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Singular perturbations of m-accretive operators

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Introduction.

Let S be a linear operator with domain $D(S)$ and range $R(S)$ in a Hilbert space H. Assume that

(♯) for all $\alpha>0$, $(1+\alpha S)^{-1}$ exists, $R(1+\alpha S)=H$ and there is a constant $M>0$ such that $\|(1+\alpha S)^{-1}\|\leqq M$.

The following three theorems are recently established by Professor T. Kato.

THEOREM 0.1. *Let A be a linear accretive operator in H. Let S be a linear operator in H, satisfying condition (♯), with $D(S)\subset D(A)$. Assume that there exist nonnegative constants a and b such that for all $u\in D(S)$,*

$$\mathrm{Re}(Au, Su)\geqq -a\|u\|^2-b\|Su\|\|u\|. \tag{0.1}$$

Then the closure $\tilde{A}$ of A is m-accretive and $D(S)$ is a core of $\tilde{A}$.

THEOREM 0.2. *In Theorem 0.1 assume further that S is m-accretive, i.e., $M=1$ in condition (♯). Then for $n=1, 2, \cdots$, $A_n=\frac{1}{n}S+A$ is also m-accretive and*

$$(\tilde{A}+\zeta)^{-1}=\operatorname*{s-lim}_{n\to\infty}\left(\frac{1}{n}S+A+\zeta\right)^{-1}, \qquad \mathrm{Re}\,\zeta>0.$$

Furthermore, A_n converges to $\tilde{A}$ strongly in the generalized sense.

For the notion of generalized strong convergence of closed linear operators we refer to Kato [**11**].

THEOREM 0.3. *Let A be a linear accretive operator in H. Let S be a nonnegative selfadjoint operator in H, with $D(S)\subset D(A)$. Assume that there is a constant $b\geqq 0$ such that for all $u\in D(S)$,*

$$\mathrm{Re}(Au, Su)\geqq -b(u, Su).$$

Let $0\leqq h\leqq 1/2$. Then $D(S^h)$ is invariant under $(\tilde{A}+\xi)^{-1}$, $\xi>2hb$, and for all $v\in D(S^h)$,

$$\|S^h(\tilde{A}+\xi)^{-1}v\|\leqq(\xi-2hb)^{-1}\|S^h v\|.$$

The purpose of this paper is to generalize the theorems stated above. In §1 we give a sufficient condition for the well-behaved singular perturbation of linear operators in a Banach space. As a simple consequence we obtain a perturbation theorem of Chernoff [5] and Okazawa [17]. § 2 is concerned with the regular perturbation of linear m-accretive operators in a Banach space. We shall generalize the theorems in Yoshikawa [22] and Okazawa [16], [18]. The result extends that of Gustafson [7]. Using the results in § 1 and § 2, we shall prove in § 3 two generalizations of Theorem 0.2 to the case of Banach space. To generalize the inequality (0.1) we need the notion of duality map. Let F be the duality map on a Banach space X to its adjoint X^*. Roughly speaking, we assume instead of (0.1),

$$\mathrm{Re}(Au, F(Su)) \geqq -a\|u\|^2 - b\|Su\|\|u\|, \tag{0.2}$$

or

$$\mathrm{Re}(Su, F(Au)) \geqq -a\|u\|^2 - b\|Au\|\|u\|. \tag{0.3}$$

Since $D(S)$ is included in $D(A)$, (0.3) is weaker than (0.2) if X is a Hilbert space, i. e., F is the identity. But, we can find the example in which (0.3), rather than (0.2), is satisfied. The purpose of § 4 is to give another proof of Theorem 0.1 and to show that A_n converges to $\tilde{A}$ strongly in the generalized sense without assuming the m-accretiveness of S. The result is closely related to a nice criterion for selfadjointness obtained by Faris-Lavine [6]. Assuming slightly more, we can generalize a theorem in Okazawa [20]. As an application we consider in § 5 some differential operators in L^p $(1<p<\infty)$. In particular, we can treat the Legendre operator A in $L^2(-1, 1)$:

$$Au(x) = -\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right]$$

with $D(A)=\{u\in H^1(-1, 1); (1-x^2)u(x)\in H^2(-1, 1)\}$. We shall show that A is the reasonable limit of a sequence of Sturm-Liouville operators

$$\left(\frac{1}{n}S+A\right)u(x) = -\frac{d}{dx}\left[\left(\frac{1}{n}+1-x^2\right)\frac{du}{dx}\right], \qquad n=1, 2, \cdots,$$

where $Su(x)=-u''(x)$ with $D(S)=\{u\in H^2(-1, 1); u'(-1)=u'(1)=0\}$. Here, it should be noted that the following inclusion holds:

$$D(S)\subset D(A)\subset D(S^{1/2})=H^1(-1, 1).$$

§ 6 is concerned with the regular perturbation of nonlinear m-accretive operators in a *real* Banach space with uniformly convex dual. We shall mention some criteria for the m-accretiveness of the sum of two m-accretive operators. Finally, in § 7 we shall try to generalize Theorem 0.3 to the case of nonlinear m-accretive A in a *real* Hilbert space. A simple example will be given at the end of this section.

The writer would like to thank Professor T. Kato for giving him a chance to learn the suggestive result before publication.

§1. Singular perturbation of linear operators.

Let X be a Banach space and X^* be the adjoint space of X. Let S be a linear operator with domain $D(S)$ and range $R(S)$ in X. We denote by S^* the adjoint operator of S when $D(S)$ is dense in X. Let A be a linear operator in X, with $D(A)\supset D(S)$.

Here we introduce two fundamental assumptions:

(I) There exists a complex number ξ such that

$$R\left(\frac{1}{n}S+A+\xi\right)=X, \qquad n=1, 2, \cdots;$$

hence for every $v\in X$ there exists a sequence $\{u_n\}$ in $D(S)$ such that

$$\frac{1}{n}Su_n+Au_n+\xi u_n=v. \tag{1.1}$$

(II) For every $v\in X$ both sequences $\{\|u_n\|\}$ and $\{\|Au_n\|\}$ are bounded. (Note that $\frac{1}{n}S+A+\xi$ is not necessarily invertible.)

Then we have

PROPOSITION 1.1. *Let A be a linear operator in X. Let S be a densely defined linear operator in X, with $D(S)\subset D(A)$ and $D(S^*)$ dense in X^*. Suppose that two assumptions* (I) *and* (II) *are satisfied. Then $R(A+\xi)$ is dense in X.*

PROOF. We shall show that $(A+\xi)D(S)$ is dense in X. To this end, let f be an element in X^* such that for all $u\in D(S)$, $((A+\xi)u, f)=0$. Then it follows from (1.1) that

$$(v, f)=\left(\frac{1}{n}Su_n, f\right)+((A+\xi)u_n, f)$$

$$=\frac{1}{n}(Su_n, f)\rightarrow 0 \qquad (n\rightarrow\infty).$$

In fact, $\left\|\frac{1}{n}Su_n\right\|$ is also bounded by assumption (II) and for every $f\in D(S^*)$,

$$\left|\frac{1}{n}(Su_n, f)\right|=\frac{1}{n}|(u_n, S^*f)|\leqq\frac{1}{n}\|u_n\|\|S^*f\|.$$

Thus, we obtain $(v, f)=0$ for all $v\in X$ and hence $f=0$. Q. E. D.

Now, a linear operator A in X is said to be *accretive* if

$$\|(A+\xi)u\|\geqq\xi\|u\| \qquad \text{for all} \quad u\in D(A) \quad \text{and} \quad \xi>0.$$

If in particular $R(A+\xi)=X$ for some (and hence for every) $\xi>0$, we say that A is *m-accretive*. Accordingly, an m-accretive operator is necessarily closed.

In this connection, let A be an arbitrary closed linear operator in X.

Then a linear manifold D contained in $D(A)$ is called a *core* of A if the closure of the restriction of A to D is again A.

THEOREM 1.2. *In Proposition* 1.1 *assume further that* A *is accretive and* $\xi>0$. *Then* A *is closable and its closure* $\tilde{A}$ *is m-accretive, and* $D(S)$ *is a core of* $\tilde{A}$.

PROOF. Since $D(A)$ is dense in X, A is closable (see Lumer-Phillips [**13**], Lemma 3.3) and $R(\tilde{A}+\xi)$ is a closed linear subspace in X. So, Proposition 1.1 implies $R(\tilde{A}+\xi)=X$. Namely, $\tilde{A}$ is m-accretive. To see that $D(S)$ is a core of $\tilde{A}$, it suffices to show that $(A+\xi)D(S)$ is dense in X (see Kato [**11**], III-§5.3). But, this fact is the key point in the proof of Proposition 1.1.

Q. E. D.

COROLLARY 1.3. *In Theorem* 1.2 *assume further that for* $n=1, 2, \cdots$, $\frac{1}{n}S+A+\xi$ *is invertible. Then* $\frac{1}{n}S+A$ *converges to* $\tilde{A}$ *strongly in the generalized sense.*

PROOF. By the invertibility we obtain from (1.1)

$$u_n=\left(\frac{1}{n}S+A+\xi\right)^{-1}v.$$

Since $\|u_n\|$ is bounded, $\left\|\left(\frac{1}{n}S+A+\xi\right)^{-1}\right\|$ is also bounded by the principle of uniform boundedness. Therefore, $\frac{1}{n}S+A$ converges strongly to $\tilde{A}$ in the generalized sense (see Kato [**11**], Theorem VIII-1.5). Q. E. D.

