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Superposition of diffusion processes

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§ 1. Introduction.

Let Ω be a domain in R^d and S^i be hyper surfaces in R^d, $i=1, 2, \cdots$. Denote by $\mathcal{E}^i$ a symmetric form on $L^2(S^i; d\sigma^i)$, where $d\sigma^i$ is the surface element of S^i. Assume that surfaces don't meet each other and each $\mathcal{E}^i$ is expressed in local coordinates as an integro-differential form. We shall be concerned with a Dirichlet form $\mathcal{E}$ of local type on $L^2(\Omega)$ such that

$$(1.1)\qquad \begin{cases} C_0^\infty(\Omega) \text{ is a dense subspace of } \mathcal{D}[\mathcal{E}], \\ \mathcal{E}(u, v)=\sum_{p,q=1}^{d}\int_\Omega \frac{\partial u}{\partial x_p}\frac{\partial v}{\partial x_q}a_{pq}dx+\sum_{i=1}^{\infty}\mathcal{E}^i(u^i, v^i) \qquad u, v\in C_0^\infty(\Omega).^{(1)} \end{cases}$$

Here a_{pq} are Borel measurable, symmetric, locally bounded and locally uniformly elliptic. u^i denotes the restriction to S^i of u. The purpose of this paper is to construct a diffusion process on Ω, whose resolvent G_λ satisfies the equation

$$(1.2)\qquad \mathcal{E}_\lambda(G_\lambda f, \varphi)=(f, \varphi)_{L^2(\Omega)} \qquad \varphi\in\mathcal{D}[\mathcal{E}],$$

where $\mathcal{E}_\lambda(\ ,\ )=\mathcal{E}(\ ,\ )+\lambda(\ ,\ )_{L^2(\Omega)}$. With the aid of results by M. Fukushima [**5**], there are d-dimensional diffusion and d-1-dimensional diffusions associated with the first integro-differential term expressing the form $\mathcal{E}$ and the remaining $\mathcal{E}^i$'s respectively. So our diffusion process is considered to be superposition of such diffusions. By using the same results in [**5**] again, we have a diffusion process whose Dirichlet form is $\mathcal{E}$ and whose state space is however outside some set of zero capacity in general. Our present assertion is much stronger in that we get a nice diffusion on the entire space Ω.

In [**10**] N. Ikeda and S. Watanabe formulated a class of diffusions whose infinitesimal generators are not necessarily described by differential operators. Such diffusions are characterized by the system of measures called the system of generators. It should be noticed that some examples in [**10**] correspond to

(1) The definition of a Dirichlet form is referred to [6] (or see Proposition 4.1). $\mathcal{E}$ is called local type if $\mathcal{E}(u, v)=0$ for $u, v\in\mathcal{D}[\mathcal{E}]$ such that $\mathrm{Supp}[u]\cap\mathrm{Supp}[v]=\emptyset$. For a set E $L^2(E)=L^2(E; dx)$, dx being the Lebesgue measure on E, and $C_0^\infty(E)=$the space of infinitely differentiable functions with supports in E.

the special case in our context and the associated diffusions were constructed from Brownian motions by the method of skew product.

On the other hand, the contrary case that all $\mathcal{E}^i$ vanish reduces to the study of the self-adjoint differential operator $\sum\frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial}{\partial x_q}\right)$ with discontinuous coefficients a_{pq}. In this connection, extensive study has been made by de Giogi [8], J. Nash [14], G. Stampacchia [16] and particularly the Hölder continuity of resolvents has been shown. M. Kanda [11] and H. Kunita [12] could then construct a diffusion by making use of those analytical results.

Our problem is to show the continuity of solutions of (1.2) near the hyper surfaces. To this end we extend L. Nirenberg's methods [15] which were used to prove the differentiability of weak solutions of generalized Dirichlet problems. Then we can follow Kunita's arguments to construct a diffusion. In §2 we formulate our assumptions and main results and prove them in the subsequent three sections. Namely, in §3 we introduce some modified Sobolev spaces and study basic properties of those spaces and their duals. Moreover we establish generalized Sobolev's inequalities. They will play important roles later. In §4 in the same way as [16] we obtain a priori global estimates and local estimates for solutions of (1.2). Then by the method similar to [15] we prove that those solutions are Hölder continuous. §5 is devoted to the construction of diffusion processes. In §6 we see typical examples that all S^i are compact or noncompact.

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§2. Assumptions and results.

We define Radon measures $\nu_{pq}(dx)=\nu_{pq}(dx_1\cdots dx_d)$ on R^d as follows:

$$\nu_{pq}(dx)=\nu_{qp}(dx)=\begin{cases} dx_1\cdots dx_{d-1}[dx_d+\delta_{(0)}(dx_d)] & \text{if } 1\leqq p\leqq q\leqq d-1\,, \\ dx_1\cdots dx_{d-1}dx_d & \text{if } 1\leqq p\leqq q=d\,, \end{cases} \tag{2.1}$$

where dx_p, $p=1,\cdots,d$ are one-dimensional Lebesgue measures and $\delta_{(0)}(dx_d)$ is the δ-measure. Let $V=\{x\in R^d\,;\,|x|<1\}$ and $\mathcal{E}$ be a Dirichlet form on $L^2(V)$ satisfying the assumptions:

$$C_0^\infty(V) \text{ is densely included in } \mathcal{D}[\mathcal{E}]. \tag{2.2}$$

$$\mathcal{E}(u,v)=\sum_{p,q=1}^{d}\int_V\frac{\partial u}{\partial x_p}\frac{\partial v}{\partial x_q}a_{pq}d\nu_{pq}\qquad u,\,v\in C_0^\infty(V)\,, \tag{2.3}$$

where a_{pq} are Borel measurable, symmetric, bounded and satisfy the following:

(2.4) There exists a constant $\gamma\geqq 1$ such that

$$\gamma^{-1}\sum_{p=1}^{d}\xi_p^2 d\nu_{pp}\leqq\sum_{p,q=1}^{d}\xi_p\xi_q a_{pq}d\nu_{pq}\leqq\gamma\sum_{p=1}^{d}\xi_p^2 d\nu_{pp}\quad\text{on } V$$

for every $\xi=(\xi_1, \cdots, \xi_d)\in R^d$.

(2.5) $\qquad a_{pq}\in C_u^{[d/2]}(V_+)\cap C_u^{[d/2]}(V_-)$ and $a_{pq}(\cdot, 0)\in C_u^{[d/2]}(V_0)$, $\qquad 1\leqq p, q\leqq d$.[(2)]

Note that a_{pq} need not be continuous on V. Put $\mathcal{E}_\lambda(\,,\,)=\mathcal{E}(\,,\,)+\lambda(\,,\,)_{L^2(V)}$. Then we have the following:

THEOREM 1. *Under the conditions* (2.2), (2.3), (2.4) *and* (2.5) *there exists a unique diffusion process* X *on* V *whose resolvent* G_λ *satisfies*

$$\text{(2.6)}\qquad \begin{cases} G_\lambda f\in\mathcal{D}[\mathcal{E}] \\ \mathcal{E}_\lambda(G_\lambda f, \varphi)=(f, \varphi)_{L^2(V)} \qquad \varphi\in\mathcal{D}[\mathcal{E}], \end{cases}$$

where $f\in L^2(V)$ *and* $\lambda>0$.

Making use of Theorem 1, we can get a diffusion associated with $\mathcal{E}$ defined by (1.1). We shall formulate our statement more precisely. Let Ω be an arbitrary domain in R^d and S^i, $i=1, 2, \cdots$ be hyper surfaces in R^d with following properties:

(2.7) For each i there are a locally finite open covering $\{U^{ij}\}_{j=1}^{\infty}$ of $S^i\cap\Omega$ and one-to-one transformations Φ^{ij}, $j\geqq 1$ such that (i) $U^{i\cdot}\cap U^{k\cdot}=\emptyset$ if $i\neq k$, (ii) $\Phi^{ij}(U^{ij})=V$, $\Phi^{ij}(S^i\cap U^{ij})=\{(X_1, \cdots, X_d)\in V; X_d=0\}$, and (iii) Φ^{ij} and its inverse Ψ^{ij} are transformations of class $C^{[d/2]+1}$. Moreover there are bounded subdomains Ω^i, $i\geqq 1$ such that $\Omega^i\subset\Omega-\bigcup_{i\geq 1}S^i$ and $\bigcup_{i\geq 1}\Omega^i=\Omega-\bigcup_{i\geq 1}S^i$.

