J. Math. Soc. Japan
Vol. 31, No. 4, 1979

A counterexample to a conjecture of Whitehead and Volodin-Kuznetsov-Fomenko

By Mitsuyuki OCHIAI

(Received Feb. 13, 1978)
(Revised Sept. 11, 1978)

In the study of 3-manifolds, to contruct an algorithm of recognizing the standard 3-sphere S^3 among all 3-manifolds is a very important problem. The first basic work of this problem was done by Whitehead in 1936 [6], who discovered that certain (but not all) Heegaard diagrams for S^3 had a rather special geometric property (, see Conjecture A in the paper). Later Volodin-Kuznetsov-Fomenko conjectured that Heegaard diagrams for S^3 are reducible except for the canonical one. But Birman states in [2] that "nobody has succeeded in verifying such an assertion between 1935 and 1977, or producing a counterexample". Most recently Homma-Ochiai-Takahashi [3] proved that the conjecture is really true for the case of genus two. But in this paper we give a counterexample for the case of genus four. The Volodin-Kuznetsov-Fomenko-Whitehead algorithm is closely related with the algorithm to determine whether a knot is trivial or not and so our counterexample is constructed as a branched covering space over a trivial 5-bridge knot.

The author wishes to express his hearty thanks to Prof. T. Homma and Prof. H. Terasaka.

1. Reducible Heegaard diagrams.

Let M be a closed orientable 3-manifold and W_1, W_2 solid tori of genus n and $h:\partial W_2 \to \partial W_1$ a homeomorphism of the boundary surfaces. Then the triple $(W_1, W_2; h)$ is called a Heegaard splitting of genus n for M when $M = W_1 \cup_h W_2$.

A properly embedded disk D in a solid torus W of genus n is called a meridian-disk of W if $cl(W - N(D, W))$ is a solid torus of genus $n-1$, and a collection of mutually disjoint n meridian-disks $D_1, \cdots, D_n$ in W is called a *complete system of meridian-disks of* W if $cl(W - \bigcup_{i=1}^{n} N(D_i, W))$ is a 3-ball. We call a collection of mutually disjoint $(n+1)$ meridian-disks in W an *extended* complete system of meridian-disks of W provided that any n subcollection is a complete system of meridiam-disks of W.

Let $\{D_{i1}, D_{i2}, \cdots, D_{in}\}$ (resp. $\{D_{i1}, D_{i2}, \cdots, D_{in}, D_{in+1}\}$) be a complete system of meridian-disks (resp. an extended complete system of meridian-disks) of W_i, $i=1, 2$; and let $u_j=\partial D_{1j}$, $v'_j=\partial D_{2j}$ for $j=1, \cdots, n, n+1$. Let h be an attaching homeomorphism from ∂W_2 onto ∂W_1. Then the manifold $M=W_1 \underset{h}{\cup} W_2$ is determined up to homeomorphisms by the collection of circles $v_1, v_2, \cdots, v_n$ on ∂W_1 with $v_k=h(v'_k)$, $k=1, \cdots n$. We call the triad $(F; u, v)$ *a Heegaard diagram for M*, where $F=\partial W_1$ and $u=u_1 \cup \cdots \cup u_n$, $v=v_1 \cup \cdots \cup v_n$. Moreover we will call the triad $(F; \tilde{u}, \tilde{v})$ *an extended Heegaard diagram for M*, where $\tilde{u}=u \cup u_{n+1}$, $\tilde{v}=v \cup v_{n+1}$. The following Figure 1 illustrates the canonical (extended) Heegaard diagram for S^3.

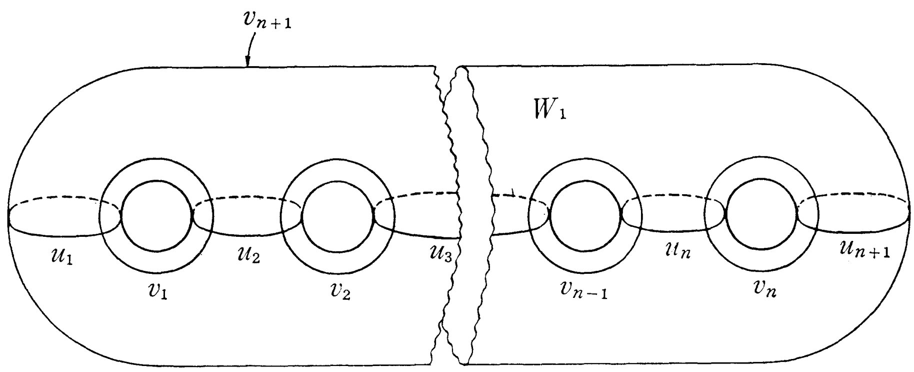

Figure 1. The canonical (extended) Heegaard diagram of genus n for S^3

Now the orientations of the circles $u_1, u_2, \cdots, u_n, (u_{n+1})$ and $v_1, v_2, \cdots, v_n, (v_{n+1})$ are supposed to be given. The $u \cup v$ gives rise to a partition of F into a set Γ of domains. Let U be a domain contained in Γ. Then each component of $\partial U \cap u_k$ and $\partial U \cap v_k$ for any k ($k=1, \cdots, n$) is called *an edge* of the domain U. A domain $U \in \Gamma$ is said to be *distinguished* if among the edges that form its boundary there are edges a_1, a_2 belonging to a single circle and if their orientations agree in any circuit around the boundary of U. The edges a_1, a_2 are also said to be *distinguished.* Furthermore the Heegaard diagram $(F; u, v)$ with the set Γ of domains is said to be W_1-*reducible* if Γ contains a distinguished domain with distinguished edges belonging to u, also W_2-*reducible* if they belong to v, and also *reducible* if it is W_1-reducible or W_2-reducible.