Let F be the duality map on X to X^*, i.e., for each $w\in X$, $F(w)=\{f\in X^*; (w, f)=\|w\|^2=\|f\|^2\}$. Then a linear operator A in X is accretive if and only if for every $u\in D(A)$ there is $f\in F(u)$ such that $\mathrm{Re}(Au, f)\geqq 0$ (see [**9**]). In this connection, we note that if A is m-accretive then $\mathrm{Re}(Au, f)\geqq 0$ for *all* $f\in F(u)$.

Applying Theorem 1.2, we obtain a result in [**5**] and [**17**].

COROLLARY 1.4. *Let* S *be a densely defined linear m-accretive operator in* X, *with* $D(S^*)$ *dense in* X^*. *Let* B *be a linear accretive operator in* X, *with* $D(B)\supset D(S)$. *Assume that there exists a constant* $a>0$ *such that for all* $u\in D(S)$,

$$\|Bu\|\leqq a\|u\|+\|Su\|.$$

Then $(S+B)^{\sim}$ *is also m-accretive.*

PROOF. Since $S+B$ is accretive, it suffices to show that two assumptions (I) and (II) with $A=S+B$ are satisfied. Let $t>0$. Then we have

$$\|Bu\|\leqq a\|u\|+(1+t)^{-1}\|(1+t)Su\|.$$

Since $(1+t)^{-1}<1$, it follows that $(1+t)S+B=tS+(S+B)$ is also m-accretive (see Gustafson [**7**]). Consequently, for every $v\in X$ there exists a family $\{u(t)\}$ in

$D(S)$ such that

$$tSu(t)+(S+B)u(t)+u(t)=v\,.$$

Now, in addition to $\|u(t)\|\leqq\|v\|$, we have

$$\begin{aligned}\|tSu(t)\|&=(1+t)\|Su(t)\|-\|Su(t)\|\\ &\leqq(1+t)\|Su(t)\|-(\|Bu(t)\|-a\|u(t)\|)\\ &\leqq\|(1+t)Su(t)+Bu(t)\|+a\|u(t)\|\\ &\leqq\|v\|+(1+a)\|u(t)\|\,.\end{aligned}$$

Therefore, $\|(S+B)u(t)\|\leqq(4+a)\|v\|$. Q. E. D.

REMARK 1.5. Starting from the m-accretiveness of $S+tB$ $(0<t<1)$, we can again obtain the same conclusion under the assumption that $D(B^*)$, rather than $D(S^*)$, is dense in X^*; see the proofs in [**5**] and [**17**]. For the case of reflexive Banach space see Lemma 3.2 below.

§ 2. Regular perturbation of linear m-accretive operators.

The result in this section will be used in the next section to show that the assumption (I) in § 1 is satisfied by the operators involved.

Let X be a Banach space and F be the duality map on X to X^*.

LEMMA 2.1. *Let A be a linear accretive operator in X. Let S be a densely defined linear m-accretive operator in X, with $D(S)\subset D(A)$. Assume that for every $u\in D(S)$ there exists $h\in F(Su)$ such that*

$$\mathrm{Re}(Au,\,h)\geqq-a\|u\|^2\,, \tag{2.1}$$

where $a\geqq 0$ is a constant. Then $S+A$ is also m-accretive.

PROOF. Since A is closable and $D(A)\supset D(S)$, there exists a constant $c>0$ such that for all $u\in D(S)$,

$$\|Au\|\leqq c(\|u\|+\|Su\|)$$

(see Kato [**11**], IV-§ 1.1). Let $[c]$ be the integral part of c, and set $\varepsilon=([c]+1)^{-1}$. Then we have

$$\|\varepsilon Au\|\leqq c\varepsilon\|u\|+c\varepsilon\|Su\|\,,\qquad c\varepsilon<1\,. \tag{2.2}$$

Therefore, by the theorem of Gustafson quoted in the proof of Corollary 1.4, $S+\varepsilon A$ is m-accretive.

Now we can prove that for all $u\in D(S)$,

$$\|Su\|\leqq\|(S+k\varepsilon A)u\|+(k\varepsilon a)^{1/2}\|u\|\,,\qquad k=1,\,2,\,\cdots.$$

In fact, we have by (2.1)

$$\|Su\|^2=(Su,\ h)\leqq(Su,\ h)+\mathrm{Re}(k\varepsilon Au,\ h)+k\varepsilon a\|u\|^2$$
$$=\mathrm{Re}((S+k\varepsilon A)u,\ h)+k\varepsilon a\|u\|^2$$
$$\leqq\|(S+k\varepsilon A)u\|\|Su\|+k\varepsilon a\|u\|^2.$$

So, we obtain from (2.2)

$$\|\varepsilon Au\|\leqq c\varepsilon\|u\|+c\varepsilon[\|(S+k\varepsilon A)u\|+(k\varepsilon a)^{1/2}\|u\|] \tag{2.3}$$
$$=c\varepsilon[1+(k\varepsilon a)^{1/2}]\|u\|+c\varepsilon\|(S+k\varepsilon A)u\|.$$

It follows from (2.3) with $k=1$ that $(S+\varepsilon A)+\varepsilon A=S+2\varepsilon A$ is m-accretive. Thus, we can prove inductively that $S+k\varepsilon A$ is m-accretive. In particular, $S+A=S+([c]+1)\varepsilon A$ is m-accretive. Q. E. D.

Now, a perturbation theorem in [18] and [22] is generalized as follows:

THEOREM 2.2. *Let A be a linear accretive operator in X. Let S be a densely defined linear m-accretive operator in X, with $D(S)\subset D(A)$. Assume that for every $u\in D(S)$ there exists $h\in F(Su)$ such that*

$$\mathrm{Re}(Au,\ h)\geqq-a\|u\|^2-b\|Su\|^2, \tag{2.4}$$

where a and $b<1$ are nonnegative constants. Then $S+A$ is also m-accretive.

PROOF. (2.4) can be written as

$$\mathrm{Re}((bS+A)u,\ h)\geqq-a\|u\|^2. \tag{2.5}$$

Multiplying (2.5) by $1-b$, we obtain

$$\mathrm{Re}((bS+A)u,\ h')\geqq-(1-b)a\|u\|^2,$$

where $h'\in F((1-b)Su)$. Since $(1-b)S$ is m-accretive and $bS+A$ is accretive, with $D((1-b)S)=D(bS+A)$, it follows from Lemma 2.1 that $(1-b)S+(bS+A)=S+A$ is m-accretive. Q. E. D.

REMARK 2.3. Let A and S be as in Theorem 2.2. Assume that there exist nonnegative constants a_1 and $b_1<1$ such that for all $u\in D(S)$,

$$\|Au\|\leqq a_1\|u\|+b_1\|Su\|.$$

Then (2.4) holds, with $b=b_1+\varepsilon<1$. In fact, for every $h\in F(Su)$,

$$\mathrm{Re}(Au,\ h)\geqq-\|Au\|\|Su\|\geqq-a_1\|u\|\|Su\|-b_1\|Su\|^2$$
$$\geqq-C_\varepsilon\|u\|^2-(b_1+\varepsilon)\|Su\|^2.$$

Therefore, $S+A$ is m-accretive. This means that Theorem 2.2 extends the result of Gustafson [7].

PROPOSITION 2.4. *Let A be a linear accretive operator in X. Let S be a densely defined linear m-accretive operator in X, with $D(S)\subset D(A)$. Assume that $D(A^*)$ is dense in X^* and that for every $u\in D(S)$ there exists $g\in F(Au)$ such that*

$$\text{(2.6)} \qquad \mathrm{Re}(Su, g) \geqq -a\|u\|^2,$$

where $a \geqq 0$ is a constant. Then $S+A$ is also m-accretive.

PROOF. As in the proof of Lemma 2.1, $S+\varepsilon A$ is m-accretive when $\varepsilon=([c]+1)^{-1}$.

Instead of (2.3) we can prove that for all $u \in D(S)$,

$$\text{(2.7)} \qquad \|\varepsilon Au\| \leqq (a\varepsilon)^{1/2}\|u\| + \|(S+k\varepsilon A)u\|,$$

where $k=1, 2, \cdots$. In fact, by the inequality (2.6) we have

$$\begin{aligned}\varepsilon\|Au\|^2 &\leqq k\varepsilon(Au, g) \leqq (k\varepsilon Au, g) + \mathrm{Re}(Su, g) + a\|u\|^2 \\ &= \mathrm{Re}((S+k\varepsilon A)u, g) + a\|u\|^2 \\ &\leqq \|(S+k\varepsilon A)u\|\|Au\| + a\|u\|^2.\end{aligned}$$

Solving this inequality, we obtain (2.7). Now, it follows from (2.7) with $k=1$ that the closure of $(S+\varepsilon A)+\varepsilon A$ is m-accretive (see Corollary 1.4 and Remark 1.5). But, (2.7) implies further that $S+k\varepsilon A$ is closed for $k=1, 2, \cdots$. Thus, we can prove inductively that $S+k\varepsilon A$ is m-accretive particularly when $k=[c]+1$. Q. E. D.

REMARK 2.5. When X is a Hilbert space, then (2.6) is the same as (2.1). In this case we note that $A(1+\alpha S)^{-1}+\alpha a$ is accretive (see Lemma 4.2 below).

§ 3. Singular perturbation of linear m-accretive operators.

In this section we shall prove two theorems on the singular perturbation of linear m-accretive operators in a Banach space.

Let X be a Banach space and F be the duality map on X to X^*.