Let us introduce a Dirichlet form $\mathcal{E}$ of local type on $L^2(\Omega)$ and assume that:

(2.8) $\qquad C_0^\infty(\Omega)$ is a dense subset of $\mathcal{D}[\mathcal{E}]$.

$$\text{(2.9)}\qquad \mathcal{E}(u, v)=\begin{cases} \displaystyle\sum_{p,q=1}^{d}\int_\Omega\frac{\partial u}{\partial x_p}\frac{\partial v}{\partial x_q}\alpha_{pq}dx & \text{if } \mathrm{Supp}[u]\cap S^i=\emptyset \text{ for any } i, \\ \displaystyle\sum_{p,q=1}^{d}\int_V\frac{\partial(u\circ\Psi^{ij})}{\partial X_p}\frac{\partial(v\circ\Psi^{ij})}{\partial X_q}a_{pq}^{ij}\nu_{pq}(dX) & \\ & \text{if } \mathrm{Supp}[u]\subset U^{ij} \text{ for some } i, j, \end{cases}$$

for $u, v\in C_0^\infty(\Omega)$, where α_{pq} are Borel measurable, symmetric, locally bounded, locally uniformly elliptic, i.e. for any compact set $K\subset\Omega$, there is a constant $\gamma=\gamma(K)\geqq 1$ such that $\gamma^{-1}|\xi|^2\leqq\sum_{1\leq p,q\leq d}\xi_p\xi_q\alpha_{pq}\leqq\gamma|\xi|^2$ on K for every $\xi\in R^d$. a_{pq}^{ij} satisfy (2.4), (2.5) for each i, j.

(2) For a set E $C^k(E)=$the space of k times continuously differentiable functions in E, $C_u^k(E)=\{v\in C^k(E)$; all derivatives up to order k are uniformly continuous in $E\}$, $E_{+(-)}=\{(x_1, \cdots, x_d)\in E;\ x_d>(<)0\}$, $E_0=\{(x_1, \cdots, x_d)\in R^{d-1};\ (x_1, \cdots, x_{d-1}, 0)\in E\}$.

THEOREM 2. *For a Dirichlet form $\mathcal{E}$ of local type satisfying* (2.7), (2.8) *and* (2.9) *there exists a unique diffusion process X on Ω such that*

$$\begin{cases} G_\lambda f \in \mathcal{D}[\mathcal{E}] \\ \mathcal{E}_\lambda(G_\lambda f, \varphi) = (f, \varphi)_{L^2(\Omega)} \qquad \varphi \in \mathcal{D}[\mathcal{E}]\,, \end{cases} \tag{2.10}$$

where G_λ is the resolvent and $f \in L^2(\Omega)$, $\lambda > 0$.

These theorems will be proved in § 5.

§ 3. Some function spaces, their duals and a generalized Sobolev's inequality.

Let Ω be a domain in R^d and ν_{pq} be Radon measures defined by (2.1) ($p, q = 1, \cdots, d$). We consider the following space for $1 < s < \infty$.

$\mathcal{V}^s(\Omega) = \{u \in L^s(\Omega\,; d\nu_{11})$; for each p $(1 \leqq p \leqq d)$ u have a version $u^{(p)}$ such that $u = u^{(p)}$ ν_{pp}-a.e., $u^{(p)}$ is absolutely continuous in x_p for a.e. $(x_1, \cdots, x_{p-1}, x_{p+1}, \cdots, x_d)$ and $\dfrac{\partial u^{(p)}}{\partial x_p} \in L^s(\Omega)$ and if $p \neq d$, then $u^{(p)}(\cdot, 0)$ is absolutely continuous in x_p for a.e. $(x_1, \cdots, x_{p-1}, x_{p+1}, \cdots, x_{d-1})$ and $\dfrac{\partial u^{(p)}}{\partial x_p}(\cdot, 0) \in L^s(\Omega_0)\}$.

Every $u \in \mathcal{V}^s(\Omega)$ has the *weak partial derivative* v:

$$\int_\Omega u \frac{\partial \varphi}{\partial x_p} d\nu_{pp} = -\int_\Omega v \varphi \, d\nu_{pp} \qquad \varphi \in C_0^\infty(\Omega)\,.$$

We denote such v by $\dfrac{\partial u}{\partial x_p}$. Notice that a version $u^{(p)}$ can be chosen to be $\dfrac{\partial u^{(p)}}{\partial x_p} = \dfrac{\partial u}{\partial x_p}$.

The following is easily verified by the same method as in Sobolev spaces.

PROPOSITION 3.1. (i) *$\mathcal{V}^s(\Omega)$ is complete with respect to the norm*

$$|||u|||_{s,\Omega} = \Big(\sum_{p=1}^d \int_\Omega \Big| \frac{\partial u}{\partial x_p} \Big|^s d\nu_{pp} + \int_\Omega |u|^s d\nu_{11} \Big)^{1/s}. \tag{3.1}$$

(ii) *$\mathcal{V}^s(\Omega)$ is reflexive.*

(iii) *Let E be a subdomain of Ω with closure $\bar{E} \subset \Omega$. For $u \in \mathcal{V}^2(\Omega)$ and for p $(1 \leqq p \leqq d)$ the difference quotient*

$$\Delta_h^{(p)} u(x) = \frac{1}{h} \{u(x_1, \cdots, x_{p-1}, x_p + h, x_{p+1}, \cdots, x_d) - u(x_1, \cdots, x_d)\}$$

belongs to $\mathcal{V}^2(E)$ for sufficiently small h and satisfies

$$\|\Delta_h^{(p)} u\|_{L^2(E;\, d\nu_{pp})} \leqq |||u|||_{2,\Omega} \quad and \quad \lim_{h \to 0} \Big\| \Delta_h^{(p)} u - \frac{\partial u}{\partial x_p} \Big\|_{L^2(E;\, d\nu_{pp})} = 0\,. \tag{3.2}$$

Moreover we introduce two function spaces.

$\mathcal{F}^s(\Omega)$=the closure of $\{u\in C^1(\Omega);\ |||u|||_{s,\Omega}<\infty\}$ in the space $\mathcal{U}^s(\Omega)$.

$\mathcal{F}_0^s(\Omega)$=the closure of $C_0^\infty(\Omega)$ in the space $\mathcal{U}^s(\Omega)$.

The following is obvious.

PROPOSITION 3.2. *$\mathcal{F}^s(\Omega)$ and $\mathcal{F}_0^s(\Omega)$ are complete and reflexive.*

Here we shall assume that:

(3.3) $\{(x_1,\cdots,x_d)\in R^d;\ (x_1,\cdots,x_{d-1})\in\Omega_0,\ |x_d|<\delta\}\subset\Omega$ for some $\delta>0$.

In the presence of this assumption we make two remarks on $\mathcal{F}^2(\Omega)$. Let us define the norm $|\cdot|_{s,\Omega}$ by

$$|u|_{s,\Omega}=\Big(\sum_{p=1}^d\int_\Omega\Big|\frac{\partial u}{\partial x_p}\Big|^s d\nu_{pp}+\int_\Omega|u|^s dx\Big)^{1/s}. \tag{3.4}$$

PROPOSITION 3.3. *Under the assumption* (3.3), (i) *$|\cdot|_{2,\Omega}$ is equivalent with $|||\cdot|||_{2,\Omega}$ in the space $\mathcal{F}^2(\Omega)$, and* (ii) *each $u\in\mathcal{F}^2(\Omega)$ has a version $\tilde u$ such that $u=\tilde u$ a.e. in Ω, $u(\cdot,0)=\tilde u(\cdot,0)$ a.e. in Ω_0 and $\tilde u(x',x_d)$ is absolutely continuous in x_d for a.e. $x'=(x_1,\cdots,x_{d-1})\in\Omega_0$.*

PROOF. (i) Let α be a function belonging to $C_0^\infty(R^1)$ such that $\alpha(0)=1$ and $\alpha(-\delta)=0$. Then

$$\begin{aligned}\int_{\Omega_0}u(x',0)^2dx'&=\int_{\Omega_0}\Big(\int_{-\delta}^0\frac{d}{dx_d}\{\alpha(x_d)u(x',x_d)\}\,dx_d\Big)^2dx'\\&\leqq c\Big(\sum_{p=1}^d\int_\Omega\Big(\frac{\partial u}{\partial x_p}\Big)^2dx+\int_\Omega u^2dx\Big)\end{aligned} \tag{3.5}$$

for every $u\in C^1(\Omega)$ with $|u|_{2,\Omega}<\infty$, $c=c(\delta)$ being some constant. Hence $\{u\in C^1(\Omega);\ |u|_{2,\Omega}<\infty\}$ coincides with $\{u\in C^1(\Omega);\ |||u|||_{2,\Omega}<\infty\}$, and the inequality (3.5) extends by continuity to all u of $\mathcal{F}^2(\Omega)$. This implies the assertion of (i).

