-strongly complete separable metrizable space. Hence there is a countable set S dense in $N(\mathcal{A})$. Let $u(\mathcal{A})$ be the group of all unitaries in $\mathcal{A}$ and $\mathcal{K}$ the group generated by S and $u(\mathcal{A})$. Let $\mathcal{G}$ be the countable factor group $\mathcal{K}/u(\mathcal{A})$ of $\mathcal{K}$ by the normal subgroup $u(\mathcal{A})$. For each g in $\mathcal{G}$, let $\rho(g)$ be a representative element of g. Then for each pair g, h in $\mathcal{G}$, there is a unitary $v(g, h)$ in $\mathcal{A}$ such that

$$\rho(g)\rho(h) = \rho(gh)v(g, h).$$

For each g in $\mathcal{G}$, let α_g be an automorphism of $\mathcal{A}$ such that

$$\alpha_g(a) = \rho(g)a\rho(g)^* \qquad (a \in \mathcal{A}).$$

Then, for each $g\ (\neq 1)$ in $\mathcal{G}$, α_g is an outer automorphism of $\mathcal{A}$. In fact, if α_g is inner, there is a unitary v in $\mathcal{A}$ such that $\alpha_g(a)=vav^*$ for all a in $\mathcal{A}$. Then $v^*\rho(g)$ is contained in the relative commutant of $\mathcal{A}$ in $\mathcal{M}$. By the condition (9), $\rho(g)=\mu v$ for some scalar μ, so that $\rho(g)$ belongs to $\mathcal{A}$. Hence g is the unit of $\mathcal{G}$.

Thus α is a semirepresentation of $\mathcal{G}$ into $Out(\mathcal{A})$ and $v(g, h)$ is a factor set associated with $(\mathcal{G}, \alpha)$. Since, for each $g\ (\neq 1)$ in $\mathcal{G}$, α_g is an outer automorphism, it follows that $e(\rho(g))=0$ for each $g\ (\neq 1)$ in $\mathcal{G}$. Therefore, by [**1**; Theorem 7], $\mathcal{M}$ is isomorphic to $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$.

The condition (9) is equivalent to that $\mathcal{A}'\cap\mathcal{M}\subset\mathcal{A}$. In the case where $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{M}$, J. Feldman and C.C. Moore obtained a result similar to Theorem 4 by giving a construction of a von Neumann algebra from an abelian von Neumann algebra [**2**; Theorem 1].

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