THEOREM 3.1. *Let A be a linear accretive operator in X. Let S be a linear m-accretive operator in X, with $D(S) \subset D(A)$. Assume that*

(i) *for every $u \in D(S)$ there is $h \in F(Su)$ such that*

$$\text{(3.1)} \qquad \mathrm{Re}(Au, h) \geqq -a_1\|u\|^2 - c\|Su\|\|u\|,$$

where a_1 and c are nonnegative constants;

(ii) *$D(S)$ is dense in X and $D(S^*)$ is dense in X^*.*

Then the closure $\tilde{A}$ of A is m-accretive and $D(S)$ is a core of $\tilde{A}$. For $n=1, 2, \cdots, \frac{1}{n}S+A$ is m-accretive and

$$\text{(3.2)} \qquad (\tilde{A}+\zeta)^{-1} = \operatorname*{s-lim}_{n\to\infty}\left(\frac{1}{n}S+A+\zeta\right)^{-1}, \qquad \mathrm{Re}\,\zeta>0.$$

Furthermore, $\frac{1}{n}S+A$ converges to $\tilde{A}$ strongly in the generalized sense.

If in particular X is reflexive, then condition (ii) *is redundant.*

PROOF. By (3.1) we can find positive constants a_2 and $b<1$ such that for $n=1, 2, \cdots,$

$$\mathrm{Re}(Au,\ h)\geqq -a_1\|u\|^2-\Big(a_2 n\|u\|^2+\frac{b}{n}\|Su\|^2\Big). \tag{3.3}$$

Multiplying (3.3) by n^{-1}, we obtain

$$\mathrm{Re}(Au,\ h')\geqq -\Big(\frac{a_1}{n}+a_2\Big)\|u\|^2-b\Big\|\frac{1}{n}Su\Big\|^2,$$

where $h'\in F\Big(\frac{1}{n}Su\Big)$. So, it follows from Theorem 2.2 that $\frac{1}{n}S+A$ is m-accretive:

$$R\Big(\frac{1}{n}S+A+1\Big)=X, \qquad n=1, 2, \cdots.$$

Let u_n be a unique solution of the equation

$$\frac{1}{n}Su_n+Au_n+u_n=v. \tag{3.4}$$

Then, in addition to $\|u_n\|\leqq\|v\|$, we can show that

$$\Big\|\frac{1}{n}Su_n\Big\|^2\leqq(2+c)\Big\|\frac{1}{n}Su_n\Big\|\|v\|+\frac{a_1}{n}\|v\|^2. \tag{3.5}$$

In fact, by virtue of (3.1) there is $h_n\in F(Su_n)$ such that

$$\begin{aligned}\Big\|\frac{1}{n}Su_n\Big\|^2&=\frac{1}{n}\Big(\frac{1}{n}Su_n,\ h_n\Big)\\&\leqq\frac{1}{n}\mathrm{Re}\Big(\Big(\frac{1}{n}S+A\Big)u_n,\ h_n\Big)+\frac{a_1}{n}\|u_n\|^2+c\Big\|\frac{1}{n}Su_n\Big\|\|u_n\|\\&\leqq\Big\|\Big(\frac{1}{n}S+A\Big)u_n\Big\|\Big\|\frac{1}{n}Su_n\Big\|+\frac{a_1}{n}\|v\|^2+c\Big\|\frac{1}{n}Su_n\Big\|\|v\|.\end{aligned}$$

Noting that $\Big\|\Big(\frac{1}{n}S+A\Big)u_n\Big\|\leqq 2\|v\|$, we obtain (3.5). Solving the inequality (3.5), we have

$$\Big\|\frac{1}{n}Su_n\Big\|\leqq\Big[2+c+\Big(\frac{a_1}{n}\Big)^{1/2}\Big]\|v\|;$$

and hence $\|Au_n\|\leqq\Big[4+c+\Big(\frac{a_1}{n}\Big)^{1/2}\Big]\|v\|$. Thus, the conclusion follows from Theorem 1.2 and Corollary 1.3 (for (3.2) see Kato [**11**], VIII-§1).

The final assertion is a consequence of Lemma 3.2 below. Q. E. D.

The following important lemma is due to Kato [8] (see Yosida [**23**], VIII-§4).

LEMMA 3.2. *Let S be a linear operator in a reflexive Banach space X, satisfying*

($\sharp$) *for all* $\alpha>0$, $(1+\alpha S)^{-1}$ *exists*, $R(1+\alpha S)=X$ *and there is a constant* $M>0$ *such that* $\|(1+\alpha S)^{-1}\|\leqq M$.

Then $D(S)$ *is dense in* X. *Consequently, it follows from the closedness of* S *that* $D(S^*)$ *is also dense in* X^* (see Kato [11], Theorem III-5.29).

THEOREM 3.3. *Let* A *be a linear accretive operator in* X. *Let* S *be a linear m-accretive operator in* X, *with* $D(S)\subset D(A)$. *Assume that*

(i) *for every* $u\in D(S)$ *there is* $g\in F(Au)$ *such that*

$$\mathrm{Re}(Su,\ g)\geqq -a\|u\|^2-b\|Au\|\|u\|, \tag{3.6}$$

where a *and* b *are nonnegative constants;*

(ii) $D(S)$ *is dense in* X, *and both* $D(S^*)$ *and* $D(A^*)$ *are dense in* X^*.

Then $\tilde{A}$ *is m-accretive and* $D(S)$ *is a core of* $\tilde{A}$. *Furthermore,* (3.2) *holds and* $\frac{1}{n}S+A$ *converges to* $\tilde{A}$ *strongly in the generalized sense.*

If in particular X *is reflexive, then condition* (ii) *is redundant.*

PROOF. We see from (3.6) that for any $\varepsilon>0$ there is a constant $C_\varepsilon>0$ such that

$$\mathrm{Re}(Su,\ g)\geqq -a\|u\|^2-\Big(\frac{C_\varepsilon}{n}\|u\|^2+\varepsilon n\|Au\|^2\Big). \tag{3.7}$$

Multiplying (3.7) by $(1-\varepsilon)/n$, we obtain

$$\mathrm{Re}\Big(\Big(\frac{1}{n}S+\varepsilon A\Big)u,\ g'\Big)\geqq -(1-\varepsilon)\Big(\frac{a}{n}+\frac{C_\varepsilon}{n^2}\Big)\|u\|^2,$$

where $g'\in F((1-\varepsilon)Au)$. Since $\frac{1}{n}S+\varepsilon A$ is m-accretive (see Remark 2.3), it follows from Proposition 2.4 that $\Big(\frac{1}{n}S+\varepsilon A\Big)+(1-\varepsilon)A=\frac{1}{n}S+A$ is m-accretive.

It remains to show that $\|Au_n\|$ is bounded, where u_n is a unique solution of (3.4). By virtue of (3.6) there is $g_n\in F(Au_n)$ such that

$$\begin{aligned}\|Au_n\|^2&=(Au_n,\ g_n)\\ &\leqq \mathrm{Re}\Big(\Big(\frac{1}{n}S+A\Big)u_n,\ g_n\Big)+\frac{a}{n}\|u_n\|^2+\frac{b}{n}\|Au_n\|\|u_n\|\\ &\leqq \Big\|\Big(\frac{1}{n}S+A\Big)u_n\Big\|\|Au_n\|+\frac{a}{n}\|v\|^2+\frac{b}{n}\|Au_n\|\|v\|\\ &\leqq \Big(2+\frac{b}{n}\Big)\|Au_n\|\|v\|+\frac{a}{n}\|v\|^2.\end{aligned}$$

Solving this inequality, we see that $\|Au_n\|\leqq\Big[2+\frac{b}{n}+\Big(\frac{a}{n}\Big)^{1/2}\Big]\|v\|$. Applying Theorem 1.2 and Corollary 1.3, we obtain the same conclusion as that of Theorem 3.1. Q. E. D.

Before concluding this section, we give a remark on the contraction semigroups generated by $-\left(\frac{1}{n}S+A\right)$ and $-\tilde{A}$.

Let A and S be as in Theorem 3.1 (or Theorem 3.3). Let $U\left(t;\frac{1}{n}S+A\right)$ and $U(t;\tilde{A})$ be the semigroups generated by $-\left(\frac{1}{n}S+A\right)$ and $-\tilde{A}$, respectively. Then, as is well known, $U\left(t;\frac{1}{n}S+A\right)$ converges strongly to $U(t;\tilde{A})$:

$$U(t;\tilde{A})=\operatorname*{s-lim}_{n\to\infty} U\left(t;\frac{1}{n}S+A\right), \qquad t\geqq 0.$$

The convergence is uniform with respect to t in each finite subinterval of $[0, \infty)$.

Now $U\left(t;\frac{1}{n}S+A\right)$ is given by the Trotter product formula:

$$U\left(t;\frac{1}{n}S+A\right)=\operatorname*{s-lim}_{p\to\infty}\left[U\left(\frac{t}{p};\frac{1}{n}S\right)U\left(\frac{t}{p};\tilde{A}\right)\right]^p$$

(see e.g. Chernoff [4]). But since $U\left(\frac{t}{p};\frac{1}{n}S\right)=U\left(\frac{t}{pn};S\right)$ tends to the identity strongly as $n\to\infty$, it follows that

$$U(t;\tilde{A})=\left[U\left(\frac{t}{p};\tilde{A}\right)\right]^p=\operatorname*{s-lim}_{n\to\infty}\left[U\left(\frac{t}{p};\frac{1}{n}S\right)U\left(\frac{t}{p};\tilde{A}\right)\right]^p.$$

Thus, we obtain the equality:

$$\operatorname*{s-lim}_{n\to\infty}\left(\operatorname*{s-lim}_{p\to\infty}\left[U\left(\frac{t}{p};\frac{1}{n}S\right)U\left(\frac{t}{p};\tilde{A}\right)\right]^p\right)$$
$$=\operatorname*{s-lim}_{p\to\infty}\left(\operatorname*{s-lim}_{n\to\infty}\left[U\left(\frac{t}{p};\frac{1}{n}S\right)U\left(\frac{t}{p};\tilde{A}\right)\right]^p\right).$$

§4. Singular perturbation in a Hilbert space (linear case).