(ii) Fix any $u\in\mathcal{F}^2(\Omega)$. There is a version $\tilde u$ such that $u=\tilde u$ a.e. in Ω, $\tilde u(x',x_d)$ is absolutely continuous in x_d for a.e. x' and $\dfrac{\partial u}{\partial x_d}=\dfrac{\partial\tilde u}{\partial x_d}$ because of $\mathcal{F}^2(\Omega)\subset\mathcal{U}^2(\Omega)$. This $\tilde u$ satisfies the required property: $u(x',0)=\tilde u(x',0)$ a.e. x' in Ω_0. Indeed, choosing a sequence $\{u_n\}_{n=1}^\infty\subset C^1(\Omega)$ such that $|u_n|_{2,\Omega}<\infty$ and $|u-u_n|_{2,\Omega}\to0$ as $n\to\infty$, we see that for the same function α as in (i)

$$\begin{aligned}&\int_{\Omega_0}|u(x',0)-\tilde u(x',0)|^2dx'\\&\leqq2\Big[\int_{\Omega_0}|u(x',0)-u_n(x',0)|^2dx'\\&\qquad+\int_{\Omega_0}\Big|\int_{-\delta}^0\frac{\partial}{\partial x_d}\{\alpha(x_d)u_n(x',x_d)-\alpha(x_d)\tilde u(x',x_d)\}\,dx_d\Big|^2dx'\Big]\end{aligned}$$

$$\leqq 2\Big(c+\int_{-\delta}^{0}\alpha(x_d)^2dx_d+\int_{-\delta}^{0}\alpha'(x_d)^2dx_d\Big)|u-u_n|_{2,\Omega}^2\to 0 \quad \text{as} \quad n\to\infty.$$

Q. E. D.

Combining the way in [**9**, § 0, Example 1] with that in [**16**, Lemma 1.1], we are next led to

PROPOSITION 3.4. *Let f be a uniformly Lipschitz function on R^1 such that the derivative f' is continuous except at a finite number of points of R^1. If $u\in\mathcal{F}^s(\Omega)$, then $f(u)\in\mathcal{F}^s(\Omega)$ and for each p*

$$\frac{\partial}{\partial x_p}f(u)=f'(u)\frac{\partial u}{\partial x_p},$$

where the right hand side is understood to be zero if $f'(u)$ is not defined as a function of x. Moreover, if $f(0)=0$ and $u\in\mathcal{F}_0^s(\Omega)$, then $f(u)\in\mathcal{F}_0^s(\Omega)$.

REMARK 3.5. Taking $f(x)=|x|$ $(x\in R^1)$, we have the following: If $u, v\in\mathcal{F}^s(\Omega)$ [resp. $\mathcal{F}_0^s(\Omega)$], then $u\vee v=(u+v+|u-v|)/2\in\mathcal{F}^s(\Omega)$ [resp. $\mathcal{F}_0^s(\Omega)$] and hence if $u\in\mathcal{F}^s(\Omega)$ [resp. $\mathcal{F}_0^s(\Omega)$], then $(0\vee u)\wedge 1$ (=the unit contraction of $u)\in\mathcal{F}^s(\Omega)$ [resp. $\mathcal{F}_0^s(\Omega)$].[(3)]

Now we study the structure of the dual space of $\mathcal{F}_0^s(\Omega)$. We denote it by $\mathcal{F}^{-s'}(\Omega)$ with $1/s+1/s'=1$.

PROPOSITION 3.6. *In order that a distribution T belongs to $\mathcal{F}^{-s}(\Omega)$ it is necessary and sufficient that there exist (in general, nonunique) $f_p\in L^s(\Omega)$ $(0\leqq p\leqq d)$, $g_p\in L^s(\Omega_0)$ $(0\leqq p\leqq d-1)$ such that*

$$T=f_0-\sum_{p=1}^{d}\frac{\partial f_p}{\partial x_p}+(g_0\times\delta_{\{0\}}^{(d)})-\sum_{p=1}^{d-1}\frac{\partial(g_p\times\delta_{\{0\}}^{(d)})}{\partial x_p} \tag{3.6}$$

where $\delta_{\{0\}}^{(d)}$ is the Dirac distribution concentrated at the point $0\in R^1$. Furthermore

$$\|T\|_{\mathcal{F}^{-s}(\Omega)}=\inf\Big(\sum_{p=0}^{d}\int_{\Omega}|f_p|^s dx+\sum_{p=0}^{d-1}\int_{\Omega_0}|g_p|^s dx'\Big)^{1/s},$$

the infimum being taken over the set of all $f_p\in L^s(\Omega)$, $g_p\in L^s(\Omega_0)$ satisfying (3.6). dx' denotes the Lebesgue measure on Ω_0.

PROOF. $\mathcal{F}_0^{s'}(\Omega)$ can be identified with a closed subspace W of $[L^{s'}(\Omega)]^{d+1}\times[L^{s'}(\Omega_0)]^d$ by means of the map $P: u\to\Big(u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_d}, u(\cdot, 0), \frac{\partial u}{\partial x_1}(\cdot, 0), \cdots, \frac{\partial u}{\partial x_{d-1}}(\cdot, 0)\Big)$. The operator T^* defined by $\langle T^*, Pu\rangle=\langle T, u\rangle$ is an element of W', the dual of W. Appealing to the Hahn-Banach theorem and the Riesz theorem we find $f_p\in L^s(\Omega)$ $(0\leqq p\leqq d)$ and $g_p\in L^s(\Omega_0)$ $(0\leqq p\leqq d-1)$ such that for $u\in\mathcal{F}_0^{s'}(\Omega)$

(3) $u\vee v=\max\{u, v\}$, $u\wedge v=\min\{u, v\}$.

$$\langle T, u\rangle=\int_{\Omega}\Big(f_0u+\sum_{p=1}^{d} f_p\frac{\partial u}{\partial x_p}\Big)dx+\int_{\Omega_0}\Big(g_0u(\cdot, 0)+\sum_{p=1}^{d-1} g_p\frac{\partial u}{\partial x_p}(\cdot, 0)\Big)dx'.$$

In particular, for $u\in C_0^\infty(\Omega)\subset\mathcal{F}_0^{s'}(\Omega)$

$$\langle T, u\rangle=\langle f_0-\sum_{p=1}^{d}\frac{\partial f_p}{\partial x_p}+(g_0\times\delta_{\{0\}}^{(d)})-\sum_{p=1}^{d-1}\frac{\partial(g_p\times\delta_{\{0\}}^{(d)})}{\partial X_p}, u\rangle .$$

This extends by continuity to all $u\in\mathcal{F}_0^{s'}(\Omega)$ since $C_0^\infty(\Omega)$ is dense in $\mathcal{F}_0^{s'}(\Omega)$.

Conversely suppose that a distribution T is represented as (3.6). Then by Hölder's inequality

$$|\langle T, u\rangle|\leqq\Big(\sum_{p=0}^{d}\int_{\Omega}|f_p|^s dx+\sum_{p=0}^{d-1}\int_{\Omega_0}|g_p|^s dx'\Big)^{1/s}|||u|||_{s',\Omega}$$

for every $u\in C_0^\infty(\Omega)$, which extends by continuity to all $u\in\mathcal{F}_0^{s'}(\Omega)$ because of the denseness of $C_0^\infty(\Omega)$ in $\mathcal{F}_0^{s'}(\Omega)$. Hence $T\in\mathcal{F}^{-s}(\Omega)$. Thus the proof is completed. Q. E. D.

REMARK 3.7. When (3.3) is satisfied and for $s=2$, by virtue of Proposition 3.3 (i) we can take $g_0\equiv 0$.

Next we want to prove

PROPOSITION 3.8. *If* (3.3) *is satisfied, then* $C_0^\infty(\Omega)$ *is dense in* $\mathcal{F}^{-s}(\Omega)$.

This follows immediately from

LEMMA 3.9. *Let* $f\in L^s(\Omega_0)$ *and* $1\leqq p\leqq d-1$. *Under the condition* (3.3) *for any* $\varepsilon>0$ *there exist* φ, $\phi\in C_0^\infty(\Omega)$ *such that*

$$\|f\times\delta_{\{0\}}^{(d)}-\varphi\|_{\mathcal{F}^{-s}(\Omega)}<\varepsilon , \tag{3.7}$$

$$\Big\|\frac{\partial}{\partial x_p}(f\times\delta_{\{0\}}^{(d)})-\phi\Big\|_{\mathcal{F}^{-s}(\Omega)}<\varepsilon . \tag{3.8}$$