2. The Volodin-Kuznetsov-Fomenko-Whitehead Algorithm.

Whitehead [6] conjectured in 1936 that (Conjecture A): either the Whitehead graph of an arbitrary Heegaard diagram for S^3 has a cut-vertex or the dual graph has one (, see [2] and [6] in detail). Recently Volodin-Kuznetsov-Fomenko formulated differently his conjecture as Algorithm (A), that is, any Heegaard diagrams for S^3 are reducible except for the canonical one (, see [5] in detail). But we give a counterexample to their conjecture in the case of genus four. It will be noticed that, independently from [5], Homma conjectured that any Heegaard diagrams of genus two for S^3 except for the canonical one of genus two are reducible and recently Homma-Ochiai-Takahashi proved in [3] that Homma's conjecture is really true.

Let $(\partial W_1;\ \tilde{u}, \tilde{v})$ be an extended Heegaard diagram of genus four given by Figure 2, where $\tilde{u}=u_1\cup u_2\cup u_3\cup u_4\cup u_5$, $\tilde{v}=v_1\cup v_2\cup v_3\cup v_4\cup v_5$. It is clear

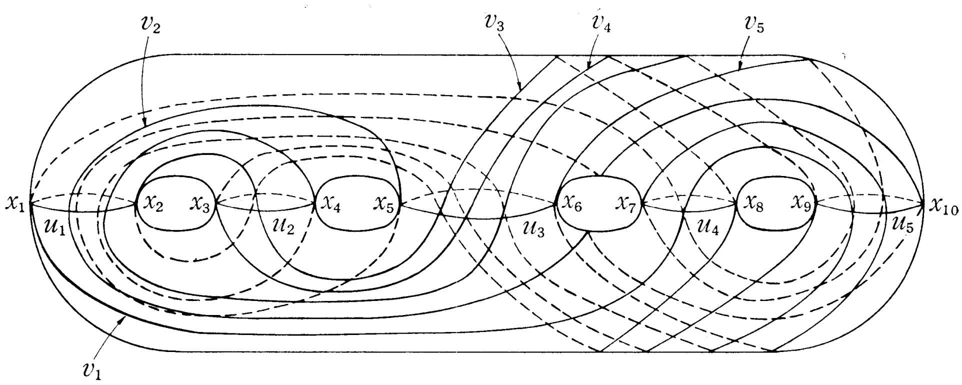

Figure 2

by the symmetry of $\tilde{v}$ with respect to $\tilde{u}$ that there is an orientation preserving involution T of W_1 with ten fixed points $x_1, x_2, \cdots, x_{10}$ on ∂W_1 such that $T(u_i)=u_i$ and $T(v_j)=v_j$ $(i, j=1, 2, \cdots, 5)$. Then, by Birman-Hilden [1] and Takahashi [4], the manifold induced by the extended Heegaard diagram is a branched covering space over the trivial 5-bridge knot illustrated in Figure 3 and so it is homeomorphic to S^3. The extended Heegaard diagram contains 25 Heegaard diagrams for S^3. Choose a Heegaard diagram $(\partial W_1;\ u, v)$ for S^3 among those diagrams, where $u=u_2\cup u_3\cup u_4\cup u_5$, $v=v_2\cup v_3\cup v_4\cup v_5$. Let Γ be the set of domains given by the Heegaard diagram. Then Γ contains nine domains $U_1, U_2, \cdots, U_9$ (, see Figure 2.1) such that by the involution T domains U_1, U_2, U_4, U_6, U_8 are mapped onto domains U_1, U_3, U_5, U_7, U_9, respectively. It is clear that all of the domains U_1, U_2, U_4, U_6, U_8 have no distinguished edges. Thus, even though the Heegaard diagram $(\partial W_1;\ u, v)$ gives S^3, it is not reducible. Hence Algorithm (A) is false in the case of genus four.

