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On the structure of polarized manifolds with total deficiency one, I

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Introduction.

A *polarized variety* is a pair (V, L) of a variety V and an ample line bundle L on it. In this paper, for the sake of simplicity, we consider only the complex analytic cases. But many of our results hold to be true also in positive characteristic cases after slight modifications (see [8]).

In [5] we defined the following three invariants of (V, L): $d(V, L)=L^n$ for $n=\dim V$, $\Delta(V, L)=n+L^n-h^0(V, L)$ which might be called the *total deficiency* or the *Δ-genus,* and $g(V, L)$ which might be called the *sectional genus* because of the following properties: a) $g(V, L)=h^1(V, \mathcal{O}_V)$ if $\dim V=1$, b) $g(D, L_D)=g(V, L)$ for any irreducible reduced member D of $|L|$, c) $2g(V, L)-2=(K^V+(n-1)L)L^{n-1}$ for any smooth V. In [5] we showed $\Delta(V, L)\geqq 0$ for any polarized variety (V, L) and we gave a complete enumeration of polarized varieties with $\Delta=0$. Here we study polarized manifold (M, L) with $\Delta=1$.

$\Delta(M, L)$, together with $g(M, L)$, might be regarded as a higher dimensional version of the genus of curves (see [6]). In particular, when $\Delta(M, L)=1$, we have: *L is very ample*$\Leftrightarrow d(M, L)\geqq 3\Rightarrow g(M, L)=1$. Moreover $K^M+(n-1)L=0\Leftrightarrow \Delta(M, L)=g(M, L)=1$. So, to consider polarized surfaces with $\Delta=g=1$ is equivalent to consider surfaces with negative canonical bundles, which are known classically as *Del Pezzo surfaces.* Generalizing this fact to higher dimensional cases, we will give a complete enumeration of all the *Del Pezzo manifolds,* that are, polarized manifolds with $\Delta=g=1$. However, in this part I, we treat only the cases in which $d(M, L)=2, 3, 4$ and $\geqq 6$. For the study of the cases $d=1, 5$ we need more complicated techniques. The result in case $d=1$ was announced in [4_{II}]. The cases $d=5$ will be treated in the second part.

Finally we give the classification table of Del Pezzo manifolds with $\dim M \geqq 3$, $d(M, L)=2, 3, 4$ or $\geqq 6$.

$d(M, L)$	$g(M, L)$	$\dim M$	$b_2(M)$	structure of M
2	$\geqq 1$	$\geqq 3$	1	two-sheeted branched covering of $\boldsymbol{P}^n$
3	1	$\geqq 3$	1	smooth hypercubic
4	1	$\geqq 3$	1	complete intersection of type (2, 2)
6	1	3	3	$\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$
			2	$\boldsymbol{P}(T)$, T being the tangent bundle of $\boldsymbol{P}^2$
		4	2	$\boldsymbol{P}^2 \times \boldsymbol{P}^2$
7	1	3	2	$Q_p(\boldsymbol{P}^3)$, p being a point on $\boldsymbol{P}^3$
8	1	3	1	$\boldsymbol{P}^3$
$\geqq 9$				not exist

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Notation, convention and terminology.

Basically we employ the notation as in [5], [6] and [7] which is analogous to that of EGA[**9**].

Variety is an irreducible reduced compact complex analytic space. *Manifold* is a smooth variety. Vector bundles and locally free sheaves are confused occasionally. Tensor products of line bundles are denoted by additive notation.

Here we show examples of symbols used in this paper.

$\{Z\}$: The homology class defined by an analytic cycle Z.

$|L|$: The complete linear system associated with a line bundle L.

$[\Lambda]$: The line bundle defined by a linear system Λ.

$Bs\Lambda$: The intersection of all the members of Λ.

ρ_Λ : The rational mapping defined by Λ.

L_T : The pull back of L to a space T by a given morphism $T \to S$. However, when there is no danger of confusion, we OFTEN write L instead of L_T by abuse of notation.

$Q_C(M)$: The blowing up of M with center C.

E_C : The exceptional divisor on $Q_C(M)$ over C.

K^M: The canonical line bundle of a manifold M.

H^E: The tautological line bundle $\mathcal{O}(1)$ on $\boldsymbol{P}(E)=E^\vee-\{0\text{-section}\}/\boldsymbol{C}^\times$ where E is a vector bundle and $E^\vee$ is the dual of it.

$H_\alpha, H_\beta, \cdots$: The line bundles defined by hyperplane sections on projective spaces $\boldsymbol{P}_\alpha, \boldsymbol{P}_\beta, \cdots$ indicated by the same suffixes.

§ 1. Generalities.

(1.0) Before applying the results in [**6**], we recall several definitions.

A *rung* of a polarized variety (V, L) is an irreducible reduced member of $|L|$. A rung D is said to be *regular* if the restriction homomorphism $\Gamma(V, L) \to \Gamma(D, L_D)$ is surjective. This condition is equivalent to $\Delta(D, L_D)=\Delta(V, L)$. A *ladder* of (V, L) is a sequence of subvarieties $V=V_n \supset V_{n-1} \supset \cdots \supset V_1$ of V with $\dim V_j=j$ such that each V_j is a rung of (V_{j+1}, L). It is said to be *regular* if each rung of it is so, or equivalently, $\Delta(V, L)=\Delta(V_1, L)$.

Let (M, L) be a polarized manifold with $\Delta(M, L)=1$. Then we have the following results.

(1.1) $Bs|L|$ is at most a finite set ([**5**, Theorem 1.9]).

(1.2) (M, L) has a ladder ([**6**, Corollary 3.6]).

(1.3) $g(M, L)\geqq 1$ ([**6**, Lemma 5.1]).

(1.4) (M, L) has a regular smooth ladder ([**6**, Corollary 4.2]).

(1.5) $Bs|L|=\emptyset$ if $d(M, L)\geqq 2$ ([**6**, Theorem 4.1, b]).

(1.6) $g(M, L)=1$ if $d(M, L)\geqq 3$ ([**6**, Theorem 4.1, c]).

(1.7) L is very ample and $\rho_{|L|}(M)$ is projectively normal if $d(M, L)\geqq 3$ ([**6**, Theorem 4.1, c]).

(1.8) $\rho_{|L|}(M)$ is defined by quadratic equations if $d(M, L)\geqq 4$ ([**6**, Theorem 4.1, d]).

The above results are used frequently, sometimes without referring explicitly.

(1.9) Now we will prove the following

THEOREM. *Let (M, L) be a polarized manifold of dimension n. Then $K^M+(n-1)L=0$ if and only if $\Delta(M, L)=g(M, L)=1$.*

(1.10) LEMMA. *Let (M, L) be a polarized manifold with* $\dim M=n$, *$d(M, L)=d$ and $K^M=-(n-1)L$. Then $\chi(M, tL)=(dt^2+d(n-1)t+n(n-1)) \cdot (\prod_{\lambda=1}^{n-2}(t+\lambda))/n!$.*

PROOF. Using the vanishing theorem of Kodaira we infer that $h^p(M, tL)=0$ for $0<p<n$, $t \in \boldsymbol{Z}$. Moreover $h^n(M, -tL)=h^0(M, K^M+tL)=h^0(M, (t-n+1)L)=0$ if $t<n-1$. Therefore $\chi(t):=\chi(M, tL)=0$ for $-(n-2) \leqq t \leqq -1$. So we put $\chi(t)=(at^2+bt+c)(\prod_{\lambda=1}^{n-2}(t+\lambda))/n!$ where a, b and c are constants. Using the Riemann-Roch-Hirzebruch Theorem we infer that $\chi(t)=$

$(dt^n/n!)-2^{-1}K^M L^{n-1}t^{n-1}/(n-1)!+\mathcal{O}(t^{n-2})$. Hence $a=d$ and $b=d(n-1)$. Now $\chi(0)=1$ implies $c=n(n-1)$.