PROOF. We may suppose $p=1$. Let us choose $h\in C_0^\infty(\Omega_0)$ and $\alpha\in C_0^\infty(R^1)$ such that $\|f-h\|_{L^s(\Omega_0)}<\varepsilon/3$ and $\alpha(0)=1$, $\alpha(-\delta)=0$. Then $g_1(x', x_d)\equiv\frac{\partial h}{\partial x_1}(x')I_{(-\delta,0)}(x_d)\alpha'(x_d)$ and $g_2(x', x_d)\equiv\frac{\partial h}{\partial x_1}(x')I_{(-\delta,0)}(x_d)\alpha(x_d)$ belong to $L^s(\Omega)$ and hence there are ϕ_1, $\phi_2\in C_0^\infty(\Omega)$ such that $\|g_i-\phi_i\|_{L^s(\Omega)}<\varepsilon/3$ for each $i=1, 2$. We shall show that $\phi\equiv\phi_1-\frac{\partial\phi_2}{\partial x_d}$ satisfies (3.8). By integration by parts

$$\Big\langle g_1-\frac{\partial g_2}{\partial x_d}, u\Big\rangle$$

$$=\int_{\Omega_0}\frac{\partial h}{\partial x_1}(x')dx'\int_{-\delta}^{0}\alpha'(x_d)u(x', x_d)dx_d+\int_{\Omega_0}\frac{\partial h}{\partial x_1}(x')dx'\int_{-\delta}^{0}\alpha(x_d)\frac{\partial u}{\partial x_d}(x', x_d)dx_d$$

$$=\int_{\Omega_0}\frac{\partial h}{\partial x_1}(x')u(x', 0)dx'$$

$$=\left\langle \frac{\partial}{\partial x_1}(h\times\delta_{\{0\}}^{(d)}),\ u\right\rangle,$$

so that

$$\left|\left\langle \frac{\partial}{\partial x_1}(f\times\delta_{\{0\}}^{(d)})-\phi,\ u\right\rangle\right|$$

$$\leqq\left|\left\langle \frac{\partial}{\partial x_1}(f\times\delta_{\{0\}}^{(d)})-\frac{\partial}{\partial x_1}(h\times\delta_{\{0\}}^{(d)}),\ u\right\rangle\right|+|\langle g_1-\phi_1,\ u\rangle|+\left|\left\langle \frac{\partial g_2}{\partial x_d}-\frac{\partial\phi_2}{\partial x_d},\ u\right\rangle\right|$$

$$\leqq\|f-h\|_{L^s(\Omega_0)}\left\|\frac{\partial u}{\partial x_1}(\cdot,\ 0)\right\|_{L^{s'}(\Omega_0)}+\|g_1-\phi_1\|_{L^s(\Omega)}\|u\|_{L^{s'}(\Omega)}$$

$$+\|g_2-\phi_2\|_{L^s(\Omega)}\left\|\frac{\partial u}{\partial x_d}\right\|_{L^{s'}(\Omega)}$$

$$<\varepsilon|||u|||_{s',\Omega},$$

for every $u\in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $\mathcal{F}_0^{s'}(\Omega)$, we have (3.8). Moreover noting that

$$\langle f\times\delta_{\{0\}}^{(d)},\ u\rangle=\left\langle f_1-\frac{\partial f_2}{\partial x_d},\ u\right\rangle \quad \text{for}\quad u\in C_0^\infty(\Omega),$$

with $f_1(x',\ x_d)\equiv f(x')I_{(-\delta,0)}(x_d)\alpha'(x_d)$, $f_2(x',\ x_d)\equiv f(x')I_{(-\delta,0)}(x_d)\alpha(x_d)$, we obtain (3.7). Q. E. D.

REMARK 3.10. If $f\in L^s(\Omega_0)\cap L^t(\Omega_0)$, it is possible to select a function φ [resp. ϕ] satisfying both of $\|f\times\delta_{\{0\}}^{(d)}-\varphi\|_{\mathcal{F}^{-s}(\Omega)}<\varepsilon$ and $\|f\times\delta_{\{0\}}^{(d)}-\varphi\|_{\mathcal{F}^{-t}(\Omega)}<\varepsilon$ $\left[\text{resp. } \left\|\frac{\partial}{\partial x_p}(f\times\delta_{\{0\}}^{(d)})-\phi\right\|_{\mathcal{F}^{-s}(\Omega)}<\varepsilon \text{ and } \left\|\frac{\partial}{\partial x_p}(f\times\delta_{\{0\}}^{(d)})-\phi\right\|_{\mathcal{F}^{-t}(\Omega)}<\varepsilon\right]$.

In the above proof we may take $\bar{g}_1(x',\ x_d)\equiv-\frac{\partial h}{\partial x_1}(x')I_{(0,\delta)}(x_d)\beta'(x_d)$, $\bar{g}_2(x',\ x_d)\equiv-\frac{\partial h}{\partial x_1}(x')I_{(0,\delta)}(x_d)\beta(x_d)$, $\bar{f}_1(x',\ x_d)\equiv-f(x')I_{(0,\delta)}(x_d)\beta'(x_d)$, $\bar{f}_2(x',\ x_d)$ $\equiv-f(x')I_{(0,\delta)}(x_d)\beta(x_d)$ instead of g_1, g_2, f_1, f_2 respectively. Here β is a function in $C_0^\infty(R^1)$ such that $\beta(0)=1$, $\beta(\delta)=0$. Therefore if $\Omega=R^d$ and $f(x')=0$, $x'\in B_0$, B being an open ball in R^d, then φ, $\phi\in C_0^\infty(R^d)$ in (3.7), (3.8) can be chosen to be $\varphi=\phi=0$ in B. Thus we are led to

PROPOSITION 3.11. *Let B be an open ball in R^d and assume that $T\in\mathcal{F}^{-s}(R^d)\cap\mathcal{F}^{-2}(R^d)$ is represented as* (3.6) *by means of some $f_p\in L^s(R^d)\cap L^2(R^d)$ $(0\leqq p\leqq d)$, $g_p\in L^s(R^{d-1})\cap L^2(R^{d-1})$ $(0\leqq p\leqq d-1)$ such that $f_p=0$ on B, $g_p=0$ on B_0. Then there exists a sequence $\{\varphi_n\}_{n=1}^\infty\subset C_0^\infty(R^d)$ such that $\lim_{n\to\infty}\|T-\varphi_n\|_{\mathcal{F}^{-s}(R^d)}$ $=\lim_{n\to\infty}\|T-\varphi_n\|_{\mathcal{F}^{-2}(R^d)}=0$ and $\varphi_n=0$ on B $(n\geqq1)$.*

Now we shall conclude this section with inequalities which are analogous to Sobolev's one.

THEOREM 3.12. *For any s,* $1/2-1/d<1/s\leqq 1/2$, *there is a constant* $C_1=C_1(d, s)$ *such that*

$$\|u\|_{L^s(\Omega;\, d\nu_{11})}\leqq C_1 \|\!|u|\!\|_{2,\Omega} \qquad u\in \mathcal{F}_0^2(\Omega). \tag{3.9}$$

Moreover if Ω *is bounded, then*

$$\|u\|_{L^s(\Omega;\, d\nu_{11})}\leqq C_2\left\{\sum_{p=1}^{d}\int_\Omega\left(\frac{\partial u}{\partial x_p}\right)^2 d\nu_{pp}\right\}^{1/2} \qquad u\in\mathcal{F}_0^2(\Omega), \tag{3.10}$$

where C_2 *is a constant depending on d, s and* diam $\Omega\equiv \sup_{x,y\in\Omega}|x-y|$.

PROOF. It follows from [**2**; § 10] that

$$\|u\|_{L^s(\Omega)}\leqq c_1\left\{\sum_{p=1}^{d}\int_\Omega\left(\frac{\partial u}{\partial x_p}\right)^2 dx+\int_\Omega u^2 dx\right\}^{1/2} \qquad u\in C_0^\infty(\Omega)$$

with $c_1=c_1(d, s)$. Applying this to $u(\cdot, 0)$, we get

$$\|u(\cdot, 0)\|_{L^s(\Omega_0)}\leqq c_1(d-1, s)\left\{\sum_{p=1}^{d-1}\int_{\Omega_0}\left(\frac{\partial u}{\partial x_p}(x', 0)\right)^2 dx'+\int_{\Omega_0} u(x', 0)^2 dx'\right\}^{1/2}$$

where $x'=(x_1, \cdots, x_{d-1})\in\Omega_0$. Therefore (3.9) is valid for $u\in C_0^\infty(\Omega)$ and so it is for all $u\in\mathcal{F}_0^2(\Omega)$.

When diam $\Omega\equiv\rho<\infty$, it is deduced by the same method as in [**13**; Lemma 2] that

$$\|u\|_{L^s(\Omega)}\leqq c_2\left\{\sum_{p=1}^{d}\int_\Omega\left(\frac{\partial u}{\partial x_p}\right)^2 dx\right\}^{1/2}$$

for some constant $c_2=c_2(d, s, \rho)$ and for every $u\in C_0^\infty(\Omega)$, and hence

$$\|u\|_{L^s(\Omega;\, d\nu_{11})}\leqq \{c_2(d, s, \rho)+c_2(d-1, s, \rho)\}\left\{\sum_{p=1}^{d}\int_\Omega\left(\frac{\partial u}{\partial x_p}\right)^2 d\nu_{pp}\right\}^{1/2},$$

which completes the proof. Q. E. D.