In this section we first consider a slightly more general class of linear operators including the class of linear m-accretive operators.

Let H be a Hilbert space. Then we have

THEOREM 4.1. *Let A be a linear accretive operator in H. Let S be a linear operator in H, satisfying condition (♯) in Introduction, with $D(S)\subset D(A)$. Assume that there exist nonnegative constants a and b such that for all $u\in D(S)$,*

$$\text{(4.1)}\qquad \operatorname{Re}(Au, Su)\geqq -a\|u\|^2-b\|Su\|\|u\|.$$

Then A is m-accretive and $D(S)$ is a core of $\tilde{A}$. Furthermore, for sufficiently large $\xi>0$,

$$(\tilde{A}+\xi)^{-1}=\text{s-}\lim_{n\to\infty}\left(\frac{1}{n}S+A+\xi\right)^{-1},$$

and $\frac{1}{n}S+A$ *converges to* $\tilde{A}$ *strongly in the generalized sense.*

The proof of this theorem is based on the following

LEMMA 4.2. *Let* $\varepsilon\in(0, 1]$. *Then under the assumption of Theorem* 4.1,

$$A\left(1+\frac{\varepsilon}{n}S\right)^{-1}+(a+b)M^2+bM, \qquad n=1, 2, \cdots,$$

is m-accretive.

PROOF. Since $D(S)$ is dense in H (see Lemma 3.2), it follows that A is closable (see Kato [**11**], Theorem V-3.4). Consequently, $A\left(1+\frac{\varepsilon}{n}S\right)^{-1}$ is a bounded linear operator on H. So, it suffices to show that for all $v\in H$,

$$\operatorname{Re}\left(A\left(1+\frac{\varepsilon}{n}S\right)^{-1}v,\ v\right)\geqq-[(a+b)M+b]M\|v\|^2.$$

Since A is accretive, it follows from (4.1) that

$$\operatorname{Re}\left(Au, \left(1+\frac{\varepsilon}{n}S\right)u\right)\geqq-\frac{\varepsilon}{n}a\|u\|^2-\frac{\varepsilon}{n}b\|Su\|\|u\|. \tag{4.2}$$

Let $v\in H$. Then $u_n=\left(1+\frac{\varepsilon}{n}S\right)^{-1}v\in D(S)$. Setting $u=u_n$ in (4.2), we obtain

$$\begin{aligned}&\operatorname{Re}\left(A\left(1+\frac{\varepsilon}{n}S\right)^{-1}v,\ v\right)\\ &\quad\geqq-\frac{\varepsilon}{n}a\left\|\left(1+\frac{\varepsilon}{n}S\right)^{-1}v\right\|^2-\frac{\varepsilon}{n}b\left\|S\left(1+\frac{\varepsilon}{n}S\right)^{-1}v\right\|\left\|\left(1+\frac{\varepsilon}{n}S\right)^{-1}v\right\|\\ &\quad\geqq-M^2a\|v\|^2-bM(1+M)\|v\|^2,\end{aligned}$$

where we have used condition (♯) and $\varepsilon\leqq n$. Q.E.D.

PROOF OF THEOREM 4.1. Let $\xi>[(a+b)M+b]M$. Taking $\varepsilon=\xi^{-1}$, we can write

$$\frac{1}{n}S+A+\xi=\left[\xi+A\left(1+\frac{\varepsilon}{n}S\right)^{-1}\right]\left(1+\frac{\varepsilon}{n}S\right).$$

Therefore, we see from condition (♯) and Lemma 4.2 that

$$R\left(\frac{1}{n}S+A+\xi\right)=H, \qquad n=1, 2, \cdots.$$

Now, let $u_n(\xi)$ be a unique solution of the equation $\frac{1}{n}Su_n(\xi)+Au_n(\xi)+\xi u_n(\xi)=v$. Then by the equality

$$\left(\frac{1}{n}S+A+\xi\right)^{-1}=\left(1+\frac{\varepsilon}{n}S\right)^{-1}\left[\xi+A\left(1+\frac{\varepsilon}{n}S\right)^{-1}\right]^{-1},$$

we obtain

$$\|u_n(\xi)\|\leqq M[\xi-(a+b)M^2-bM]^{-1}\|v\|.$$

It remains to show that for every $v\in H$, $\left\|\frac{1}{n}Su_n(\xi)\right\|$ is bounded. By virtue of (4.1) we have

$$\left\|\frac{1}{n}Su_n(\xi)\right\|^2=\left(\frac{1}{u}Su_n(\xi),\ \frac{1}{n}Su_n(\xi)\right)$$

$$\leqq \mathrm{Re}\left(\left(\frac{1}{n}S+A\right)u_n(\xi),\ \frac{1}{n}Su_n(\xi)\right)+\frac{a}{n}\|u_n(\xi)\|^2+\frac{b}{n}\|Su_n(\xi)\|\|u_n(\xi)\|$$

$$\leqq[\|v\|+(\xi+b)\|u_n(\xi)\|]\left\|\frac{1}{n}Su_n(\xi)\right\|+\frac{a}{n}\|u_n(\xi)\|^2.$$

This implies that

$$\left\|\frac{1}{n}Su_n(\xi)\right\|\leqq\|v\|+\left[\xi+b+\left(\frac{a}{n}\right)^{1/2}\right]\|u_n(\xi)\|.$$

Thus, the conclusion follows from Theorem 1.2 and Corollary 1.3. Q. E. D.

REMARK 4.3. In order to see that $\tilde{A}$ is m-accretive, it is easier to show that A^* is accretive (see [**19**], § 1). In fact, it follows from (4.1) that for all $u\in D(S)$ and $\alpha>0$,

$$\mathrm{Re}\,(Au,\ (1+\alpha S)u)\geqq-\alpha a\|u\|^2-b\|\alpha Su\|\|u\|.$$

Setting $u=(1+\alpha S)^{-1}v$, $v\in D(A^*)$, we have

$$\mathrm{Re}\,((1+\alpha S)^{-1}v,\ A^*v)\geqq-\alpha a\|(1+\alpha S)^{-1}v\|^2$$
$$-b\|v-(1+\alpha S)^{-1}v\|\|(1+\alpha S)^{-1}v\|.$$

Going to the limit $\alpha\to+0$, we obtain $\mathrm{Re}(v,\ A^*v)\geqq 0$ for all $v\in D(A^*)$.

Now, we give a generalization of an approximation theorem obtained in [**20**].

THEOREM 4.4. *Let A be a linear accretive operator in H. Let S be a nonnegative selfadjoint operator in H, with $D(S)\subset D(A)$. Assume that*

(i) *there exist nonnegative constants a and b such that for all $u\in D(S)$ the inequality* (4.1) *holds;*

(ii) $D(\tilde{A})\subset D(S^{1/2})$, *where $S^{1/2}$ denotes the square root of S.*

Then, in addition to the conclusion of Theorem 3.1, *for every ζ with* $\mathrm{Re}\,\zeta>0$ *there is a constant $c(\zeta)>0$ such that*

$$\left\|(\tilde{A}+\zeta)^{-1}-\left(\frac{1}{n}S+A+\zeta\right)^{-1}\right\|\leqq\frac{c(\zeta)}{\sqrt{n}},\qquad n=1,\ 2,\ \cdots;\tag{4.3}$$

hence the compactness of $\left(\frac{1}{n}S+A+\zeta\right)^{-1}$ *implies that of* $(A+\zeta)^{-1}$.

PROOF. Since S is m-accretive, the conclusion of Theorem 3.1 follows from condition (i). Let ζ be a complex number with $\operatorname{Re}\zeta>0$. Then for every $v\in H$ there are $u(\zeta)\in D(\tilde{A})$ and $u_n(\zeta)\in D(S)$ such that $\tilde{A}u(\zeta)+\zeta u(\zeta)=v$ and

$$\frac{1}{n}Su_n(\zeta)+Au_n(\zeta)+\zeta u_n(\zeta)=v, \qquad n=1, 2, \cdots.$$

Hence we can write

$$\zeta[u(\zeta)-u_n(\zeta)]=-[\tilde{A}u(\zeta)-Au_n(\zeta)]+\frac{1}{n}Su_n(\zeta).$$

So, we have

$$\begin{aligned}\operatorname{Re}\zeta\|u(\zeta)-u_n(\zeta)\|^2 &\leqq \frac{1}{n}\operatorname{Re}(Su_n(\zeta),\ u(\zeta)-u_n(\zeta))\\ &\leqq \frac{1}{2n}\|S^{1/2}u(\zeta)\|^2-\frac{1}{2n}\|S^{1/2}u_n(\zeta)\|^2.\end{aligned}$$

Now, by virtue of condition (ii) there is a constant $c>0$ such that for all $u\in D(\tilde{A})$, $\|S^{1/2}u\|\leqq c(\|u\|+\|\tilde{A}u\|)$. Consequently, $\|S^{1/2}u(\zeta)\|$ is estimated as follows:

$$\begin{aligned}\|S^{1/2}u(\zeta)\| &\leqq c\|u(\zeta)\|+c\|(\tilde{A}+i\operatorname{Im}\zeta)u(\zeta)\|+c|\operatorname{Im}\zeta|\,\|u(\zeta)\|\\ &\leqq c(1+|\operatorname{Im}\zeta|)\|u(\zeta)\|+c\|(\tilde{A}+\zeta)u(\zeta)\|\\ &\leqq c\left(1+\frac{1+|\operatorname{Im}\zeta|}{\operatorname{Re}\zeta}\right)\|v\|.\end{aligned}$$

Setting $c(\zeta)=\dfrac{c}{\sqrt{2\operatorname{Re}\zeta}}\left(1+\dfrac{1+|\operatorname{Im}\zeta|}{\operatorname{Re}\zeta}\right)$, we obtain (4.3). Q. E. D.