Next let us consider the Whitehead graph G_u of the diagram $(\partial W_1;\ u, v)$ and the dual graph G_v (, see the definition of Whitehead graphs and dual graphs in [2]). It is easily checked that G_u is the graph illustrated in

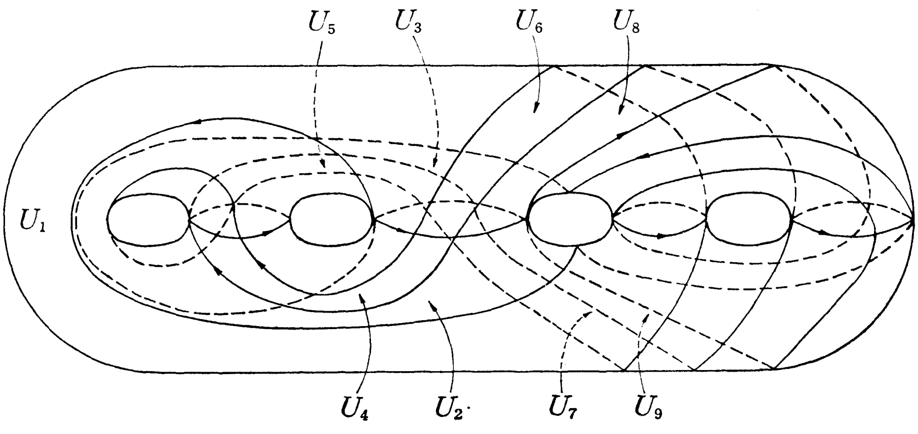

Figure 2.1

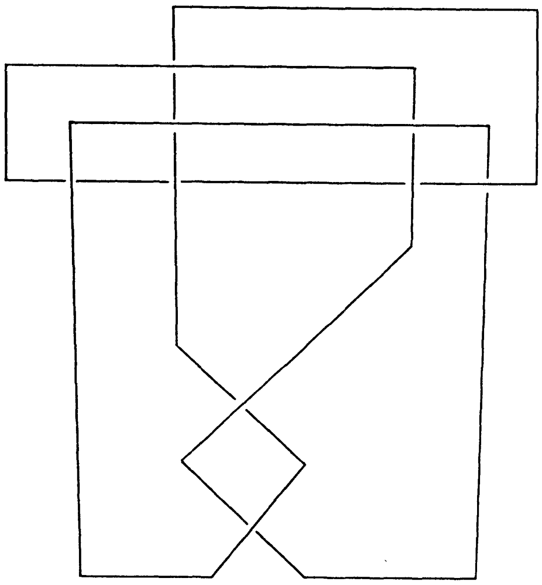

Figure 3

Figure 4. Moreover the Heegaard diagram $(\partial W_1;\, u,\, v)$ induces the following dual presentation for the fundamental group $\pi_1(S^3)$ of S^3;

$$\pi_1(S^3)=\langle v_2,\, v_3,\, v_4,\, v_5 \,|\, v_3^2 v_4 = v_2 v_3^{-1} v_4^{-1} v_5 v_4^{-1} v_3^{-1} = v_3 v_5 = v_4 v_5^{-1} v_2 v_5^{-1} = 1\rangle\,.$$

Thus G_v is the graph illustrated in Figure 5. But both of the graphs G_u, G_v have no cut-vertices (, see the definition of cut-vertices in [**2**]). Consequently, Conjecture A is false in the case of genus four. It will be noticed that the set of four words in the above presentation of $\pi_1(S^3)$ is not a simple set of words (, see [**6**]).

Remark that it remains an open question to determine whether Algorithm (A) is necessary in the case of genus three.

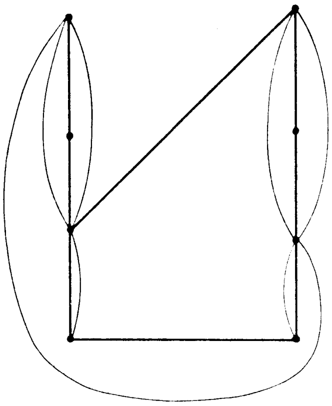

Figure 4

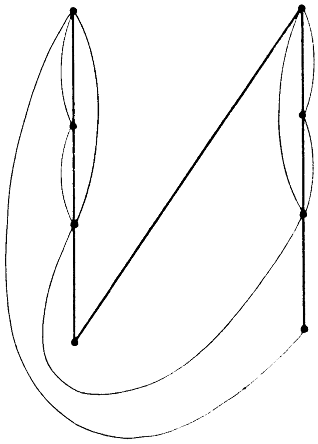

Figure 5

References

[1] J.S. Birman and H.M. Hilden, Heegaard splittings of branched coverings of S^3, Trans. Amer. Math. Soc., **213** (1975), 315-352.

[2] J.S. Birman, Heegaard splittings, Diagrams and Sewings for Closed, Orientable 3-manifolds, Lecture notes for CBMS conference at Oct. 8-12, 1977.

[3] T. Homma, M. Ochiai and M. Takahashi, An algorithm for recognizing S^3 in 3-manifolds with Heegaard splittings of genus two, to appear.

[4] M. Takahashi, A alternative proof of Birman-Hilden-Viro's theorem, to appear in Tsukuba Math. J.

[5] I.A. Volodin, V.E. Kuznetsov and A.T. Fomenko, The Problem of Discriminating algorithmically the Standard Three-Dimensional Sphere, Russian Math. Surveys, **29**: 5 (1974) 71-172.

[6] J.H.C. Whitehead, On certain sets of elements in a free group, London Math. Soc., (2) **41** (1936), 48-56.

Mitsuyuki OCHIAI

Department of Information Sciences
Faculty of Science
Tokyo Institute of Technology
Meguro-ku, Tokyo 152
Japan