REMARK 3.13. If Ω satisfies the condition (3.3), in view of Proposition 3.3 (i) we have that for some $C_3=C_3(d, s, \delta)$

$$\|u\|_{L^s(\Omega;\, d\nu_{11})}\leqq C_3 |u|_{2,\Omega} \qquad u\in\mathcal{F}_0^2(\Omega). \tag{3.11}$$

§ 4. Global and local estimates, Hölder continuity.

Let Ω be a domain in R^d. Throughout this section we shall assume (3.3). We define a symmetric bilinear form $\mathcal{E}_\Omega$ on $\mathcal{F}^2(\Omega)\times\mathcal{F}^2(\Omega)$ by

$$\mathcal{E}_\Omega(u, v)=\sum_{p,q=1}^{d}\int_\Omega \frac{\partial u}{\partial x_p}(x)\frac{\partial v}{\partial x_p}(x) a_{pq}\nu_{pq}(dx), \tag{4.1}$$

where derivatives are taken in the weak sense, ν_{pq} are defined by (2.1), and

a_{pq} are Borel measurable, symmetric, bounded and satisfies the following:

$$\gamma^{-1}\sum_{p=1}^{d}\xi_p^2 d\nu_{pp}\leqq\sum_{p,q=1}^{d}\xi_p\xi_q a_{pq}d\nu_{pq}\leqq\gamma\sum_{p=1}^{d}\xi_p^2 d\nu_{pp}\qquad\text{on }\Omega,\tag{4.2}$$

for every $\xi=(\xi_1,\cdots,\xi_d)\in R^d$, γ ($\geqq 1$) being a constant. Denote $\mathcal{E}_\Omega(\,,\,)+\lambda(\,,\,)_{L^2(\Omega)}$ by $\mathcal{E}_{\Omega,\lambda}$ for $\lambda\geqq 0$. We simply write $\mathcal{E}$ and $\mathcal{E}_\lambda$ instead of $\mathcal{E}_\Omega$ and $\mathcal{E}_{\Omega,\lambda}$ respectively. From the condition (4.2) the form $\mathcal{E}_\lambda$ with $\lambda>0$ is bounded and coercive on $\mathcal{F}^2(\Omega)\times\mathcal{F}^2(\Omega)$, that is,

$$|\mathcal{E}_\lambda(u,v)|\leqq(\gamma\vee\lambda)|u|_{2,\Omega}|v|_{2,\Omega},\tag{4.3}$$

$$\mathcal{E}_\lambda(u,u)\geqq(\gamma^{-1}\wedge\lambda)|u|_{2,\Omega}^2,\tag{4.4}$$

for every $u\in\mathcal{F}^2(\Omega)$. Of course (4.3) holds for $\lambda=0$. In view of Proposition 3.4 we obtain

PROPOSITION 4.1. *Both of $(\mathcal{F}^2(\Omega),\mathcal{E})$ and $(\mathcal{F}_0^2(\Omega),\mathcal{E})$ are Dirichlet spaces relative to $L^2(\Omega)$. Namely, for $\mathcal{F}=\mathcal{F}^2(\Omega)$ or $\mathcal{F}_0^2(\Omega)$,* (i) *$\mathcal{F}$ is dense in $L^2(\Omega)$,* (ii) *$\mathcal{E}$ is symmetric, bilinear and $\mathcal{E}(u,u)\geqq 0$, $u\in\mathcal{F}$,* (iii) *$\mathcal{F}$ is a Hilbert space with the inner product $\mathcal{E}_1(\,,\,)$, and* (iv) *the unit contraction operates to the form $\mathcal{E}$: if $u\in\mathcal{F}$, then $v\equiv(0\vee u)\wedge 1\in\mathcal{F}$ and $\mathcal{E}(v,v)\leqq\mathcal{E}(u,u)$.*

Now we shall show a priori global estimate for solutions of the following equation.

$$\begin{cases} u\in\mathcal{F}_0^2(\Omega)\\ \mathcal{E}_\lambda(u,\varphi)=\langle T,\varphi\rangle\qquad\varphi\in\mathcal{F}_0^2(\Omega),\end{cases}\tag{4.5}$$

where $\lambda\geqq 0$ and $T\in\mathcal{F}^{-2}(\Omega)$. It should be noticed that for $\lambda>0$ and $T\in\mathcal{F}^{-2}(\Omega)$ there exists a unique solution of (4.5) by the Riesz representation theorem. Moreover in the case when Ω is bounded and diam $\Omega=\rho$, by virtue of (3.10)

$$\mathcal{E}(u,u)\geqq D_1|u|_{2,\Omega}^2\qquad u\in\mathcal{F}_0^2(\Omega)\tag{4.6}$$

with some constant $D_1=D_1(d,\rho,\gamma)$ and hence a solution of (4.5) exists uniquely for $\lambda\geqq 0$ and $T\in\mathcal{F}^{-2}(\Omega)$.

THEOREM 4.2. *Let T be in $\mathcal{F}^{-s}(\Omega)\cap\mathcal{F}^{-2}(\Omega)$ with $s>d$. Then for any $\lambda>0$ there is a constant $D_2=D_2(d,s,\delta,\gamma,\lambda)$ such that the solution u of* (4.5) *satisfies*

$$\|u\|_{L^\infty(\Omega;\,d\nu_{11})}\leqq D_2\|T\|_{\mathcal{F}^{-s}(\Omega)}+\|u\|_{L^2(\Omega;\,d\nu_{11})}.\tag{4.7}$$

Furthermore if diam $\Omega\equiv\rho$ *is finite, the solution u of* (4.5) *with $\lambda\geqq 0$ satisfies*

$$\|u\|_{L^\infty(\Omega;\,d\nu_{11})}\leqq D_3\|T\|_{\mathcal{F}^{-s}(\Omega)},\tag{4.8}$$

where $D_3=D_3(d,s,\delta,\rho,\gamma)$.

PROOF. We use the same method as in [**16**; Theorem 4.2]. By Remark 3.7 we are given T represented as (3.6) by $g_0\equiv 0$ and some $f_p\in L^s(\Omega)\cap L^2(\Omega)$

$(0 \leqq p \leqq d)$, $g_p \in L^s(\Omega_0) \cap L^2(\Omega_0)$ $(1 \leqq p \leqq d-1)$. For $k>0$ we define

$$v(x)=\begin{cases} u(x)+k & \text{if } u(x)<-k \\ 0 & \text{if } -k \leqq u(x) \leqq k \\ u(x)-k & \text{if } k<u(x). \end{cases}$$

It follows from Proposition 3.4 that $v \in \mathcal{F}_0^2(\Omega)$, $\dfrac{\partial v}{\partial x_p}=\dfrac{\partial u}{\partial x_p}$ on $A(k)$ and $v=\dfrac{\partial v}{\partial x_p}=0$ on $A(k)^c$, where $A(k)=\{x \in \Omega\,;\, |u(x)|>k\}$. Hence by (4.4) we have

$$(\gamma^{-1} \wedge \lambda)|v|_{2,\Omega}^2 \leqq \mathcal{E}_\lambda(v, v) \leqq \mathcal{E}_\lambda(u, v)=\langle T, v\rangle .$$

On the other hand

$$\begin{aligned}\langle T, v\rangle &=\int_{A(k)} f_0 v\, dx+\sum_{p=1}^d \int_{A(k)} f_p \frac{\partial v}{\partial x_p} dx+\sum_{p=1}^{d-1} \int_{A(k)_0} g_p \frac{\partial v}{\partial x_p}(\cdot, 0) dx' \\ &\leqq\Big(\sum_{p=0}^d \int_{A(k)} f_p^2 dx+\sum_{p=1}^{d-1} \int_{A(k)_0} g_p^2 dx'\Big)^{1/2} |v|_{2,\Omega} .\end{aligned}$$

Therefore (3.11) and Hölder's inequality lead us to

$$\begin{aligned}\|v\|_{L^t(\Omega;\, d\nu_{11})} &\leqq c_1 |v|_{2,\Omega} \leqq c_1(\gamma \vee \lambda^{-1})\Big(\sum_{p=0}^d \int_{A(k)} f_p^2 dx+\sum_{p=1}^{d-1} \int_{A(k)_0} g_p^2 dx'\Big)^{1/2} \\ &\leqq c_1(\gamma \vee \lambda^{-1})(d+1)\Big(\sum_{p=0}^d \int_\Omega |f_p|^s dx+\sum_{p=1}^{d-1} \int_{\Omega_0} |g_p|^s dx'\Big)^{1/s} \nu_{11}(A(k))^{1/2-1/s},\end{aligned}$$

where $1/t=(1-1/d-1/s)/2$ and $c_1=c_1(d, s, \delta)$. Since

$$(h-k)\nu_{11}(A(h))^{1/t} \leqq \|v\|_{L^t(\Omega;\, d\nu_{11})} \qquad \text{for } h>k ,$$

we get

$$\nu_{11}(A(h)) \leqq c_2 \|T\|_{\mathcal{F}^{-s}(\Omega)}^t (h-k)^{-t} \nu_{11}(A(k))^{t(1/2-1/s)} \tag{4.9}$$

with $h>k$ and $c_2=c_2(d, s, \delta, \gamma, \lambda)$.