As is well known, a criterion for m-accretiveness includes a criterion for selfadjointness. So, we can deduce from Theorem 4.4 a generalization of a result obtained by Faris-Lavine (see [6], Theorem 1).

THEOREM 4.5. *Let A be a (Hermitian) symmetric operator in H. Let S be a nonnegative selfadjoint operator in H, with $D(S)\subset D(A)$. Assume that there exist nonnegative constants a and b such that for all $u\in D(S)$,*

$$\pm i[(Au,\ Su)-(Su,\ Au)]\leqq 2a\|u\|^2+2b\|Su\|\|u\|. \tag{4.4}$$

Then A is essentially selfadjoint on $D(S)$, i.e., $\tilde{A}$ is selfadjoint and $D(S)$ is a core of $\tilde{A}$.

PROOF. It suffices to show that $R(1\pm iA)$ are dense in H. The left-hand side of (4.4) is equal to

$$(\pm iAu,\ Su)+(Su,\ \pm iAu)=2\operatorname{Re}(\pm iAu,\ Su).$$

So, we obtain from (4.4)

(4.5) $$\mathrm{Re}(\pm iAu,\, Su) \geqq -a\|u\|^2 - b\|Su\|\|u\|.$$

Since $\pm iA$ are accretive, it follows from Theorem 4.4 that the closures of $\pm iA$ are m-accretive. Namely, the closures of $R(1\pm iA)$ are equal to H. Q. E. D.

REMARK 4.6. Let A and S be as in Theorem 4.5. Then it follows from (4.5) that

$$\mathrm{Re}\left(\pm i\left(\frac{1}{n}S+A\right)u,\, Su\right) \geqq -a\|u\|^2 - b\|Su\|\|u\|.$$

Therefore, $\left(\frac{1}{n}S+A\right)^{\sim}$ is also selfadjoint. Consequently, for every nonreal ζ,

$$(\tilde{A}-\zeta)^{-1} = \operatorname*{s-lim}_{n\to\infty}\left[\left(\frac{1}{n}S+A\right)^{\sim}-\zeta\right]^{-1},$$

and $\left(\frac{1}{n}S+A\right)^{\sim}$ converges to $\tilde{A}$ strongly in the generalized sense (see Kato [11], VIII-§ 1).

§ 5. Applications.

This section is devided into three subsections.

5.1. A degenerate elliptic operator in $L^p(\Omega)$, $1<p<\infty$.

Let Ω be a bounded domain in R^N which lies locally on one side of its boundary Γ, which we assume is a compact C^∞-manifold. We denote by $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ the usual Sobolev spaces: $W^{0,p}(\Omega)=L^p(\Omega)$. But we restrict ourselves to the case of $p\in(1,\infty)$.

Let $a(x)\geqq 0$ be a function of class $C^\infty(\bar{\Omega})$, and set

$$D(A)=\{u\in W_0^{1,p}(\Omega)\,;\ a(x)u(x)\in W^{2,p}(\Omega)\}.$$

Then we can define a linear operator A in $X=L^p(\Omega)$ by

$$Au(x) = -a(x)\Delta u(x), \qquad u\in D(A),$$

where Δ is the Laplacian. We want to show that $A+(M/p)$ is accretive when we set $M=\max\{\Delta a(x)\,;\ x\in\bar{\Omega}\}$.

To see this, we first note that the duality map F on $L^p(\Omega)$ is given by

$$F(u)=\|u\|^{2-p}u(x)|u(x)|^{p-2}, \qquad u\in L^p(\Omega).$$

So, we have

$$(Au,\, F(u)) = -\|u\|^{2-p}\int_\Omega a(x)\Delta u(x)\overline{u(x)}|u(x)|^{p-2}dx.$$

By a simple calculation we obtain

$$\|u\|^{p-2}\operatorname{Re}(Au,\ F(u))=\frac{1}{p}\sum_{i=1}^{N}\int_{\Omega}\frac{\partial a}{\partial x_i}\frac{\partial}{\partial x_i}|u(x)|^p dx$$

$$+\sum_{i=1}^{N}\int_{\Omega}a(x)\left[\left|\frac{\partial u}{\partial x_i}\right|^2|u(x)|^2+(p-2)\left\{\operatorname{Re}\frac{\partial u}{\partial x_i}\overline{u(x)}\right\}^2\right]|u(x)|^{p-4}dx\,.$$

The second term on the right-hand side is obviously nonnegative for $p\geqq 2$. If $p<2$ then it is larger than

$$(p-1)\sum_{i=1}^{N}\int_{\Omega}a(x)\left|\frac{\partial u}{\partial x_i}\right|^2|u(x)|^{p-2}dx\,.$$

Consequently, we obtain

$$\operatorname{Re}(Au,\ F(u))\geqq-\frac{\|u\|^{2-p}}{p}\int_{\Omega}|u(x)|^p\Delta a(x)dx$$

$$\geqq-\frac{M}{p}\|u\|^2,\qquad u\in D(A)\,.$$

Now let S be the minus Laplacian with Dirichlet condition: $Su(x)=-\Delta u(x)$ for $u\in D(S)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Then, as was shown above, S is accretive and it is well known that S is m-accretive in $L^p(\Omega)$. Furthermore, we see that $D(S)\subset D(A)$ and for all $u\in D(S)$,

$$\operatorname{Re}(Au,\ F(Su))=\int_{\Omega}a(x)|\Delta u(x)|^p dx\geqq 0\,.$$

So, we obtain

$$\operatorname{Re}\left(Au+\frac{M}{p}u,\ F(Su)\right)\geqq\frac{M}{p}\operatorname{Re}(u,\ F(Su))$$

$$\geqq-\frac{M}{p}\|u\|\|Su\|\,.$$

Since X is reflexive, the assumption of Theorem 3.1 is satisfied.

REMARK 5.1. The above fact is related to an observation stated in Shimakura [**21**].

5.2. An ordinary differential operator in $L^p(0,1)$, $1<p<\infty$.

It is easy to find the examples of ordinary differential operators satisfying the assumption of Theorem 3.3.

To do this, we choose two linear operators A and S in $X=L^p(0,1)$ as follows:

$$Au(x)=u'(x)\qquad\text{with}\quad u(0)=u(1)\,;$$

$$Su(x)=-u''(x)\qquad\text{with}\quad u(0)=u(1)\quad\text{and}\quad u'(0)=u'(1)\,.$$

Then both A and S are accretive, with $D(S)\subset D(A)$. Furthermore, we have

$$\operatorname{Re}(Su,\ F(Au))=-\frac{\|Au\|^{2-p}}{2}\int_0^1|u'(x)|^{p-2}\frac{d}{dx}|u'(x)|^2dx$$

$$=-\frac{\|Au\|^{2-p}}{p}\int_0^1\frac{d}{dx}|u'(x)|^p dx=0.$$

Since S is m-accretive (see e. g. Martin [14]), the conclusion of Theorem 3.3 holds good.

REMARK 5.2. If in particular $p=2$ in the above example, it can be shown that for every ζ with $\mathrm{Re}\,\zeta>0$,

$$\left\|(A+\zeta)^{-1}-\left(\frac{1}{n}S+A+\zeta\right)^{-1}\right\|=O(n^{-1/2}), \qquad n\to\infty.$$

In fact, $S=A^*A$ in this case and so we can apply a result in [20].

5.3. A degenerate elliptic operator in $L^2(\Omega)$.

Let Ω be a bounded domain in R^N with smooth boundary Γ as in §5.1. We shall use the abbreviation: $H^k(\Omega)=W^{k,2}(\Omega)$, $H_0^k(\Omega)=W_0^{k,2}(\Omega)$.

Let $\phi(x)\geqq 0$ be a function of class $C^\infty(\bar{\Omega})$ such that $\phi(x)=0$ for $x\in\Gamma$ and therefore

$$\frac{\partial\phi}{\partial\nu}(x)\leqq 0 \qquad \text{for} \quad x\in\Gamma, \tag{5.1}$$

where ν denotes the unit outward normal on Γ. Set

$$D(A)=\{u\in H^1(\Omega)\,;\,\phi(x)u(x)\in H^2(\Omega)\}.$$

Then we can define a linear operator A in $H=L^2(\Omega)$ by

$$Au(x)=-\sum_{j=1}^N\frac{\partial}{\partial x_j}\left[\phi(x)\frac{\partial u}{\partial x_j}\right].$$

By assumption A is accretive and symmetric.