(1.11) PROPOSITION. *Let M be a smooth surface which contains an elliptic curve C as an ample divisor. Then $(M, [C])\cong(\boldsymbol{P}(E), H^E)$ for some ample vector bundle E on C unless $H^1(M)=0$.*

PROOF. Suppose $H^1(M)\neq 0$ and let $\iota: C\to M$ be the inclusion. $Alb(\iota): C\to Alb(M)$ is surjective and any general fiber of it is connected ([**7**], §1, Theorem VII*). Hence $Alb(\iota)$ is an isomorphism. So we have the Albanese mapping $a: M\to C$. ι defines a section of a. So $CY=1$ for any fiber Y of a. Hence Y is irreducible and reduced because C is ample. Note that $\deg N_{M/C}=C^2>0$ for the normal bundle $N_{M/C}$ of C in M. Hence $H^1(C, N_{M/C})=0$ and $H^0(C, N_{M/C})\neq 0$. In view of [**14**] we infer that C is contained in a non-trivial deformation family of submanifolds in M. Let C' be a member of this family which is different from C. C' is homologically equivalent to C and we find $F\in Pic_0(Alb(M))\cong Pic_0(M)$ such that $a^*F=[C'-C]$. Therefore $[C'-C]_X=0$ for a general fiber X of a. This implies $X\cong\boldsymbol{P}^1$ since C'_X and C_X are different points on X. Now, [**5**, Corollary 5.4] proves our assertion.

(1.12) PROOF OF THEOREM (1.9). First we consider the 'only if' part. By (1.10) we have $\chi(M, L)=d+n-1$. Using the vanishing theorem of Kodaira we infer that $h^0(M, L)=\chi(M, L)$. Combining them we obtain $\Delta(M, L)=1$. $g(M, L)=1$ follows from the property c) of the sectional genus stated in the introduction.

In order to prove the 'if' part, we use the induction on n. The assertion for $n=1$ follows from the property a) of g. Second we consider the case in which $n=2$. Let C be a smooth member of $|L|$. Then $g(C)=g(C, L)=g(M, L)=1$ and C is an elliptic curve. Suppose that $H^1(M)\neq 0$. Then, by (1.11), we have an ample vector bundle E on C such that $(M, [C])\cong(\boldsymbol{P}(E), H^E)$. The result of Atiyah [**1**], Lemma 15 implies that $h^0(M, L)=h^0(C, E)=\deg(\det E)=(H^E)^2=d(M, L)$. So $\Delta(M, L)=2$, which contradicts our assumption. Therefore $H^1(M)=0$. Now we have the following exact sequence: $0=H^1(M, \mathcal{O}_M)\to H^1(C, \mathcal{O}_C)\to H^2(M, -L)$. Hence $1=h^1(C)\leqq h^2(M, -L)=h^0(M, K^M+L)$. So we have a member D of $|K^M+L|$. Then $LD=L(K^M+L)=2g(M, L)-2=0$ and $D=0$ since L is ample. Thus we prove $K^M+L=0$. Finally we consider the case in which $n\geqq 3$. By (1.4) we take a smooth regular rung A of (M, L). Then we have $\Delta(A, L)=\Delta(M, L)=1$ and $g(A, L)=g(M, L)=1$. So $[K^M+(n-1)L]_A=K^A+(n-2)L_A=0$ by the induction hypothesis. On the other hand $Pic(M)\to Pic(A)$ is injective ([**7**, (2.5)]). Combining them we prove the assertion.

(1.13) DEFINITION. A polarized manifold of the type (1.9) shall be called a *Del Pezzo manifold.*

§ 2. The cases in which $d=2$, 3, 4.

(2.0) Throughout this section let (M, L) be a polarized manifold with $\Delta(M, L)=1$, $d(M, L)=d$ and $\dim M=n$.

(2.1) THEOREM. *If $d=3$, then M is isomorphic to a hypercubic in $\boldsymbol{P}^{n+1}$.*

PROOF. Obvious by (1.7).

(2.2) THEOREM. *If $d=4$, then M is isomorphic to a complete intersection of type* (2, 2) *in $\boldsymbol{P}^{n+2}$.*

PROOF. L is very ample and $\rho_{|L|}: M \to \boldsymbol{P}^{n+2}$ is an embedding. Using (1.10) we infer that $h^0(M, 2L)=\chi(M, 2L)=(n^2+7n+8)/2=h^0(\boldsymbol{P}^{n+2}, 2H)-2$. Hence there are two distinct hyperquadrics Q_1, Q_2 in $\boldsymbol{P}^{n+2}$ which contain $\rho_{|L|}(M)$. Now (1.8) proves the assertion.

(2.3) Let F be a line bundle on a variety V and let B be a member of $|kF|$ for a positive integer k. Take a sufficiently fine covering $\{U_\alpha\}$ of V such that the restriction F_α of F to U_α is trivial. Let ζ_α be the coordinate of F_α along the fiber and $\{f_{\alpha\beta}\}$ be the cocycle defined by F. Namely $f_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_V^\times)$ and $\zeta_\alpha=f_{\alpha\beta}\zeta_\beta$ on $F_\alpha \cap F_\beta$. Let $b_\alpha \in \Gamma(U_\alpha, \mathcal{O}_V)$ be the equation defining B. Note that $b_\alpha=f_{\alpha\beta}^k b_\beta$ on $U_\alpha \cap U_\beta$. Let $D_\alpha \subset F_\alpha$ be the divisor defined by $\zeta_\alpha^k=b_\alpha$. Then $D_\alpha=D_\beta$ on $F_\alpha \cap F_\beta$. Thus we obtain a divisor $D=\bigcup_\alpha D_\alpha$ on F. D is determined uniquely by the triple (V, F, B). So we denote D by ${}_kR_{B,F}(V)$. When $k=2$ and there is no danger of confusion, we write $R_{B,F}(V)$ or even $R_B(V)$ instead of ${}_2R_{B,F}(V)$. Note that the projection $F \to V$ makes D a k-sheeted cyclic branched covering of V with branch locus B. D is smooth if both V and B are smooth.

(2.4) THEOREM. *Let $f: N' \to N$ be a finite morphism of degree two from a manifold N' onto another manifold N. Then there is a line bundle F on N and a smooth member B of $|2F|$ such that $N' \cong R_{B,F}(N)$.*

For a proof, see Wavrik [**19**].

(2.5) THEOREM. *If $d=2$ and $g(M, L)=g$, then $(M, L) \cong (R_B(\boldsymbol{P}^n), H_M)$ where $H=\mathcal{O}_{\boldsymbol{P}^n}(1)$ and B is a smooth member of $|(2g+2)H|$.*

PROOF. By (1.5) we obtain a finite morphism $\rho=\rho_{|L|}: M \to \boldsymbol{P}^n$ such that $L=H_M$. Moreover $\deg \rho=d=2$. Hence (2.4) implies that $M \cong R_{B,kH}(\boldsymbol{P}^n)$ for some $B \in |2kH|$ and a positive integer k. Then $K^M=K^{\boldsymbol{P}^n}+kH=(k-n-1)H_M$ and $2g-2=(K^M+(n-1)H)H^{n-1}\{M\}=(k-2)H^n\{M\}=2(k-2)$. So $k=g+1$.

§ 3. The cases in which $n=2$.

In this section we recall the classical theory on Del Pezzo surfaces.

(3.1) Any surface S with $-K^S$ being ample is rational ([**16**], Chapter III).

(3.2) A rational surface S contains an exceptional curve of the first kind unless $S \cong \boldsymbol{P}^2$ or S is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ ([**16**], Chapter V).

(3.3) If S is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ and if $-K^S$ is ample, then $S\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$ or $\cong Q_p(\boldsymbol{P}^2)$.