If $\rho=\operatorname{diam}\Omega<\infty$, by means of (3.10) there are constants $c_3=c_3(d, \rho)$, $c_4=c_4(d, s, \rho)$ such that

$$|v|_{2,\Omega} \leqq c_3\Big(\sum_{p=1}^d \int_{A(k)}\Big(\frac{\partial v}{\partial x_p}\Big)^2 d\nu_{pp}\Big)^{1/2}, \quad \|v\|_{L^t(\Omega;\, d\nu_{11})} \leqq c_4\Big(\sum_{p=1}^d \int_{A(k)}\Big(\frac{\partial v}{\partial x_p}\Big)^2 d\nu_{pp}\Big)^{1/2}$$

and hence

$$\|v\|_{L^t(\Omega;\, d\nu_{11})} \leqq c_3 c_4 \gamma(d+1)\Big(\sum_{p=0}^d \int_\Omega |f_p|^s dx+\sum_{p=1}^{d-1} \int_{\Omega_0} |g_p|^s dx'\Big)^{1/s} \nu_{11}(A(k))^{1/2-1/s}.$$

Thus we arrive at (4.9) with $c_5=c_5(d, s, \delta, \rho, \gamma)$ instead of c_2 in this case, too.

Since $t(1/2-1/s)>1$, appealing to [**16**; Lemma 4.1] we obtain the following estimate:

$$|u|\leqq k_0+c_6\|T\|_{\mathcal{F}^{-s}(\Omega)}\nu_{11}(A(k_0))^{1/2-1/s-1/t} \qquad \nu_{11}\text{-a. e. in } \Omega.$$

Here $k_0\geqq 0$ and c_6 is a constant depending on c_2 or c_5. Put $k_0=\|u\|_{L^2(\Omega;\, d\nu_{11})}$ or 0 according to diam $\Omega=\infty$ or $<\infty$, which establishes the desired estimates.

Q. E. D.

Next we turn to the following local estimates. From now on we are concerned with the estimates in the case that Ω is a cube $Q(a, \rho)$: $Q(a, \rho)=\{(x_1, \cdots, x_d)\in R^d;\ |x_p-a_p|<\rho/2,\ 1\leqq p\leqq d\}$, where $a=(a_1, \cdots, a_d)$.

THEOREM 4.3. *If u is a solution of*

$$(4.10)\qquad \begin{cases} u\in\mathcal{F}^2(Q(a, \rho)) \\ \mathcal{E}_\lambda(u, \varphi)=0 \qquad \varphi\in\mathcal{F}_0^2(Q(a, \rho)) \end{cases}$$

for some $\lambda\geqq 0$, then there is a constant $D_4=D_4(d, \rho, \gamma)$ such that

$$\|u\|_{L^\infty(Q(a,\rho/2);\, d\nu_{11})}\leqq D_4\|u\|_{L^2(Q(a,\rho);\, d\nu_{11})}\,.$$

In the course of the proof of Theorem 4.3, we still utilize the idea of Stampacchia [**16**; Theorems 5.1 and 5.2]. A function $u\in\mathcal{F}^2(\Omega)$ [resp. $\mathcal{F}_0^2(\Omega)$] is called to be nonnegative in a *generalized sense* if there is a sequence $\{u_n\}_{n=1}^\infty\subset\{u\in C^1(\Omega);\ |u|_{2,\Omega}<\infty\}$ [resp. $C_0^\infty(\Omega)$] of nonnegative functions such that $\lim_{n\to\infty}|u_n-u|_{2,\Omega}=0$. Observing Remark 3.5, (4.3) and (4.6) we obtain the following by the same method as [**16**; Theorem 3.4].

LEMMA 4.4. *Fix $\lambda\geqq 0$. If u satisfies*

$$(4.11)\qquad \begin{cases} u\in\mathcal{F}^2(Q(a, \rho)) \\ \mathcal{E}_\lambda(u, \varphi)\leqq 0 \qquad \varphi\geqq 0,\ \in C_0^\infty(Q(a, \rho)), \end{cases}$$

then $(u-k)\vee 0$ does so for every $k\geqq 0$.

LEMMA 4.5. *Let u be a solution of* (4.11) *for some $\lambda\geqq 0$. Assume that u is nonnegative in a generalized sense. Then for each s, $1/2-1/d<1/s\leqq 1/2$, there is a constant $D_5=D_5(d, s, \rho, \gamma)$ and we get*

$$\int_{Q(a,\rho)}(\varphi u)^2 d\nu_{11}\leqq D_5[\nu_{11}(Q(a, \rho)\cap\{\varphi u\neq 0\})]^{1-2/s}\sum_{p=1}^d\int_{Q(a,\rho)}\left(u\frac{\partial\varphi}{\partial x_p}\right)^2 d\nu_{pp}$$

for every $\varphi\in C_0^\infty(Q(a, \rho))$.

PROOF. Put $\Omega=Q(a, \rho)$. Since $\varphi^2 u$ belongs to $\mathcal{F}_0^2(\Omega)$ and $\varphi^2 u\geqq 0$ in a generalized sense, $\mathcal{E}_\lambda(u, \varphi^2 u)\leqq 0$ and so $\mathcal{E}(u, \varphi^2 u)\leqq 0$. Hence

$$\sum_{p,q=1}^d\int_\Omega\varphi^2\frac{\partial u}{\partial x_p}\frac{\partial u}{\partial x_q}a_{pq}d\nu_{pq}$$

$$\leqq -2\sum_{p,q=1}^{d}\int_{\Omega}\varphi\frac{\partial u}{\partial x_p}u\frac{\partial\varphi}{\partial x_q}a_{pq}d\nu_{pq}$$

$$\leqq 2\Big(\sum_{p,q=1}^{d}\int_{\Omega}\varphi^2\frac{\partial u}{\partial x_p}\frac{\partial u}{\partial x_q}a_{pq}d\nu_{pq}\Big)^{1/2}\Big(\sum_{p,q=1}^{d}\int_{\Omega}u^2\frac{\partial\varphi}{\partial x_p}\frac{\partial\varphi}{\partial x_q}a_{pq}d\nu_{pq}\Big)^{1/2},$$

that is,

$$\sum_{p,q=1}^{d}\int_{\Omega}\varphi^2\frac{\partial u}{\partial x_p}\frac{\partial u}{\partial x_q}a_{pq}d\nu_{pq}\leqq 4\sum_{p,q=1}^{d}\int_{\Omega}u^2\frac{\partial\varphi}{\partial x_p}\frac{\partial\varphi}{\partial x_q}a_{pq}d\nu_{pq}.$$

By condition (4.2)

$$\sum_{p=1}^{d}\int_{\Omega}\Big(\varphi\frac{\partial u}{\partial x_p}\Big)^2 d\nu_{pp}\leqq 4\gamma^2\sum_{p=1}^{d}\int_{\Omega}\Big(u\frac{\partial\varphi}{\partial x_p}\Big)^2 d\nu_{pp}.$$

Therefore by using Hölder's inequality and (3.10), we obtain

$$\int_{\Omega}(\varphi u)^2 d\nu_{11}$$

$$\leqq\Big\{\int_{\Omega}(\varphi u)^s d\nu_{11}\Big\}^{2/s}[\nu_{11}(\Omega\cap\{\varphi u\neq 0\})]^{1-2/s}$$

$$\leqq c[\nu_{11}(\Omega\cap\{\varphi u\neq 0\})]^{1-2/s}\sum_{p=1}^{d}\int_{\Omega}\Big(\frac{\partial\varphi}{\partial x_p}u+\varphi\frac{\partial u}{\partial x_q}\Big)^2 d\nu_{pp}$$

$$\leqq 2c(1+4\gamma^2)[\nu_{11}(\Omega\cap\{\varphi u\neq 0\})]^{1-2/s}\sum_{p=1}^{d}\int_{\Omega}\Big(u\frac{\partial\varphi}{\partial x_p}\Big)^2 d\nu_{pp},$$

where $c=c(d, s, \rho)$, completing the proof. Q. E. D.