Now let S be the minus Laplacian with Neumann condition: $Su(x)=-\Delta u(x)$ for $u\in D(S)=\{u\in H^2(\Omega)\,;\,\partial u/\partial\nu=0 \text{ on } \Gamma\}$. Then S is a nonnegative selfadjoint operator in $H=L^2(\Omega)$, with $D(S^{1/2})=H^1(\Omega)$. Set

$$M=\max\left\{\left|\frac{\partial^2\phi}{\partial x_i\partial x_j}\right|\,;\,x\in\bar{\Omega},\ 1\leqq i,\ j\leqq N\right\}.$$

Since $D(S)\subset D(A)$, we can show that for all $u\in D(S)$,

$$\mathrm{Re}\,(Au,\,Su)\geqq-\frac{3N}{2}M(u,\,Su). \tag{5.2}$$

In fact, let $u(x)$ be a function of class $C^2(\bar{\Omega})$ such that $\partial u/\partial\nu=0$ on Γ. Then we have

$$(Au,\,Su)=\int_\Omega\phi(x)|\Delta u(x)|^2dx+\int_\Omega\sum_{j=1}^N\frac{\partial\phi}{\partial x_j}\frac{\partial u}{\partial x_j}\overline{\Delta u(x)}dx$$

$$=\int_\Omega\phi(x)|\Delta u(x)|^2dx-\sum_{i,j=1}^N\int_\Omega\frac{\partial^2\phi}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_j}\overline{\frac{\partial u}{\partial x_i}}dx$$

$$-\sum_{i,j=1}^{N}\int_{\Omega}\frac{\partial\phi}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\overline{\partial u}}{\partial x_i}dx\,.$$

So, we obtain

$$\mathrm{Re}(Au,\ Su)\geqq-\sum_{i,j=1}^{N}\int_{\Omega}\frac{\partial^2\phi}{\partial x_i\partial x_j}\mathrm{Re}\frac{\partial u}{\partial x_j}\frac{\overline{\partial u}}{\partial x_i}dx$$

$$-\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}\sum_{j=1}^{N}\frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}\left|\frac{\partial u}{\partial x_i}\right|^2dx\,.$$

The second term on the right-hand side is equal to

$$\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}\Delta\phi(x)\left|\frac{\partial u}{\partial x_i}\right|^2dx-\frac{1}{2}\sum_{i=1}^{N}\int_{\Gamma}\frac{\partial\phi}{\partial\nu}\left|\frac{\partial u}{\partial x_i}\right|^2d\sigma\,.$$

We see from (5.1) that for all $u\in D(S)$,

$$\mathrm{Re}(Au,\ Su)\geqq-\sum_{i,j=1}^{N}\int_{\Omega}\left|\frac{\partial^2\phi}{\partial x_i\partial x_j}\right|\left|\frac{\partial u}{\partial x_j}\right|\left|\frac{\partial u}{\partial x_i}\right|dx$$

$$-\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}|\Delta\phi(x)|\left|\frac{\partial u}{\partial x_i}\right|^2dx$$

$$\geqq-\left(NM+\frac{N}{2}M\right)\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^2dx\,.$$

Namely, (5.2) holds. Therefore, it follows from Theorem 3.1 that $\tilde{A}$ is m-accretive. Furthermore, since A is symmetric, $\frac{1}{n}S+A$ $(n=1,2,\cdots)$ and $\tilde{A}$ are selfadjoint operators in $H=L^2(\Omega)$.

In the rest of this subsection assume further that $\phi(x)>0$ for $x\in\Omega$ and $\partial\phi/\partial\nu<0$ on Γ. Then A itself is a nonnegative selfadjoint operator in $H=L^2(\Omega)$ (see e. g. Baouendi-Goulaouic [**1**]). Consequently, we see that

$$D(A)=D(\tilde{A})\subset D(S^{1/2})=H^1(\Omega)\,.$$

Thus, the assumption of Theorem 4.4 is satisfied.

Set $\phi_n(x)=\frac{1}{n}+\phi(x)$ for $x\in\bar{\Omega}$ $(n\geqq1)$. Then we have

$$\left(\frac{1}{n}S+A\right)u(x)=-\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left[\phi_n(x)\frac{\partial u}{\partial x_j}\right]$$

with Neumann condition. The properties of this operator are well known. In particular, for every ζ with $\arg\zeta\neq0$, $\left(\frac{1}{n}S+A+\zeta\right)^{-1}$ is compact. Consequently, it follows from Theorem 4.4 that $(A+\zeta)^{-1}$ is also compact. Thus, A has a discrete spectrum consisting entirely of nonnegative eigenvalues with finite multiplicities.

Finally, let $N=1$, $\Omega=(-1,1)$ and $\phi(x)=1-x^2$. Then A becomes the Legendre operator in $H=L^2(-1, 1)$. In this case, the above result is supported by the well known fact that the spectrum of A consists of simple eigenvalues alone: $\lambda_l=l(l+1)$, $l=0, 1, 2, \cdots$.

REMARK 5.3. Let $c(x)\geqq 0$ be a function of class $C^2(\bar{\Omega})$ such that $\partial c/\partial\nu\geqq 0$ on Γ. Then we define a linear operator B in $H=L^2(\Omega)$ by

$$Bu(x)=c(x)u(x), \qquad u\in L^2(\Omega).$$

Obviously, B is a bounded accretive operator on $L^2(\Omega)$. Furthermore, for all $u\in D(S)$ we have

$$\begin{aligned}(Bu, Su)&=-\int_\Omega c(x)u(x)\overline{\Delta u(x)}\,dx\\ &=\int_\Omega \sum_{i=1}^N c(x)\left|\frac{\partial u}{\partial u_i}\right|^2 dx+\int_\Omega \sum_{i=1}^N \frac{\partial c}{\partial x_i}\frac{\partial}{\partial x_i}|u(x)|^2 dx\\ &\geqq\int_\Gamma \frac{\partial c}{\partial \nu}|u(x)|^2 d\sigma-\int_\Omega |u(x)|^2\Delta c(x)\,dx.\end{aligned}$$

Setting $M_1=\max\{\Delta c(x)\,;\, x\in\bar{\Omega}\}$, we obtain

$$\mathrm{Re}(Bu, Su)\geqq -M_1\|u\|^2, \qquad u\in D(S).$$

Therefore, it follows from (5.2) that for all $u\in D(S)$,

$$\mathrm{Re}((A+B)u, Su)\geqq -\frac{3N}{2}M(u, Su)-M_1\|u\|^2.$$

If in particular $c(x)>0$ on $\bar{\Omega}$ then we have $R(A+B)=H$.

§ 6. Regular perturbation of nonlinear m-accretive operators.

First we give a definition of nonlinear (multi-valued) m-accretive operators in a real Banach space X.

An operator A in X is said to be *accretive* if for each $\lambda>0$ and $u, v\in D(A)$,

$$\|x-y\|\geqq\|u-v\| \qquad \text{whenever} \quad x\in(1+\lambda A)u, \quad y\in(1+\lambda A)v.$$

Let F be the duality map of X into its dual space X^*. Then A is accretive if and only if for each $u, v\in D(A)$ there exists $f\in F(u-v)$ such that $(x-y, f)\geqq 0$ for each $x\in Au$, $y\in Av$. We say that an accretive operator A is *m-accretive* if $R(1+\lambda A)=X$ for some (and hence for every) $\lambda>0$.

Now let A and B be two m-accretive operators in X. Let B_ε be the *Yosida approximation* of B:

$$B_\varepsilon=\varepsilon^{-1}[1-(1+\varepsilon B)^{-1}], \qquad \varepsilon>0.$$

Then it is well known that $A+B_\varepsilon$ is again m-accretive. Accordingly, for each

$v \in X$ there exists a unique solution u_ε of the equation

$$u_\varepsilon + y_\varepsilon + B_\varepsilon u_\varepsilon = v\,, \qquad y_\varepsilon \in A u_\varepsilon\,. \tag{6.1}$$

The following lemma is fundamental (see Barbu [**2**], II-§3.2, or Konishi [**12**]).

LEMMA 6.1. *If $D(A) \cap D(B)$ is non-empty, then $\{\|u_\varepsilon\|\}$ is bounded as $\varepsilon \to +0$. Assume further that X^* is uniformly convex and that for each $v \in X$, $\{\|B_\varepsilon u_\varepsilon\|\}$ is bounded as $\varepsilon \to +0$. Then $A+B$ is also m-accretive.*

Next we give a simple sufficient condition for $\|B_\varepsilon u_\varepsilon\|$ to be bounded.

LEMMA 6.2. *Let X^* be uniformly convex. Let A and B be m-accretive operators in X, with $D(A) \cap D(B)$ non-empty. Assume that there exist a nonnegative constant $b<1$ and a nondecreasing function $\phi(r) \geqq 0$ of $r \geqq 0$ such that for each $u \in D(A)$,*

$$(y, F(B_\varepsilon u)) \geqq -\phi(\|u\|) - b\|B_\varepsilon u\|^2 \qquad \textit{whenever} \quad y \in Au\,. \tag{6.2}$$

Then $A+B$ is also m-accretive.