(3.4) THEOREM. *A Del Pezzo surface is a blowing-up of $\boldsymbol{P}^2$ with center being finite points unless $S\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$.*

PROOF. S is rational by (3.1). If S is relatively minimal, then $S\cong\boldsymbol{P}^2$ or $\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$ by (3.2) and (3.3). If S contains an exceptional curve E of the first kind, then $S\cong Q_p(N)$ and $E=E_p$ for some surface N. Using [7, (5.7)] we infer that $-K^N$ is ample. Thus, repeating this process if necessary, we have $S\cong Q_{p_r}(Q_{p_{r-1}}(\cdots(Q_{p_1}(\boldsymbol{P}^2))\cdots))$ unless $S\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$, since $Q_p(\boldsymbol{P}^1\times\boldsymbol{P}^1)\cong Q_{q_1\cup q_2}(\boldsymbol{P}^2)$ for two points q_1 and q_2 on $\boldsymbol{P}^2$. Suppose that p_i lies on the proper transform of E_{p_j} for some $i>j$. Then we have $E_j'K^S\geqq 0$ for the proper transform E_j' of E_{p_j} on S. This contradicts the ampleness of $-K^S$. Consequently the centers $p_1, \cdots, p_r$ must map to different points on $\boldsymbol{P}^2$.

REMARK. No three points of $p_1, \cdots, p_r$ are colinear. Indeed, if otherwise, $C'K^S\geqq 0$ for the proper transform C' on S of the line which contains three of $p_1, \cdots, p_r$. This contradicts the ampleness of $-K^S$. Similarly no six points lie on a common quadric curve.

(3.5) COROLLARY. *There is no polarized manifold (M, L) with $\Delta(M, L)=1$, $d(M, L)\geqq 10$ and $\dim M\geqq 2$.*

PROOF. Suppose that such a polarized manifold exists. Applying (1.4), we let S be the two dimensional rung of a regular smooth ladder of (M, L). Then $d(S, L)=d(M, L)\geqq 10$ and $\Delta(S, L)=\Delta(M, L)=1$. Moreover (1.6) says $g(S, L)=1$ and S is a Del Pezzo surface. This contradicts (3.4) since $d(S, L)=(c_1(S))^2=12-c_2(S)\leqq 9$.

§ 4. From surfaces to threefolds.

(4.1) We will study the structure of a given polarized manifold (M, L) by induction on $n=\dim M$. Suppose A to be a smooth rung of it. In order to relate the structure of M to that of A, it is very important to study the restriction mapping $r: Pic(M)\to Pic(A)$, because, roughly speaking, the structure of manifolds are described by certain morphisms defined on them, and the morphisms are given by linear systems, and the linear systems are associated to line bundles. If $n\geqq 4$, then r is bijective by the Lefschetz Theorem. However, for $n=3$, we have only the following results.

(4.2) *r is injective and $Coker(r)$ is torsion free.*

A proof can be found in [7, § 1]. More precisely one has $Coker(r)\subset Coker(H^2(M; \boldsymbol{Z})\to H^2(A; \boldsymbol{Z}))$ and the latter is torsion free. Consequently we have

(4.3) *$F\in Pic(A)$ comes from $Pic(M)$ if and only if $c_1(F)\in H^2(A; \boldsymbol{Z})$ comes from $H^2(M; \boldsymbol{Z})$. $\alpha\in H^2(A; \boldsymbol{Z})$ comes from $H^2(M; \boldsymbol{Z})$ if and only if $\alpha_{\boldsymbol{Q}}\in H^2(A; \boldsymbol{Q})$ comes from $H^2(M; \boldsymbol{Q})$.*

(4.4) We will study $Im(r)$ more precisely in case $n=3$. From now on L is assumed to be very ample. So we regard M to be a submanifold in $P\cong \boldsymbol{P}^N$ ($N=\dim|L|$) by $\rho_{|L|}$.

(4.5) Let P^* be the dual of P, namely, the Grassmann variety of hyperplanes on P. Let $F=\{(x, h)|x\in h\in P^*\}\subset P\times P^*$ and let $M^*=M\times_P F$. Let $\pi: M^*\to M$ and $\psi: M^*\to F\to P^*$ be the natural morphisms. Then we have a bijection $f: P^*\to|L|$ such that for any $h\in P^*$ the restriction of π to $\psi^{-1}(h)$ is an isomorphism onto $f(h)$. Note that π makes M^* a $\boldsymbol{P}^{N-1}$-bundle over M and that $H^{\cdot}(M^*; \boldsymbol{Z})$ is generated by $c_1(H^*)$ as an $H^{\cdot}(M; \boldsymbol{Z})$*-algebra,* where $H^*=\mathcal{O}_{P^*}(1)$.

(4.6) We have an open dense subset U of P^* such that $\psi^{-1}(h)$ is smooth for any $h\in U$. $R^2\psi_*(\boldsymbol{Z}_{M^*})$ defines a local system on U, where $\boldsymbol{Z}_{M^*}$ is the constant sheaf on M^* with fiber $\boldsymbol{Z}$. Now we fix a point $o\in U$ and let $S=\psi^{-1}(o)\cong f(o)\in|L|$. The above local system is defined by the action λ of $\pi_1(U, o)$ on $H^2(S; \boldsymbol{Z})$. λ is called the *monodromy action* induced by $|L|$. This action is compatible with the intersection pairing defined by cup product on $H^2(S; \boldsymbol{Z})$.

(4.7) LEMMA. *$\alpha\in H^2(S; \boldsymbol{Z})$ comes from $H^2(M; \boldsymbol{Z})$ if and only if α is stabilized by the above monodromy action λ.*

PROOF. The 'only if' part is obvious. Suppose that α is stabilized by λ. This implies that α comes from $\beta\in\Gamma(U, R^2\psi_*\boldsymbol{Z}_{M^*})$. In view of Deligne [3, Theorem 4-1-1], we infer that $\beta_{\boldsymbol{Q}}$ comes from $H^2(M^*; \boldsymbol{Q})$. By (4.5), $H^2(M^*; \boldsymbol{Q})$ is generated by $H^2(M; \boldsymbol{Q})$ and $c_1(H^*)$. Putting things together, we infer that $\alpha_{\boldsymbol{Q}}$ comes from $H^2(M; \boldsymbol{Q})$, since $c_1(H^*)_S=0$. So (4.3) applies.

(4.8) The above lemma tells us that we should study the monodromy action λ induced by $|L|$ in order to study $r: Pic(M)\to Pic(S)$. λ is an action on $H^2(S; \boldsymbol{Z})$ which is compatible with the cup product pairing and which stabilizes $c_1(S)$. So we make the following

(4.9) DEFINITION. A triple $\mathfrak{L}=(\Lambda, q, c)$ of a $\boldsymbol{Z}$-module Λ, $\boldsymbol{Z}$-valued symmetric bilinear form q on Λ and an element c of Λ is called a *marked Lorentz lattice.* An automorphism of $\mathfrak{L}$ is a linear bijection $\sigma: \Lambda\to\Lambda$ such that $\sigma(c)=c$ and $q(\sigma(x), \sigma(y))=q(x, y)$ for any $x, y\in\Lambda$. $\mathrm{Aut}(\mathfrak{L})$ denotes the group of all the automorphisms of $\mathfrak{L}$.

(4.10) For any surface S the triple ($H^2(S; \boldsymbol{Z})$, cup product paing, $c_1(S)$) is a marked Lorentz lattice. We denote it by $\mathfrak{L}(S)$.

(4.11) If (M, L) is a Del Pezzo threefold, then a general member S of $|L|$ is a Del Pezzo surface. In view of (3.4), we let S_r be the blowing up of $\boldsymbol{P}^2$ with center being r points $p_1, p_2, \cdots, p_r$ and we will study the structure of $\mathfrak{L}_r=\mathfrak{L}(S_r)$ and $\mathrm{Aut}(\mathfrak{L}_r)$.

(4.12) A free base $\{h, e_1, \cdots, e_r\}$ of $\Lambda_r=H^2(S_r; \boldsymbol{Z})$ is called *a normal base* of $\mathfrak{L}_r$ if $(h, h)=1$, $(h, e_j)=0$, $(e_i, e_j)=-\delta_{ij}$ and $c=c_1(S_r)=3h-\sum_{j=1}^{r} e_j$. A normal base can be constructed by putting $h=c_1(\mathcal{O}_{P^2}(1))$ and $e_j=c_1(E_{p_j})$.