PROOF OF THEOREM 4.3. Put $Q(\rho)=Q(a, \rho)$. Let u be a solution of (4.10) with some $\lambda\geqq 0$. For each $0<\rho_2<\rho_1\leqq\rho$ we shall choose a function $\varphi\in C_0^\infty(Q(\rho))$ such that $\varphi=1$ on $Q(\rho_2)$, $\varphi=0$ on $Q(\rho_1)^c$ and $|\varphi|\leqq 1$, $\left|\frac{\partial\varphi}{\partial x_p}\right|\leqq 4/(\rho_1-\rho_2)$. Since for all $k\geqq 0$, $(u-k)\vee 0$ satisfies (4.11) and is nonnegative in a generalized sense, we see by applying Lemma 4.5 to $s=2d/(d-1)$,

$$\int_{Q(\rho_2)\cap\{u>h\}}(u-h)^2 d\nu_{11}$$

$$\leqq\int_{Q(\rho)}\varphi^2((u-h)\vee 0)^2 d\nu_{11}$$

$$\leqq c_1(\rho_1-\rho_2)^{-2}[\nu_{11}(Q(\rho_1)\cap\{u>h\})]^{1/d}\int_{Q(\rho_1)\cap\{u>h\}}(u-h)^2 d\nu_{11}$$

$$\leqq c_1(\rho_1-\rho_2)^{-2}(h-k)^{-2/d}\Big(\int_{Q(\rho_1)\cap\{u>k\}}(u-k)^2 d\nu_{11}\Big)^{1+1/d},$$

for $h>k$, with $c_1=c_1(d, \rho, \gamma)$. Taking $\int_{Q(\rho)\cap\{u>h\}}(u-h)^2 d\nu_{11}$ as $\varphi(h, \rho)$ in [**16**; Lemma 5.1], we get

$$u \leqq c_2\|u\|_{L^2(Q(\rho);\, d\nu_{11})} \qquad \nu_{11}\text{-a.e. in } Q(\rho/2), \tag{4.12}$$

where $c_2=c_2(d, \rho, \gamma)$. The above argument may be repeated for $-u$. Thus the theorem is proved. Q. E. D.

We proceed to the Hölder continuity of solutions. In the rest of this section, besides the assumptions stated before the following is presented:

(4.13) $a_{pq}\in C_u^{[d/2]}(Q(a, \rho)_+)\cap C_u^{[d/2]}(Q(a, \rho)_-)$ and $a_{pq}(\cdot, 0)\in C_u^{[d/2]}(Q(a, \rho)_0)$ for every p, q.

THEOREM 4.6. *Assume* (4.13). *Let* $f\in W^{[d/2]-1}(Q(a, \rho))$ *and* $\lambda\geqq 0$.(4) *If* $u\in \mathcal{F}^2(Q(a, \rho))$ *satisfies* $\mathcal{E}_\lambda(u, \varphi)=(f, \varphi)_{L^2(Q(a,\rho))}$, $\varphi\in\mathcal{F}_0^2(Q(a, \rho))$, *then* u *is Hölder continuous on* $\overline{Q(a, r)}$ $(r<\rho)$ *with exponent* l, *where* $0<l\leqq 1/2$ *or* $0<l<1$ *according to* d *is odd or even.*

PROOF. The method of the proof is based on an idea of Nirenberg [**15**; §5]. Put $Q=Q(a, \rho)$. In view of the Sobolev imbedding theorem (cf. [**1**; Theorem 5.4]) and Proposition 3.3 (ii), it is sufficient to verify that αu belongs to $W^{[d/2]+1}(Q_+)\cap W^{[d/2]+1}(Q_-)$ for every $\alpha\in C_0^\infty(Q)$. Fix arbitrary $\alpha\in C_0^\infty(Q)$ and $\varphi\in\mathcal{F}^2(Q)$. Since $\alpha\varphi\in\mathcal{F}_0^2(Q)$, $\mathcal{E}_\lambda(u, \alpha\varphi)=(f, \alpha\varphi)_{L^2(Q)}$. The left quantity equals to $\mathcal{E}_\lambda(\alpha u, \varphi)+E(u, \varphi)$:

$$E(u, \varphi)=\sum_{p,q=1}^{d}\left(\int_Q \frac{\partial u}{\partial x_p}\frac{\partial \alpha}{\partial x_q}\varphi a_{pq}d\nu_{pq}-\int_Q \frac{\partial \alpha}{\partial x_p}u\frac{\partial \varphi}{\partial x_q}a_{pq}d\nu_{pq}\right).$$

Consider the difference quotient of φ:

$$\Delta_h^{(r)}\varphi(x)=\frac{1}{h}\{\varphi(x_1, \cdots, x_{r-1}, x_r+h, x_{r+1}, \cdots, x_d)-\varphi(x_1, \cdots, x_d)\},$$

where $1\leqq r\leqq d-1$. Because $\alpha\Delta_h^{(r)}\varphi\in\mathcal{F}_0^2(Q)$ for sufficiently small h, we get

$$\mathcal{E}_\lambda(\Delta_{-h}^{(r)}(\alpha u), \varphi)=-(f, \alpha\Delta_h^{(r)}\varphi)_{L^2(Q)}+F_h(u, \varphi)+E(u, \Delta_h^{(r)}\varphi),$$

where

$$\begin{aligned}F_h(u, \varphi)&=\mathcal{E}_\lambda(\Delta_{-h}^{(r)}(\alpha u), \varphi)+\mathcal{E}_\lambda(\alpha u, \Delta_h^{(r)}\varphi)\\ &=-\sum_{p,q=1}^{d}\int_Q \frac{\partial(\alpha u)}{\partial x_p}(x)\frac{\partial\varphi}{\partial x_q}(x_1, \cdots, x_r+h, \cdots, x_d)\Delta_h^{(r)}a_{pq}(x)\nu_{pq}(dx).\end{aligned}$$

An application of (3.2), (4.2), (4.13) and Proposition 3.3 (i) gives

(4) For a domain E $W^k(E)$ denotes the Sobolev space $W^{k,2}(E)=\{u\in L^2(E)\,;\, D^\alpha u\in L^2(E)$ for $0\leqq|\alpha|\leqq k\}$, where $D^\alpha u$ is the distributional partial derivative.

$$|\mathcal{E}_\lambda(\Delta_{-h}^{(r)}(\alpha u),\ \varphi)|\leqq c_1(|||\varphi|||_{2,Q}+\sum_{p=1}^{d}\left\|\frac{\partial\varphi}{\partial x_p}\right\|_{L^2(Q;\ d\nu_{pp})})\leqq c_2|\varphi|_{2,Q}\,,$$

c_1, c_2 being positive constants independent of h and φ. Setting $\varphi=\Delta_{-h}^{(r)}(\alpha u)\in\mathcal{F}_0^2(Q)$, we obtain by (4.6) that $|\Delta_{-h}^{(r)}(\alpha u)|_{2,Q}$ is bounded by a constant independent of h. Since by (3.2) $\Delta_{-h}^{(r)}(\alpha u)\rightarrow\frac{\partial}{\partial x_r}(\alpha u)$ in $L^2(Q\,;\,d\nu_{11})$ as $h\rightarrow 0$, we see that $\Delta_{-h}^{(r)}(\alpha u)$, by choosing a subsequence if necessary, converges weakly to $\frac{\partial}{\partial x_r}(\alpha u)$. We thus can utilize a theorem of Banach-Saks to get a subsequence $\{\Delta_{-h_n}^{(r)}(\alpha u)\}$ whose arithmetic means converge strongly to $\frac{\partial}{\partial x_r}(\alpha u)$, which implies $\frac{\partial}{\partial x_r}(\alpha u)\in\mathcal{F}_0^2(Q)$. Since $f\in W^{[d/2]-1}(Q)$, $a_{pq}\in C_u^{[d/2]}(Q_+)\cap C_u^{[d/2]}(Q_-)$ and $a_{pq}(\cdot,\,0)\in C_u^{[d/2]}(Q_0)$, repeating the above argument we obtain that for every $\alpha\in C_0^\infty(Q)$ and $0\leqq|j|\leqq[d/2]$

$$D_{x'}^j(\alpha u)\in\mathcal{F}_0^2(Q)\ \text{ or equivalently }\ \alpha D_{x'}^j u\in\mathcal{F}_0^2(Q)\,, \tag{4.14}$$

where $D_{x'}^j$ denotes the partial differential operator $\frac{\partial^{|j|}}{\partial x_1^{j_1}\cdots\partial x_{d-1}^{j_{d-1}}}$ with multi-index $j=(j_1,\,\cdots,\,j_{d-1})$ of nonnegative integers, $|j|=j_1+\cdots+j_{d-1}$.