PROOF. It suffices by Lemma 6.1 to show that $\|B_\varepsilon u_\varepsilon\|$ is bounded. It follows from (6.1) and (6.2) that

$$\begin{aligned}(v-u_\varepsilon, F(B_\varepsilon u_\varepsilon)) &= (y_\varepsilon, F(B_\varepsilon u_\varepsilon)) + \|B_\varepsilon u_\varepsilon\|^2 \\ &\geqq (1-b)\|B_\varepsilon u_\varepsilon\|^2 - \phi(\|u_\varepsilon\|)\,.\end{aligned}$$

Thus, we obtain

$$(1-b)\|B_\varepsilon u_\varepsilon\|^2 \leqq (\|v\| + \|u_\varepsilon\|)\|B_\varepsilon u_\varepsilon\| + \phi(\|u_\varepsilon\|)$$

and hence

$$(1-b)\|B_\varepsilon u_\varepsilon\| \leqq \|v\| + \|u_\varepsilon\| + [(1-b)\phi(\|u_\varepsilon\|)]^{1/2}\,.$$

Since $\|u_\varepsilon\|$ is bounded (see Lemma 6.1), so is $\|B_\varepsilon u_\varepsilon\|$, too. Q. E. D.

REMARK 6.3. Lemma 6.2 is a slight generalization of a result obtained by Barbu (see [**2**], Theorem II-3.6). If in particular A and B are linear m-accretive operators in a Hilbert space, then we can mention about the case of $b=1$ (see [**19**], Theorem 2.1).

The following theorem will be used in the next section.

THEOREM 6.4. *Let X^* be uniformly convex. Let A be a nonlinear m-accretive operator in X. Let S be a linear m-accretive operator in X, with $D(S) \subset D(A)$. Assume that there exist a nonnegative constant $b<1$ and a non-decreasing function $\phi(r)$ of $r \geqq 0$ such that for each $u \in D(S)$,*

$$(y, F(Su)) \geqq -\phi(\|u\|) - b\|Su\|^2 \qquad \textit{whenever} \quad y \in Au\,. \tag{6.3}$$

Then $S+A$ is also m-accretive.

PROOF. Let S_ε be the Yosida approximation of S. Then, since S is linear, we have

$$S_\varepsilon = \varepsilon^{-1}[1-(1+\varepsilon S)^{-1}] = S(1+\varepsilon S)^{-1}\,.$$

Now let $u\in D(A)$. Then for $y\in Au$ and $z\in A(1+\varepsilon S)^{-1}u$ we can write

$$\begin{aligned}\varepsilon(y, F(S_\varepsilon u))&=(y, F(\varepsilon S_\varepsilon u))=(y, F(u-(1+\varepsilon S)^{-1}u))\\&=(y-z, F(u-(1+\varepsilon S)^{-1}u))+(z, F(\varepsilon S(1+\varepsilon S)^{-1}u))\\&\geqq\varepsilon(z, F(S(1+\varepsilon S)^{-1}u)),\end{aligned}$$

where we have used the accretiveness of A. By virtue of (6.3), we obtain

$$\begin{aligned}(y, F(S_\varepsilon u))&\geqq(z, F(S(1+\varepsilon S)^{-1}u))\\&\geqq-\phi(\|(1+\varepsilon S)^{-1}u\|)-b\|S(1+\varepsilon S)^{-1}u\|^2\\&\geqq-\phi(\|u\|)-b\|S_\varepsilon u\|^2.\end{aligned}$$

This is nothing but the inequality (6.2) with $B=S$. Therefore, $S+A$ is m-accretive by Lemma 6.2. Q. E. D.

REMARK 6.5. Theorem 6.4 is a "semi-linear" version of Theorem 2.2.

§ 7. Singular perturbation in a Hilbert space (semi-linear case).

Let H be a real Hilbert space. The following theorem is a semi-linear version of Theorem 0.3.

THEOREM 7.1. *Let A be a nonlinear m-accretive operator in H. Let S be a nonnegative selfadjoint operator in H, with $D(S)\subset D(A)$. Assume that there exist a nonnegative constant b and a non-decreasing fnnction $\phi(r)\geqq 0$ of $r\geqq 0$ such that for each $u\in D(S)$,*

$$(7.1)\qquad (w, Su)\geqq-\phi(\|u\|)-b(u, Su)\quad \textit{whenever}\quad w\in Au.$$

Then $\frac{1}{n}S+A$ $(n=1, 2, \cdots)$ is m-accretive and for $\lambda<b^{-1}$, $\left[1+\lambda\left(\frac{1}{n}S+A\right)\right]^{-1}$ converges strongly to $(1+\lambda A)^{-1}$ as $n\rightarrow\infty$. Let $S^{1/2}$ be the square root of S. Then $D(S^{1/2})$ is invariant under $(1+\lambda A)^{-1}$, $b\lambda<1$. Furthermore, if $\phi(r)\equiv 0$, then for each $v\in D(S^{1/2})$,

$$(7.2)\qquad \|S^{1/2}(1+\lambda A)^{-1}v\|\leqq(1-b\lambda)^{-1}\|S^{1/2}v\|$$

PROOF. We see from Theorem 6.4 that $\frac{1}{n}S+A$ is m-accretive. Let u_n be a unique solution of the equation

$$(7.3)\qquad u_n+\lambda\left(\frac{1}{n}Su_n+w_n\right)=v,\qquad w_n\in Au_n,\quad b\lambda<1,$$

where $v\in D(S^{1/2})$. Then by (7.1) we have

$$(S^{1/2}v, S^{1/2}u_n)=(v, Su_n)$$

$$=(u_n, Su_n)+\lambda\left(\frac{1}{n}Su_n, Su_n\right)+\lambda(w_n, Su_n)$$
$$\geqq\|S^{1/2}u_n\|^2-b\lambda(u_n, Su_n)-\lambda\phi(\|u_n\|)$$
$$=(1-b\lambda)\|S^{1/2}u_n\|^2-\lambda\phi(\|u_n\|).$$

So, we obtain

$$\|S^{1/2}u_n\|\leqq(1-b\lambda)^{-1}\|S^{1/2}v\|+\left[\frac{\lambda\phi(\|u_n\|)}{1-b\lambda}\right]^{1/2}, \qquad n\geqq 1. \tag{7.4}$$

Next we show that for each $v\in D(S^{1/2})$, $\{u_n\}$ forms a Cauchy sequence. To this end, we note that

$$\|u_n-u_m\|^2\leqq(u_n-u_m+\lambda(w_n-w_m), u_n-u_m)$$
$$=-\lambda\left(\frac{1}{n}Su_n-\frac{1}{m}Su_m, u_n-u_m\right)$$
$$\leqq\frac{\lambda}{2}\left(\frac{1}{m}-\frac{1}{n}\right)\|S^{1/2}u_n\|^2+\frac{\lambda}{2}\left(\frac{1}{n}-\frac{1}{m}\right)\|S^{1/2}u_m\|^2.$$

By virtue of (7.4) we see that for $m\leqq n$,

$$\|u_n-u_m\|\leqq\left(\frac{\lambda}{2}\right)^{1/2}\left(\frac{1}{m}-\frac{1}{n}\right)^{1/2}\left(\frac{\|S^{1/2}v\|}{1-b\lambda}+\left[\frac{\lambda\phi(\|u_n\|)}{1-b\lambda}\right]^{1/2}\right) \tag{7.5}$$

and hence $u=\lim_{n\to\infty}u_n$ exists; note that $\|u_n\|$ is bounded.

We want to show that $u=(1+\lambda A)^{-1}v$. It follows from (7.3) and (7.1) that

$$\left(v-u_n, \frac{1}{n}Su_n\right)=\lambda\left\|\frac{1}{n}Su_n\right\|^2+\frac{\lambda}{n}(w_n, Su_n)$$
$$\geqq\lambda\left\|\frac{1}{n}Su_n\right\|^2-\frac{\lambda}{n}\phi(\|u_n\|)-b\lambda\left(u_n, \frac{1}{n}Su_n\right).$$

So we obtain

$$\lambda\left\|\frac{1}{n}Su_n\right\|^2\leqq\frac{\lambda}{n}\phi(\|u_n\|)+(\|v\|+(1-b\lambda)\|u_n\|)\left\|\frac{1}{n}Su_n\right\|.$$

Consequently, $\left\|\frac{1}{n}Su_n\right\|$ is bounded:

$$\left\|\frac{1}{n}Su_n\right\|\leqq\lambda^{-1}[\|v\|+(1-b\lambda)\|u_n\|]+[\phi(\|u_n\|)/n]^{1/2}.$$

Now let $f\in D(S)$. Then we have

$$\left|\frac{1}{n}(Su_n, f)\right|=\frac{1}{n}|(u_n, Sf)|\leqq\frac{1}{n}\|u_n\|\|Sf\|\to 0 \qquad (n\to\infty).$$

Since $D(S)$ is dense in H, $\left\{\frac{1}{n}Su_n\right\}$ converges to zero weakly. Noting that

$w_n=\lambda^{-1}(v-u_n)-\frac{1}{n}Su_n$, we see that $\{w_n\}$ also converges to $\lambda^{-1}(v-u)$ weakly. Therefore, it follows from the demi-closedness of A that $u\in D(A)$ and $\lambda^{-1}(v-u)\in Au$ (see Kato [10]). Namely, we have

$$u=(1+\lambda A)^{-1}v.$$

Since $D(S^{1/2})$ is dense in H, $\left[1+\lambda\left(\frac{1}{n}S+A\right)\right]^{-1}$ converges to $(1+\lambda A)^{-1}$ strongly; note that both $\left[1+\lambda\left(\frac{1}{n}S+A\right)\right]^{-1}$ and $(1+\lambda A)^{-1}$ are nonexpansive.