(4.13) $\alpha\in\Lambda_r$ is called a *root* of $\mathfrak{L}_r$ if $(\alpha, \alpha)=-2$ and $(\alpha, c)=0$. The set of all the roots of $\mathfrak{L}_r$ is denoted by $\mathfrak{R}_r$ or $\mathfrak{R}(\mathfrak{L}_r)$.

(4.14) LEMMA. *Let* $\{h, e_1, \cdots, e_r\}$ *be a normal base of* $\mathfrak{L}_r$. *Then* $\mathfrak{R}_r=\emptyset$ *if* $r\leqq 1$, $=\{\pm(e_1-e_2)\}$ *if* $r=2$, $=\{\pm(h-e_i-e_j-e_k)\}_{i<j<k}\cup\{\pm(e_i-e_j)_{i<j}\}$ *if* $3\leqq r\leqq 5$.

PROOF. Consider the case in which $r=4$. Suppose $\alpha=xh-y_1e_1-y_2e_2-y_3e_3-y_4e_4$ to be a root. Then $0=(c, \alpha)=3x-\sum y_j$ and $-2=(\alpha, \alpha)=x^2-\sum y_j^2$. Schwarz's inequality gives $4(x^2+2)=4\sum y_j^2\geqq(\sum y_j)^2=9x^2$. Hence $x^2\leqq 8/5$. So $x=0$ or ± 1. If $x=0$, then $\sum y_j=0$ and $\sum y_j^2=2$. Therefore $\alpha=e_i-e_j$ for some i, j. If $x=1$, then $\sum y_j=3$ and $\sum y_j^2=3$. This implies $\alpha=h-e_i-e_j-e_k$ for some $i<j<k$. Putting things together, we prove the assertion for $r=4$. Similar argument works in other cases too.

REMARK. We can determine $\mathfrak{R}_r$ in case $r\geqq 6$ too. In fact, $\mathfrak{R}_r$ is a finite set if and only if $r\leqq 8$. But we don't need the result for $r\geqq 5$ in this study of Del Pezzo manifolds.

(4.15) We see easily that $\mathfrak{R}_r$ is isomorphic to the root system corresponding to the Dynkin diagram of type A_1 if $r=2$, $A_1\cup A_2$ if $r=3$, A_4 if $r=4$, D_5 if $r=5$, E_r if $r=6, 7, 8$. In any case $\{h-e_1-e_2-e_3, e_1-e_2, e_2-e_3, \cdots, e_{r-1}-e_r\}$ gives rise to a simple root system if $r\geqq 3$.

(4.16) For any root α we define a homomorphism $S_\alpha: \Lambda_r\to\Lambda_r$ by $S_\alpha(x)=x+(x, \alpha)\alpha$. It is easy to see that $S_\alpha\circ S_\alpha$ is the identity and that $S_\alpha\in\mathrm{Aut}(\mathfrak{L}_r)$. By W_r we denote the subgroup of $\mathrm{Aut}(\mathfrak{L}_r)$ generated by all the S_α with $\alpha\in\mathfrak{R}_r$. Of course W_r is the Weyl group corresponding to the Dynkin diagram as in (4.15).

(4.17) DEFINITION. $e\in\Lambda_r$ is called an *exceptional cycle* of $\mathfrak{L}_r$ if $(e, c)=-(e, e)=1$. $\mathfrak{E}_r=\mathfrak{E}(\mathfrak{L}_r)$ denotes the set of all the exceptional cycles of $\mathfrak{L}_r$.

(4.18) LEMMA. *Let* $\{h, e_1, \cdots, e_r\}$ *be a normal base of* $\mathfrak{L}_r$. *Then* $\mathfrak{E}_r=\emptyset$ *if* $r=0$, $=\{e_1\}$ *if* $r=1$, $=\{e_i\}_{1\leqq i\leqq r}\cup\{h-e_i-e_j\}_{1\leqq i<j\leqq r}$ *if* $r=2, 3, 4$.

PROOF. The case $r=4$. Suppose $e=xh-\sum y_je_j$ be an exceptional cycle. Then $1=(e, c)=3x-\sum y_j$ and $-1=(e, e)=x^2-\sum y_j^2$. Schwarz's inequality gives $4(x^2+1)=4(\sum y_j^2)\geqq(\sum y_j)^2=(3x-1)^2$. This implies $x=0$ or 1. If $x=0$, then $\sum y_j=-1$ and $\sum y_j^2=1$. This implies $e=e_j$ for some j. If $x=1$, then $\sum y_j=2$ and $\sum y_j^2=2$. This implies $e=h-e_i-e_j$ for some $i<j$. Similar arguments prove the lemma.

(4.19) LEMMA. W_r *acts on* $\mathfrak{E}_r$ *transitively if* $r=3, 4$.

PROOF. $\{S_{e_i-e_j}\}_{i<j}$ generate all the permutations among $\{e_j\}$. Hence it suffices to show that e_1 and $h-e_2-e_3$ are contained in the same orbit. So the conclusion follows from $S_{h-e_1-e_2-e_3}(e_1)=h-e_2-e_3$.

(4.20) LEMMA. $W_r=\mathrm{Aut}(\mathfrak{L}_r)$ *for* $r\leqq 4$.

PROOF. Our claim follows from (4.18) if $r \leqq 2$. So we use the induction on r. Let $\sigma \in \mathrm{Aut}(\mathfrak{L}_r)$. (4.19) implies that $\tau\sigma(e_r)=e_r$ for some $\tau \in W_r$. $\tau\sigma$ induces an automorphism of the orthogonal sublattice to e_r in Λ_r. This lattice is generated by $h, e_1, \cdots, e_{r-1}$ and is naturally isomorphic to Λ_{r-1}. This inclusion induces inclusions $\mathfrak{R}_{r-1} \subset \mathfrak{R}_r$, $W_{r-1} \subset W_r$ and $\mathrm{Aut}(\mathfrak{L}_{r-1}) \subset \mathrm{Aut}(\mathfrak{L}_r)$. Using the induction hypothesis we infer that $\tau\sigma \in \mathrm{Aut}(\mathfrak{L}_{r-1})=W_{r-1} \subset W_r$. Hence $\sigma=\tau^{-1}(\tau\sigma) \in W_r$. Thus we prove the lemma.

REMARK. By the above method we can prove $W_r=\mathrm{Aut}(\mathfrak{L}_r)$ for greater r too.

(4.21) DEFINITION. $\alpha \in \mathfrak{R}_r$ is said to be a *component* of $\sigma \in W_r=\mathrm{Aut}(\mathfrak{L}_r)$ if there is a reduced expression $(\alpha_1, \cdots, \alpha_l)$ of σ such that $\alpha=\alpha_j$ for some j (see Def. (A1) in Appendix). α is said to be a *component* of a subgroup G of W_r if α is a component of some $\sigma \in G$. The sublattice generated by all the components of G is called the *envelope* of G.

(4.22) LEMMA. *For $x \in \Lambda_r$, the following conditions are equivalent to each other.*

a) *x is stabilized by $G \subset W_r$.*

b) *$x\alpha=0$ for any component α of G.*

c) *$x\gamma=0$ for any γ in the envelope of G.*

PROOF. The equivalence of b) and c) is obvious. b)$\Rightarrow$a) follows from Theorem A3 in Appendix. a)$\Rightarrow$b) follows from Proposition A2.

(4.23) Now, putting things together, we obtain the following conclusion:

Let (M, L) be a Del Pezzo threefold with $d=d(M, L) \geqq 5$. Let S be a general member of $|L|$. By (3.4), S is the blowing up of $\boldsymbol{P}^2$ with center being $r=9-d$ points unless $S \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Let λ be the monodromy action on $\mathfrak{L}(S)$ induced by $|L|$ (see (4.6)). Let Γ be the envelope of $\mathrm{Im}(\lambda) \subset \mathrm{Aut}(\mathfrak{L}(S)) \cong W_r$. Then $F \in Pic(S)$ comes from $Pic(M)$ if and only if $x \cdot c_1(F)=0$ for any $x \in \Gamma$.

Hence, in order to know $Pic(M)$, it suffices to study Γ.

§5. The cases in which $d(M, L) \geqq 6$.

(5.1) THEOREM. *There is no polarized manifold (M, L) with $\Delta(M, L)=1$, $d(M, L)=9$ and $n=\dim M \geqq 3$.*

PROOF. Suppose that such a polarized manifold exists. Let S be the two-dimensional rung of a regular smooth ladder of (M, L). S is a Del Pezzo surface with $d(S, L_S)=9$. Hence $(S, L_S) \cong (\boldsymbol{P}^2, 3H)$. This contradicts [7, (3.10)].

(5.2) In order to study the cases in which $d \leqq 8$, we establish several lemmata.

LEMMA. *Let (S, L) be a Del Pezzo surface and let e be an exceptional cycle in $H^2(S; \boldsymbol{Z})$. Then there is an exceptional curve E such that $c_1(E)=e$.*

PROOF. Let $F \in Pic(S)$ such that $c_1(F)=e$. The Riemann-Roch theorem implies $\chi(S, F)=1$. $h^2(S, F)=h^0(S, K^S-F)=0$ since $(K^S-F)L=-L^2-1<0$. Hence $h^0(S, F)>0$. Let $E \in |F|$. $EL=FL=eL=1$ implies that E is irreducible and reduced since L is ample. So $(K^S+E)E=(K^S+e)e=-2$ implies that $E \cong \boldsymbol{P}^1$. Clearly $E^2=-1$ and E is an exceptional curve.

(5.3) LEMMA. *Let (S, L) be a Del Pezzo surface with $d \geqq 3$. Then a curve C on S is an exceptional curve of the first kind if and only if $\rho_{|L|}: S \to \boldsymbol{P}^N$ maps C onto a line in $\boldsymbol{P}^N$.*

PROOF. If C is exceptional, then $K^S C=-1$ and $LC=1$. So $\rho_{|L|}(C)$ is a line. If conversely $\rho_{|L|}(C)$ is a line, then $LC=1$ and $(K^S+C)C=-2$ since L is very ample. So $C^2=-1$.

(5.4) LEMMA. *Let (S, L) be a Del Pezzo surface with $d \geqq 3$ and let E be an exceptional curve on S of the first kind. Then $H^1(S, E-tL)=0$ for any $t \geqq 1$.*

PROOF. (5.3) implies that $Bs|L-E|=\emptyset$. Moreover $(L-E)^2>0$ implies that $\dim \rho_{|L-E|}(S)=2$. So our assertion follows from the vanishing theorem due to Ramanujam [**15**].

(5.5) LEMMA. *Let (M, L) be a Del Pezzo threefold with $d \geqq 3$ and let S be a smooth rung of it. Suppose that $c_1(E_S)$ is an exceptional cycle for $E \in Pic(M)$. Then there is a Del Pezzo threefold $(N, L^\flat)$ and a point p on N such that $M=Q_p(N)$, $[E_p]=E$, $L+E=L^\flat$ and $d(N, L^\flat)=d(M, L)+1$.*

PROOF. By (5.2) $|E_S|$ contains an exceptional curve. (5.4) enables us to apply [**7**, (2.4)] and we obtain a member D of $|E|$. In view of [**7**, (5.4) a) and (5.3)] we infer that $D \cong \boldsymbol{P}^2$ and D can be blown down to a smooth point p. Let N be the manifold obtained by this contraction. $L^\flat=L+E$ comes from $Pic(N)$ since $L^\flat{}_D=0$. $L^\flat$ is ample on N by [**7**, (5.7)]. Moreover one sees easily $K^N=-2L^\flat$ and $L^3=(L^\flat)^3-1$. Putting things together we prove the lemma.

(5.6) THEOREM. *Let (M, L) be a polarized manifold with $\Delta(M, L)=1$, $d(M, L)=8$ and* $\dim M=3$. *Then $(M, L) \cong (\boldsymbol{P}^3, 2H)$.*

PROOF. Let S be a smooth member of $|L|$. Since $d(S, L)=8$, $S \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ or $Q_p(\boldsymbol{P}^2)$ with $p \in \boldsymbol{P}^2$ by (3.4). Assume that $S \cong Q_p(\boldsymbol{P}^2)$. Then, using (4.14), (4.20) and (4.6), we infer that the monodromy action λ on $H^2(S; \boldsymbol{Z})$ induced by $|L|$ is trivial. So (4.7) implies that $Pic(M) \cong Pic(S)$. Let F be the line bundle on M such that $F_S=[E_p]$. Applying (5.5), we obtain a Del Pezzo threefold with $d=9$. This contradicts (5.1).

Now we conclude that $S \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. By (4.2) we infer that there is a line bundle H on M such that $L=2H$ since L_S is divisible by two. We have $-K^M=2L=4H$. Hence [**5**, Theorem 2.1, c)] proves $(M, H) \cong (\boldsymbol{P}^3, \mathcal{O}(1))$.

Q. E. D.

(5.7) THEOREM. *There is no polarized manifold (M, L) such that $\Delta(M, L)=1$, $d(M, L)=8$ and* $\dim M \geqq 4$.

For a proof, use (5.6) and [7, (3.10)].

(5.8) THEOREM. *Let* (M, L) *be a polarized manifold with* $\Delta(M, L)=1$, $d(M, L)=7$ *and* $\dim M=3$. *Then* $(M, L)\cong(Q_p(\boldsymbol{P}^3), 2H-E_p)$ *for* $p\in\boldsymbol{P}^3$.

PROOF. Let S be a smooth member of $|L|$. (3.4) implies that $S\cong Q_{p_1\cup p_2}(\boldsymbol{P}^2)$. Put $E=H-E_{p_1}-E_{p_2}\in Pic(S)$. In view of (4.20), (4.14), (4.22) and (4.23) we infer that E comes from $Pic(M)$. So, applying (5.5), we have $(M, L)\cong(Q_p(M^\flat), L^\flat-E_p)$ for a Del Pezzo threefold $(M^\flat, L^\flat)$ with $d(M^\flat, L^\flat)=d(M, L)+1=8$ and a point p on $M^\flat$. Now (5.6) proves the assertion.

(5.9) COROLLARY. *There is no polarized manifold* (M, L) *such that* $\Delta(M, L)=1$, $d(M, L)=7$ *and* $\dim M\geqq 4$.

For a proof, use [7, (5.8)].

(5.10) Finally we consider the case in which $d=6$.

Let (M, L) be a Del Pezzo threefold with $d(M, L)=6$ and let S be a smooth rung of it. (3.4) implies that $S\cong Q_{p_1\cup p_2\cup p_3}(\boldsymbol{P}^2)$ and that p_1, p_2 and p_3 are not colinear. Let $\{h, e_1, e_2, e_3\}$ be the normal base of $\mathfrak{L}(S)$ as in (4.12). We consider how is the envelope Γ of the monodromy action on $H^2(S; \boldsymbol{Z})$ induced by $|L|$ (cf. (4.23)).

(5.11) LEMMA. *Let* C *be an exceptional curve on* S. *Then there are infinitely many lines lying on* M *which meet* C.

PROOF. Let Λ be the linear subsystem of $|L|$ on M consisting of members that contain C. Let $F=\{(D, l)|D\in\Lambda,\ l$ is a line lying on $D,\ l\cap C\neq\emptyset\}$. A general member D of Λ is smooth since $S\in\Lambda$. (4.18), (5.2) and (5.3) implies that there are exactly two lines l_1, l_2 with $(D, l_j)\in F$ for each $D\in\Lambda$. This implies $\dim F=\dim\Lambda=5$. On the other hand, for any line $l\neq C$, $F_l=\{D\in\Lambda|(D, l)\in F\}$ is at most of dimension 4. Our assertion follows from these facts.