Finally, since $S^{1/2}$ is weakly closed, we see from (7.4) that $u=(1+\lambda A)^{-1}v\in D(S^{1/2})$ and $S^{1/2}u_n$ converges to $S^{1/2}u$ weakly. Therefore, we have

$$\|S^{1/2}u\|\leqq\liminf_{n\to\infty}\|S^{1/2}u_n\|.$$

Now (7.2) follows from (7.4). Q. E. D.

REMARK 7.2. If $\phi(r)$ is a continuous function, then it follows from (7.5) that

$$\left\|\left[1+\lambda\left(\frac{1}{n}S+A\right)\right]^{-1}v-(1+\lambda A)^{-1}v\right\|$$
$$\leqq\left(\frac{\lambda}{2n}\right)^{1/2}\left(\frac{\|S^{1/2}v\|}{1-b\lambda}+\left[\frac{\lambda\phi(\|(1+\lambda A)^{-1}v\|)}{1-b\lambda}\right]^{1/2}\right).$$

Let A and S be as in Theorem 7.1. Then $\frac{1}{n}S+A$ $(n=1, 2, \cdots)$ and A are m-accretive. Let $\{U_n(t)\}$ and $\{U(t)\}$ be the contraction semigroups generated by $-\left(\frac{1}{n}S+A\right)$ and $-A$, respectively. Then, since $\left[1+\lambda\left(\frac{1}{n}S+A\right)\right]^{-1}$ converges strongly to $(1+\lambda A)^{-1}$, it follows from the (nonlinear) Trotter-Kato theorem that $U_n(t)$ converges strongly to $U(t)$ (see e.g. Benilan [3] or Miyadera [15]). In this connection we have

THEOREM 7.3. *In Theorem* 7.1 *assume further that* $\phi(r)$ *is a continuous function. Let* $\{U_n(t)\}$ *and* $\{U(t)\}$ *be the contraction semigroups generated by* $-\left(\frac{1}{n}S+A\right)$ *and* $-A$, *respectively. Then for* $v\in D(S^{1/2})$ *we have the estimate*

$$\|U_n(t)v-U(t)v\| \tag{7.6}$$
$$\leqq\left(\frac{t}{n}\right)^{1/2}e^{bt}\left(\|S^{1/2}v\|+\left[2\int_0^t\phi(\|U(s)v\|)ds\right]^{1/2}\right).$$

REMARK 7.4. (7.6) is a semi-linear generalization of a rather restricted result obtained in [20].

To prove Theorem 7.3 we prepare two lemmas.

Let $u_0\in D(S)$. Then $u_n(t)=U_n(t)u_0$ is a unique solution of the equation

$$(7.7) \qquad u_n'(t)+\frac{1}{n}Su_n(t)\in -Au_n(t), \qquad \text{a. a.} \quad t\geqq 0,$$

with the initial condition $u_n(0)=u_0$. The following lemma is known in a more general form (see e. g. Barbu [**2**], IV-§ 2). But we give a direct proof.

LEMMA 7.5. *Let $u_0\in D(S)$ and $T>0$. Set $u_n(t)=U_n(t)u_0$. Then $\|S^{1/2}u_n(t)\|^2$ is absolutely continuous on $[0, T]$ and*

$$(7.8) \qquad (d/dt)\|S^{1/2}u_n(t)\|^2=2(u_n'(t), Su_n(t)) \qquad \text{a. e. on} \quad [0, T].$$

PROOF. Let S_ε be the Yosida approximation of S. Then we have (7.8) with S replaced by S_ε. Since both $\|u_n(t)\|$ and $\|u_n'(t)\|$ are bounded a. e. on $[0, T]$, we see that for $t\leqq T$,

$$(7.9) \qquad \|S_\varepsilon^{1/2}u_n(t)\|^2-\|S_\varepsilon^{1/2}u_0\|^2=2\int_0^t(u_n'(s), S_\varepsilon u_n(s))ds.$$

Consequently, by the bounded convergence theorem we obtain (7.9) with S_ε replaced by S if $\|Su_n(s)\|$ is also bounded a. e. on $[0, T]$. But it follows from (7.7) and (7.1) that a. e. on $[0, T]$,

$$(7.10) \qquad \left(-u_n'(s)-\frac{1}{n}Su_n(s), Su_n(s)\right)\geqq -\psi(\|u_n(s)\|)-b(u_n(s), Su_n(s)).$$

Namely we have

$$\frac{1}{n}\|Su_n(s)\|^2\leqq(bu_n(s)-u_n'(s), Su_n(s))+\psi(\|u_n(s)\|).$$

Hence we can conclude that $\|Su_n(s)\|$ is bounded a. e. on $[0, T]$. Q. E. D.

By virtue of Lemma 7.5 we can prove

LEMMA 7.6. *$D(S^{1/2})$ is invariant under $U(t)$, and for each $v\in D(S^{1/2})$ the following estimate holds:*

$$(7.11) \qquad \|S^{1/2}U(t)v\|^2\leqq e^{2bt}\left[\|S^{1/2}v\|^2+2\int_0^t\psi(\|U(s)v\|)ds\right].$$

PROOF. Let $u_n(t)$ be as in Lemma 7.5. Then it follows from (7.10) and (7.8) that for a. a. $s\geqq 0$,

$$(d/ds)\|S^{1/2}u_n(s)\|^2-2b\|S^{1/2}u_n(s)\|^2\leqq 2\psi(\|u_n(s)\|).$$

Integrating this inequality, we have

$$e^{-2bt}\|S^{1/2}u_n(t)\|^2\leqq\|S^{1/2}u_0\|^2+2\int_0^t\psi(\|u_n(s)\|)ds.$$

Since the sequence $\{\|u_n(s)\|\}$ is bounded, we see that $\{S^{1/2}u_n(t)\}$ converges weakly as $n\to\infty$. But since $S^{1/2}$ is weakly closed, it follows that $U(t)u_0=\lim_{n\to\infty}u_n(t)\in D(S^{1/2})$ and $S^{1/2}U(t)u_0$ is equal to the weak limit of $\{S^{1/2}u_n(t)\}$.

Therefore, we obtain

$$\|S^{1/2}U(t)u_0\| \leqq \liminf_{n\to\infty} \|S^{1/2}u_n(t)\| .$$

By the bounded convergence theorem we see that

$$e^{-2bt}\|S^{1/2}U(t)u_0\|^2 \leqq \|S^{1/2}u_0\|^2 + 2\int_0^t \phi(\|U(s)u_0\|)ds .$$

Noting further that $D(S)$ is a core of $S^{1/2}$, we can obtain (7.11). Q. E. D.

PROOF OF THEOREM 7.3. Since $D(S)$ is a core of $S^{1/2}$, it suffices to show that (7.6) holds for each v in $D(S)$.

Let $u_0 \in D(S)$. Then $u(t)=U(t)u_0$ is a unique solution of the equation

$$u'(t) \in -Au(t), \quad \text{a. a. } t \geqq 0, \tag{7.12}$$

with the initial condition $u(0)=u_0$. So, we see from (7.7) and (7.12) that

$$\begin{aligned}(d/ds)\|u_n(s)-u(s)\|^2 &= -2(u_n'(s)-u'(s),\ u_n(s)-u(s)) \\ &\leqq -2\Big(\frac{1}{n}Su_n(s),\ u_n(s)-u(s)\Big) \\ &\leqq -\frac{2}{n}\|S^{1/2}u_n(s)\|^2 + \frac{2}{n}(S^{1/2}u_n(s),\ S^{1/2}u(s)) \\ &\leqq \frac{1}{n}\|S^{1/2}u(s)\|^2 - \frac{1}{n}\|S^{1/2}u_n(s)\|^2, \quad \text{a. a. } s \geqq 0 .\end{aligned}$$

Consequently, (7.6) follows from (7.11). Q. E. D.

EXAMPLE 7.7. Let Ω be a bounded domain in R^3 with smooth boundary. Let

$$S = -\Delta \qquad (\Delta = \text{Laplacian})$$

with $D(S)=H^2(\Omega)\cap H_0^1(\Omega)$. Then S is a positive definite selfadjoint operator in $L^2(\Omega)$, with $D(S^{1/2})=H_0^1(\Omega)$. Now set

$$Au(x)=[u(x)]^3, \qquad x \in \Omega ,$$

for $u \in D(A)=\{u(x),\ [u(x)]^3 \in L^2(\Omega)\}$. Then A is a single-valued m-accretive operator in $L^2(\Omega)$. Also, we see from the Sobolev theorem that $D(S)$ is included in $D(A)$. Furthermore, we can prove (7.1) with $b=0$ and $\phi(r)\equiv 0$. In fact, we have

$$\begin{aligned}(Au,\ Su) &= -\int_\Omega [u(x)]^3 \Delta u(x)dx \\ &= 3\int_\Omega |u(x)|^2 \sum_{k=1}^3 \left|\frac{\partial u}{\partial x_k}\right|^2 dx \geqq 0.\end{aligned}$$

Thus, the conclusions of Theorems 7.1 and 7.3 hold good. Roughly speaking, for every $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ the solution of the partial differential equation

$$\frac{\partial u_n}{\partial t} + [u_n(x, t)]^3 = \frac{1}{n} \Delta u_n(x, t), \qquad (x, t) \in \Omega \times (0, \infty),$$

$$u_n(x, 0) = u_0(x)$$

converges in $L^2(\Omega)$ to the solution of the ordinary differential equation

$$\frac{du}{dt} + [u(x, t)]^3 = 0, \qquad (x, t) \in \Omega \times (0, \infty),$$

$$u(x, 0) = u_0(x).$$

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