(5.12) LEMMA. *There is a non-trivial effective divisor on* M *which is not ample.*

PROOF. Let C be the exceptional curve on S such that $c_1(C)=h-e_2-e_3$. Let T be the union of all the lines lying on M which meet C. (5.11) implies $\dim T=2$. So it suffices to show that T is not ample. Consider the exceptional curve E_{p_1} on S. Assume that $T\cap E_{p_1}\neq\emptyset$. Then there is a line l lying on M that meets both C and E_{p_1}. Let H be the hyperplane such that $H\cap M=S$. Clearly C and E_{p_1} are in H. Hence $l\subset H$. So $l\subset H\cap M=S$. Therefore, by (5.3), l is an exceptional curve on S. However, (4.18) implies that there is no exceptional cycle e on S such that $eC>0$ and $eE_{p_1}>0$. This contradiction proves $T\cap E_{p_1}=\emptyset$. So T is not ample.

(5.13) LEMMA. *One of* h, $h-e_1$, $h-e_2$, $h-e_3$ *comes from* $Pic(M)$.

PROOF. Suppose that h does not come from $Pic(M)$. Then, in view of (4.23) and (4.14), we infer that $h-e_1-e_2-e_3\in\Gamma$. Assume that two of e_1-e_2,

e_2-e_3 and e_1-e_3 are contained in Γ. Then Γ contains all the roots. By (4.23), this implies that only the integral multiples of L come from $Pic(M)$. Hence $Pic(M)$ is generated by L and any effective divisor on M is a positive multiple of L, and is consequently ample. This contradicts (5.12). Therefore at most only one of e_1-e_2, e_2-e_3 and e_1-e_3 can be contained in Γ. If neither e_2-e_3 nor e_1-e_3 is in Γ, then $h-e_3$ comes from $Pic(M)$ by (4.23) and (4.14). Similar arguments prove the assertion in other cases too.

(5.14) The case in which h comes from $Pic(M)$.

Let H be the line bundle on M such that $c_1(H)_S=h$. Put $H'=L-H$. Note that $Bs|H'_S|=\emptyset$ and $\rho_{|H'_S|}$ makes S a blowing up of $P'\cong \boldsymbol{P}^2$ with center being three points. By [**15**] we get $H^1(S, H-tL)=0$ for $t>0$. So [**7**, (2.3)] applies and we have $|H|_S=|H_S|=\emptyset$. Hence $Bs|H|$ is finite since S is ample. Similarly $Bs|H'|$ is finite. We have $H^3+H'^3=H^3+(L-H)^3=L^3-3L^2H+3LH^2=(L^2-3LH+3H^2)\{S\}=0$. Hence $H^3=H'^3=0$. So $Bs|H|=\emptyset$. $(H-L)L^2=-H'L^2<0$ proves $H^0(M, H-L)=0$ and $\dim|H|=\dim|H_S|=2$. Similarly $Bs|H'|=\emptyset$ and $\dim|H'|=2$. Now, combining $\rho_{|H|}$ and $\rho_{|H'|}$, we obtain a morphism $\rho: M\to \boldsymbol{P}^2_\alpha\times\boldsymbol{P}^2_\beta$ such that $H=\rho^*H_\alpha$ and $H'=\rho^*H_\beta$. ρ is finite since $L=\rho^*(H_\alpha+H_\beta)$ is ample. $W=\rho(M)$ is a divisor on $\boldsymbol{P}^2_\alpha\times\boldsymbol{P}^2_\beta$, so we put $[W]=aH_\alpha+bH_\beta$. Then $1=H'^2L\{M\}=\deg\rho\cdot H^2_\beta(H_\alpha+H_\beta)\{W\}=a\cdot\deg\rho$. This implies $a=\deg\rho=1$. Similarly $b=1$. Therefore W is isomorphic to either $D_0=\{((\alpha_0:\alpha_1:\alpha_2), (\beta_0:\beta_1:\beta_2))\in\boldsymbol{P}^2_\alpha\times\boldsymbol{P}^2_\beta|\alpha_0\beta_0=0\}$, $D_1=\{\alpha_0\beta_0+\alpha_1\beta_1=0\}$ or $D_2=\{\alpha_0\beta_0+\alpha_1\beta_1+\alpha_2\beta_2=0\}$. $W\cong D_1$ or D_2, since W is irreducible. In any case W is normal. Hence $\rho: M\to W$ is an isomorphism because ρ is birational and finite. So $W\cong D_2$ since it is smooth. Easily one sees $(M, L)\cong(D_2, H_\alpha+H_\beta)\cong(\boldsymbol{P}(T), H^T)$ where T is the tangent bundle of $\boldsymbol{P}^2$.

(5.15) The case in which $h-e_j$ comes from $Pic(M)$.

There is a line bundle H_ζ on M such that $c_1(H_\zeta)_S=h-e_j$. Note that $Bs|[H_\zeta]_S|=\emptyset$ and $\dim|[H_\zeta]_S|=1$. Put $H_\tau=L-H_\zeta$. Note that $Bs|[H_\tau]_S|=\emptyset$ and that $S\cong Q_{q_1\cup q_2}(\boldsymbol{P}^1_\xi\times\boldsymbol{P}^1_\eta)$ with $H_\tau=[H_\xi+H_\eta]_S$. It is easy to see $h^1(S, H_\zeta-tL)=h^1(S, H_\tau-tL)=0$ for $t>0$. Moreover, $(H_\zeta-L)L^2=-H_\tau L\{S\}<0$ implies $H^0(M, H_\zeta-L)=0$ and similarly we get $H^0(M, H_\tau-L)=0$. Applying [**7**, (2.8) and (2.3)], we obtain $\dim|H_\zeta|=1$, $\dim|H_\tau|=3$, $Bs|H_\zeta|=\emptyset$ and $Bs|H_\tau|$ is finite. Moreover $H^3_\tau=(L-H_\zeta)^3=L^3-3L^2H_\zeta=0$ proves $Bs|H_\tau|=\emptyset$ and $\dim\rho_{|H_\tau|}(M)\leqq 2$. Therefore $\rho_{|H_\tau|}(M)=\rho_{|H_\tau|}(S)\cong\boldsymbol{P}^1_\xi\times\boldsymbol{P}^1_\eta$. Combining $\rho_{|H_\zeta|}$ and $\rho_{|H_\tau|}$ we obtain a morphism $\rho: M\to\boldsymbol{P}^1_\zeta\times\boldsymbol{P}^1_\xi\times\boldsymbol{P}^1_\eta$ such that $L=\rho^*[H_\zeta+H_\xi+H_\eta]$. Hence ρ is finite. ρ is birational because $(H_\zeta+H_\xi+H_\eta)^3\{\boldsymbol{P}^1_\zeta\times\boldsymbol{P}^1_\xi\times\boldsymbol{P}^1_\eta\}=6$. Consequently ρ is an isomorphism.

(5.16) Now, combining (5.10)$\sim$(5.15), we prove the following

THEOREM. *Let (M, L) be a polarized manifold with $\Delta(M, L)=1$, $d(M, L)=6$ and $\dim M=3$. Then $(M, L)\cong(\boldsymbol{P}^1_\zeta\times\boldsymbol{P}^1_\xi\times\boldsymbol{P}^1_\eta, H_\zeta+H_\xi+H_\eta)$ or $(\boldsymbol{P}(T), H^T)$ where*

T is the tangent bundle of $\boldsymbol{P}^2$.

(5.17) THEOREM. *Let* (M, L) *be a polarized manifold with* $\Delta(M, L)=1$, $d(M, L)=6$ *and* $\dim M=4$. *Then* $(M, L)\cong(\boldsymbol{P}_\alpha^2\times\boldsymbol{P}_\beta^2, H_\alpha+H_\beta)$.

PROOF. Let D be a smooth member of L. $Pic(M)\cong Pic(D)$ by Lefschetz's theorem. Using (5.16) and [**17**, Proposition IV], we infer that $(D, L)\cong(D_2, H_\alpha+H_\beta)$ where D_2 is the same as in (5.14). It is easy to see $h^1(D, H_\alpha-tL)=h^1(D, H_\beta-tL)=0$ for $t>0$. Applying [**7**, (2.8)], we obtain a morphism ρ: $M\to\boldsymbol{P}_\alpha^2\times\boldsymbol{P}_\beta^2$ such that $L=\rho^*[H_\alpha+H_\beta]$. ρ is birational since $(H_\alpha+H_\beta)^4\{\boldsymbol{P}_\alpha^2\times\boldsymbol{P}_\beta^2\}=6$. Hence ρ is an isomorphism.

(5.18) COROLLARY. *There is no polarized manifold with* $\Delta(M, L)=1$, $d(M, L)=6$ *and* $\dim M\geqq 5$.

For a proof, use [**17**, Proposition IV].

§6. Applications.

(6.1) EXAMPLE. *Let* G *be the grassmann variety of lines on* $P=\boldsymbol{P}^4$. *Let* S *be a smooth hyperquadric in* P. *Let* M *be the subset of* G *consisting of lines contained in* S. *Then* M *is isomorphic to* $\boldsymbol{P}^3$. (Compare Tango [**18**]).

PROOF. Let $A=\{(x, l)\in S\times M\,|\,x\in l\}$. This is a subset of $F=\{(y, l)\in P\times G\,|\,y\in l\}=\boldsymbol{P}(T_P^\vee)$, where $T_P^\vee$ is the cotangent bundle of P. Let H^* denote the tautological line bundle $\mathcal{O}(1)$ on $\boldsymbol{P}(T_P^\vee)$. Let $\pi: F\to P$ be the projection and let $\pi_A: A\to S$ be the restriction of it. It is easy to see that $\pi_A^{-1}(x)$ is a smooth curve of degree two in $\pi^{-1}(x)\cong\boldsymbol{P}^3$ for any $x\in S$. So π_A makes A a $\boldsymbol{P}^1$-bundle over S. $(\pi_A)_*\mathcal{O}_A(H^*)$ defines a quotient vector bundle of rank three of $T_P^\vee|_S$, which is nothing other than the cotangent bundle $T_S^\vee$ of S. Hence $H^0(A, H^*+tH)\cong H^0(S, T_S^\vee[tH])$ for any t, where H is the line bundle defined by a hyperplane on P. Let $\psi: F\to G$ be the natural morphism and let $\psi_A: A\to M$ be its restriction. Then, for any $l\in M$, we have $\psi_A^{-1}(l)=\psi^{-1}(l)\cong l\subset P$ by π. So ψ_A makes A a $\boldsymbol{P}^1$-bundle over M. Moreover, $H\{\psi^{-1}(l)\}=1$ and $H^*\{\psi^{-1}(l)\}=-2$. This implies that $[H^*+2H]_A=L_A$ for a certain $L\in Pic(M)$.

$h^0(M, L)=h^0(A, H^*+2H)=h^0(S, T_S^\vee[2H])$. The exact sequence $0\to-2H_S\to T_P^\vee|_S\to T_S^\vee\to 0$ implies $h^0(S, T_S^\vee[2H])=h^0(S, T_P^\vee[2H])-1$. The exact sequence $0\to T_P^\vee[2H]\to H\oplus\cdots\oplus H\to 2H\to 0$ gives the exact sequence $0\to H^0(S, T_P^\vee[2H])\to H^0(S, H)\otimes H^0(S, H)\to H^0(S, 2H)$. The last homomorphism is nothing but the natural multiplication, and consequently it is surjective. Hence $h^0(S, T_P^\vee[2H])=11$. Putting things together we obtain $h^0(M, L)=10$.

Clearly M is a projective manifold with $b_2(M)=b_2(A)-1=b_2(S)=1$. So any non-trivial effective divisor on M is ample. Hence L is ample on M.

Now we calculate $d(M, L)$. We regard $B=\boldsymbol{P}(T_S^\vee)$ to be a submanifold of $\pi^{-1}(S)=\boldsymbol{P}(T_P^\vee|_S)$. Then A is a divisor on B. We may put $[A]_B=2H_B^*+aH$, since the fiber of π_A is a curve of degree two. Note that $H^{*3}+c_1(T_S)H^{*2}+c_2(T_S)H^*+c_3(T_S)=0$ in $H^6(B;\boldsymbol{Z})$. So we have $H^5=H^4H^*=0$, $H^3H^{*2}=2$, $H^2H^{*3}=-6$, $HH^{*4}=10$ and $H^{*5}=-10$ on B. On the other hand $(H^*+2H)^4(2H^*+aH)=L^4A=0$. This gives $a=2$. Now we have $d(M, L)=L^3HA=8$.

Combining the above fact, we infer that (M, L) is a polarized manifold with $\Delta(M, L)=1$, $d(M, L)=8$. So (5.6) applies.

(6.2) A construction of even dimensional rational smooth hypercubics.

Let D_α be a general member of $|2H_\alpha+H_\beta|$, a linear system on $\boldsymbol{P}_\alpha^n\times\boldsymbol{P}_\beta^n$. It is easy to see that $\rho_\alpha=\rho_{|H_\alpha|}$ makes D_α a $\boldsymbol{P}^{n-1}$-bundle over $\boldsymbol{P}_\alpha^n$. Similarly a general member D_β of $|H_\alpha+2H_\beta|$ is a $\boldsymbol{P}^{n-1}$-bundle over $\boldsymbol{P}_\beta^n$. Let A be the blowing-up of $\boldsymbol{P}_\alpha^n\times\boldsymbol{P}_\beta^n$ with center $C=D_\alpha\cap D_\beta$. Let E_α and E_β be the proper transform of D_α and D_β respectively. Then $E_\alpha\cap E_\beta=\emptyset$. It is easy to see $E_\alpha\cong D_\alpha$ and $[E_\alpha]_{E_\alpha}=[2H_\alpha+H_\beta]-[H_\alpha+2H_\beta]=H_\alpha-H_\beta$. Hence E_α can be blown down to a submanifold isomorphic to $\boldsymbol{P}_\alpha^n$. Similarly E_β can be blown down to $C_\beta\cong\boldsymbol{P}_\beta^n$. So we let $A=Q_{C_\alpha\cup C_\beta}(M)$ for a manifold M. Put $L=2H_\alpha+2H_\beta-E_C\in Pic(A)$. Restricting to fibers of $A\to M$, we easily see that L comes from $Pic(M)$. Moreover, using [7, (5.7)], we infer that L is ample on M.

We have $K^M+(n-1)E_\alpha+(n-1)E_\beta=K^A=-(n+1)H_\alpha-(n+1)H_\beta+E_C$: In view of $E_\alpha=2H_\alpha+H_\beta-E_C$ and $E_\beta=H_\alpha+2H_\beta-E_C$, we infer $K^M=-(2n-1)L$ on M. Namely, (M, L) is a Del Pezzo manifold.

Let us calculate $d(M, L)$. Put $\alpha=c_1(H_\alpha)$ and $\beta=c_1(H_\beta)$. We have $E_\alpha E_\beta=0$ in $H^4(A;\boldsymbol{Z})$ since they don't intersect. $L|_{E_\alpha}=2H_\alpha+2H_\beta-[E_C]_{E_\alpha}=2H_\alpha+2H_\beta-[H_\alpha+2H_\beta]=H_\alpha$ implies $L^tH_\alpha^sE_\alpha=H_\alpha^{t+s}E_\alpha=0$ in $H^*(A;\boldsymbol{Z})$ for $t+s>n$. In view of the above facts we make the following calculation: $d(M, L)=L^{2n}\{A\}=(E_\alpha+H_\beta)(E_\beta+H_\alpha)L^{2n-2}=(\alpha\beta)L^{2n-2}=\alpha\beta(E_\alpha+H_\beta)(E_\beta+H_\alpha)L^{2n-4}=(\alpha\beta)^2L^{2n-4}=\cdots=(\alpha\beta)^{n-1}L^2=(\alpha\beta)^n+(\alpha\beta)^{n-1}\alpha(2\alpha+\beta)+(\alpha\beta)^{n-1}\beta(\alpha+2\beta)=3$.

Now (2.1) proves that M is a hypercubic.

REMARK. C_α and C_β become n-dimensional linear subspaces disjoint to each other (more precisely, in a general position). The existence of such submanifolds of M characterizes the hypercubics of the above type. However, there is no such linear submanifold on a general hypercubic if $n\geqq 2$.

(6.3) Using our theory on Del Pezzo manifolds we can easily show that the tangent bundle of a Del Pezzo manifold (M, L) cannot be ample unless $(M, L)\cong(\boldsymbol{P}^3, 2H)$ or $(\boldsymbol{P}^2, 3H)$. Recently T. Mabuchi used this fact in order to prove the following conjecture of Hartshorne in three dimensional case: *if the tangent bundle of a manifold M is ample, then $M\cong\boldsymbol{P}^n$.*

(6.4) Any Kähler deformation of a Del Pezzo manifold (M, L) is also a Del Pezzo manifold. The results in § 3 and § 5 imply that (M, L) is rigid if $n=\dim M\geqq 2$ and $d=L^n\geqq 6$. Later we shall see that (M, L) is rigid if $n\geqq 3$ and $d=5$.

Appendix.

Here, for the convenience of the reader, we review several results on Weyl groups in the following mimeographed note:

Iwahori, N.; Lie algebras and Chevalley groups (in Japanese), Seminar Notes **12** of Dept. Math., Univ. of Tokyo, 1965.

First we fix our notation.

R: A given system of roots in a Euclidean space E. We consider not only simple roots, but the set of all the roots.

S_α: The reflection defined by $\alpha \in R$.

W: The Weyl group of R, i.e. the subgroup of $GL(E)$ generated by $\{S_\alpha\}_{\alpha\in R}$.

$E(\sigma)$: The eigenspace of $\sigma \in W$ with eigenvalue 1.

$l(\sigma) := \operatorname{codim} E(\sigma) = \dim E(\sigma)^\perp$, where $^\perp$ means the orthogonal complement in E.

DEFINITION A1. A sequence $(\alpha_1, \cdots, \alpha_l)$ of l roots is called a *reduced expression* of σ if $l = l(\sigma)$ and $\sigma = S_{\alpha_1} \cdots S_{\alpha_l}$. Clearly there is no sequence $(\beta_1, \cdots, \beta_m)$ of roots such that $\sigma = S_{\beta_1} \cdots S_{\beta_m}$ and $m < l(\sigma)$.

PROPOSITION A2. *Let $(\alpha_1, \cdots, \alpha_l)$ be a reduced expression of $\sigma \in W$. Then $\alpha_1, \cdots, \alpha_l$ are linearly independent and they form a base of $E(\sigma)^\perp$.*

PROOF. Let F be the subspace of E generated by $\alpha_1, \cdots, \alpha_l$. Clearly $F^\perp \subset E(\sigma)$. So $E(\sigma)^\perp \subset F$. Comparing the dimensions, we obtain the conclusion.

THEOREM A3. *Any $\sigma \in W$ has a reduced expression.*

First we prove several lemmata.

LEMMA A4. *$l(\sigma S_\alpha) = l(\sigma) \pm 1$ for any $\sigma \in W$, $\alpha \in R$.*

PROOF. $l(\sigma S_\alpha) \leqq l(\sigma)+1$ since $E(\sigma) \cap \alpha^\perp \subset E(\sigma S_\alpha)$. Similarly $l(\sigma) = l(\sigma S_\alpha S_\alpha) \leqq l(\sigma S_\alpha)+1$. On the other hand, $\det \tau = (-1)^{l(\tau)}$ for any $\tau \in W$ because the action of τ on E is real and orthogonal. Hence $l(\sigma S_\alpha) \neq l(\sigma)$. Combining the above facts, we prove the lemma.

LEMMA A5. *$l(\sigma S_\alpha) = l(\sigma) - 1$ if $\alpha \in E(\sigma)^\perp$.*

PROOF. Clearly $E(\sigma) \subset E(\sigma S_\alpha)$. So this follows from Lemma A4.

LEMMA A6. *σ is the identity if $E(\sigma)^\perp \cap R = \emptyset$.*

PROOF. Let $f_1, \cdots, f_{n-l(\sigma)}$ be a base of the dual space $E(\sigma)^\vee$ of $E(\sigma)$ and let $f_{n-l(\sigma)+1}, \cdots, f_n$ be a base of $(E(\sigma)^\perp)^\vee$. Then $f_1, \cdots, f_n$ define a coordinate system of E in a natural way. Let $\{\alpha_1, \cdots, \alpha_n\} \subset R$ be the set of all simple positive roots with respect to the lexicographical order defined by this coordinate system. Since $E(\sigma)^\perp \cap R = \emptyset$, there is a positive integer $k_j \leqq n - l(\sigma)$ for any $j = 1, \cdots, n$ such that $f_i(\alpha_j) = 0$ for any $i < k_j$ and $f_{k_j}(\alpha_j) > 0$. Here we regard $f_i \in E^\vee$ in view of $E = E(\sigma) \oplus E(\sigma)^\perp$. By definition of f_i, we have $f_i(x^\sigma) = f_i(x)$ for any $x \in E$ and $i \leqq n - l(\sigma)$. Hence α_j^σ is positive for each $1 \leqq j \leqq n$. This implies that σ is the identity (see [**2**, Chap. VI, 1, 5, 6 and 7]).

Now we prove Theorem A3. We use the induction on $l=l(\sigma)$ since everything is trivial when $l=0$. Suppose $l>0$. Then σ is not the identity, and $E(\sigma)^{\perp}\cap R\neq\emptyset$ by Lemma A6. Let $\alpha\in E(\sigma)^{\perp}\cap R$. Then $l(\sigma S_{\alpha})=l-1$ by Lemma A5. Hence we can find a reduced expression $(\alpha_l, \cdots, \alpha_{l-1})$ of σS_{α}. Then obviously $(\alpha_l, \cdots, \alpha_{l-1}, \alpha)$ is a reduced expression of σ.

Supplementary notes.

After the present paper was finished, I found that V. A. Iskovskih has obtained very similar results in the following paper:

[I] Fano 3-folds, I (translated by M. Reid), Izv. Akad. Nauk SSSR, AMS-translations, **11** (1977), 485-527.

Every Fano threefold of index 2 in his terminology is a Del Pezzo threefold. The converse is also true except $(\boldsymbol{P}^3, 2H)$. Therefore, many of my results are included in Theorem (4.2) of [**I**]. Nevertheless I think the present paper to be worth publishing because it deals with higher-dimensional cases too, the methods of proofs are rather different, it does not need the hypothesis (1.14) in [**I**], and the statement of [**I**], (4.2) is not quite correct in case $d=6$. Namely, he overlooks the Fano threefold $\boldsymbol{P}(T_{\boldsymbol{P}^2})$ of index 2. The error lies in the argument (6.3) (e), where he claims that a degenerated Del Pezzo surface with one ordinary double point can be considered as the anticanonical model of the blow up of $\boldsymbol{P}^2$ with center being colinear three points. But there is another type of degenerated Del Pezzo surface. Let x_1, x_2, x_3 be points on $\boldsymbol{P}^2$, which is not colinear, but x_3 is infinitely near to x_2. Let S be the blow up of these points. Precisely speaking, $S=Q_{x_3}(Q_{x_1\cup x_2}(\boldsymbol{P}^2))$ where x_3 is a point on E_{x_2} but not on the proper transform of the line on $\boldsymbol{P}^2$ passing through x_1 and x_2. Contract the proper transform of E_{x_2} to a point w. Then we get a degenerated Del Pezzo surface S_w with an ordinary double point w. We see easily that there are exactly two lines on S_w which passes through w. Hence the claim (6.3) (e) fails to be true in general. Indeed, this is really the case when $V=\boldsymbol{P}(T_{\boldsymbol{P}^2})$.

Despite of this small defect, his paper is very interesting from my viewpoint too.

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