Now we shall show $\alpha u\in W^{[d/2]+1}(Q_+)\cap W^{[d/2]+1}(Q_-)$, $\alpha\in C_0^\infty(Q)$. Fix again any $\alpha\in C_0^\infty(Q)$. Since u satisfies

$$\sum_{p,q=1}^{d}\int_{Q_+}\frac{\partial u}{\partial x_p}\frac{\partial\alpha\varphi}{\partial x_q}a_{pq}dx+\lambda\int_{Q_+}u\alpha\varphi dx=\int_{Q_+}f\alpha\varphi dx\qquad \varphi\in C_0^\infty(Q_+)\,,$$

we have

$$\left\langle D_{x'}^j\frac{\partial}{\partial x_d}\left\{a_{dd}\frac{\partial}{\partial x_d}(\alpha u)\right\},\ \varphi\right\rangle \tag{4.15}$$

$$=\sum_{p=1}^{d}\int_{Q_+}D_{x'}^j\Big(\sum_{q=1}^{d-1}A_{pq}\frac{\partial^2 u}{\partial x_p\partial x_q}+B_p\frac{\partial u}{\partial x_p}\Big)\varphi dx+\int_{Q_+}D_{x'}^j(Cu-\alpha f)\varphi dx$$

for $\varphi\in C_0^\infty(Q_+)$ and $0\leqq|j|\leqq[d/2]-1$. Here A_{pq} belongs to $C_u^{[d/2]}(Q_+)$, B_p and C belong to $C_u^{[d/2]-1}(Q_+)$ and their supports are contained in Supp$[\alpha]$. It follows from (4.14) that the right quantity of (4.15) extends to all $\varphi\in L^2(Q_+)$ and defines a continuous linear functional on $L^2(Q_+)$. Therefore

$$D_{x'}^j\frac{\partial}{\partial x_d}\left\{a_{dd}\frac{\partial}{\partial x_d}(\alpha u)\right\}\in L^2(Q_+)\qquad 0\leqq|j|\leqq\left[\frac{d}{2}\right]-1\,. \tag{4.16}$$

It is derived from (4.14) and (4.16) that

$$D_{x'}^j\left\{a_{dd}\frac{\partial^2}{\partial x_d^2}(\alpha u)\right\}=D_{x'}^j\frac{\partial}{\partial x_d}\left\{a_{dd}\frac{\partial}{\partial x_d}(\alpha u)\right\}-D_{x'}^j\left\{\frac{\partial a_{dd}}{\partial x_d}\frac{\partial(\alpha u)}{\partial x_d}\right\}$$

belongs to $L^2(Q_+)$, and so by $a_{dd} \geqq \gamma^{-1} > 0$

$$D_{x'}^j \frac{\partial^2}{\partial x_d^2}(\alpha u) \in L^2(Q_+) \qquad 0 \leqq |j| \leqq \left[\frac{d}{2}\right]-1.$$

We then obtain

$$D_{x'}^j \frac{\partial^2}{\partial x_d^2}\left\{a_{dd}\frac{\partial}{\partial x_d}(\alpha u)\right\} \in L^2(Q_+) \qquad 0 \leqq |j| \leqq \left[\frac{d}{2}\right]-2,$$

by the same method as getting (4.16). Since for $0 \leqq |j| \leqq [d/2]-2$, $D_{x'}^j\left\{a_{dd}\frac{\partial^3}{\partial x_d^3}(\alpha u)\right\}$ is given in terms of $D_{x'}^j \frac{\partial^2}{\partial x_d^2}\left\{a_{dd}\frac{\partial}{\partial x_d}(\alpha u)\right\}$ and other derivatives belonging to $L^2(Q_+)$,

$$D_{x'}^j \frac{\partial^3}{\partial x_d^3}(\alpha u) \in L^2(Q_+) \qquad 0 \leqq |j| \leqq \left[\frac{d}{2}\right]-2.$$

Repeating this argument we conclude that the derivatives of αu up to order $[d/2]+1$ are in $L^2(Q_+)$, namely $\alpha u \in W^{[d/2]+1}(Q_+)$. We can also verify $\alpha u \in W^{[d/2]+1}(Q_-)$ in the same way as above. Thus the proof is completed.

Q. E. D.

§ 5. Construction of diffusion processes.

In this section we first construct a diffusion on R^d and then prove Theorems 1 and 2. We shall be concerned with the form $\mathcal{E}$ defined by (4.1) on $\mathcal{F}_0^2(R^d) \times \mathcal{F}_0^2(R^d)$. Namely

$$(5.1) \qquad \mathcal{E}(u, v) = \sum_{p,q=1}^{d} \int_{R^d} \frac{\partial u}{\partial x_p}(x) \frac{\partial v}{\partial x_q}(x) a_{pq}(x) \nu_{pq}(dx) \qquad u, v \in \mathcal{F}_0^2(R^d).$$

(5.2) a_{pq} are Borel measurable, symmetric, bounded and satisfy (4.2) for $\Omega = R^d$.

Moreover we assume that:

(5.3) For every $a = (a_1, \cdots, a_d) \in R^d$ such that $a_d = 0$ there is a positive ρ satisfying (4.13).

For each $T \in \mathcal{F}^{-2}(R^d)$ and $\lambda > 0$ we denote by $G_\lambda T$ the unique solution of (4.5) with $\Omega = R^d$: $G_\lambda T \in \mathcal{F}_0^2(R^d)$ and $\mathcal{E}_\lambda(G_\lambda T, \varphi) = \langle T, \varphi \rangle$, $\varphi \in \mathcal{F}_0^2(R^d)$. Then we have

Proposition 5.1. *$G_\lambda f \in C_\infty(R^d)$ for every $f \in C_0^\infty(R^d)$ and $\lambda > 0$.*[5]

Proof. Fix arbitrary $f \in C_0^\infty(R^d)$ and $\lambda > 0$. In view of (5.3) and Theorem 4.6 there is an unbounded domain U such that U_0 coincides with R^{d-1} and $G_\lambda f$ is continuous in U. Let $B(a, \rho)$ be a ball with radius ρ centered at a. If

(5) $C_\infty(R^d)$ = the space of continuous functions vanishing at ∞.

$B(a, \rho)_0=\emptyset$, then

$$\sum_{p,q=1}^{d}\int_{B(a,\rho)}\frac{\partial G_\lambda f}{\partial x_p}\frac{\partial\varphi}{\partial x_q}a_{pq}dx+\lambda\int_{B(a,\rho)}G_\lambda f\varphi dx=\int_{B(a,\rho)}f\varphi dx\,,$$
$$\varphi\in C_0^\infty(B(a, \rho))\,.$$

Therefore we can apply Stampacchia's results [**16**; §6] to obtain that $G_\lambda f$ is continuous in $B(a, \rho/2)$. Thus $G_\lambda f$ is continuous in R^d. It is derived from Theorem 4.3 that $G_\lambda f$ vanishes at infinity. Indeed, $\mathcal{E}_\lambda(G_\lambda f, \varphi)=0$ for all $\varphi\in\mathcal{F}_0^2(Q(a, 2))$ provided $Q(a, 2)\cap\mathrm{Supp}[f]=\emptyset$. By virtue of Theorem 4.3

$$\sup_{Q(a,1)}|G_\lambda f|\leqq D_4(d, 2, \gamma)\left\{\int_{Q(a,2)}(G_\lambda f)^2d\nu_{11}\right\}^{1/2}.$$

For any $\varepsilon>0$, taking a sufficiently large l, we get

$$\sup_{a\in Q(0,l+2)^c}|G_\lambda f(a)|\leqq\sup_{a\in Q(0,l+2)^c}\ \sup_{x\in Q(a,1)}|G_\lambda f(x)|$$
$$\leqq D_4(d, 2, \gamma)\left\{\int_{Q(0,l)^c}(G_\lambda f)^2d\nu_{11}\right\}^{1/2}<\varepsilon\,,$$

which finishes the proof. Q. E. D.

PROPOSITION 5.2. *Let $s>d$. Then G_λ is a continuous linear operator from $\mathcal{F}^{-s}(R^d)\cap\mathcal{F}^{-2}(R^d)$ into $C_\infty(R^d)$ for each $\lambda>0$.*

PROOF. First we remark the following: For every $T\in\mathcal{F}^{-2}(R^d)$

$$(\gamma^{-1}\wedge\lambda)|G_\lambda T|_{2,R^d}^2\leqq\mathcal{E}_\lambda(G_\lambda T, G_\lambda T)=\langle T, G_\lambda T\rangle\leqq\|T\|_{\mathcal{F}^{-2}(R^d)}|G_\lambda T|_{2,R^d}\,,$$

so that

$$\|G_\lambda T\|_{L^2(R^d;\,d\nu_{11})}\leqq c_1|G_\lambda T|_{2,R^d}\leqq c_1(\gamma\vee\lambda^{-1})\|T\|_{\mathcal{F}^{-2}(R^d)} \tag{5.4}$$

for some $c_1>0$.

Now by using Proposition 3.8 and Remark 3.10 for any $T\in\mathcal{F}^{-s}(R^d)\cap\mathcal{F}^{-2}(R^d)$ there is a sequence $\{\varphi_n\}_{n=1}^\infty\subset C_0^\infty(R^d)$ such that $\|T-\varphi_n\|_{\mathcal{F}^{-s}(R^d)}\to0$ and $\|T-\varphi_n\|_{\mathcal{F}^{-2}(R^d)}\to0$ as $n\to\infty$. Proposition 5.1, (4.7) and (5.4) assure that $\{G_\lambda\varphi_n\}_{n=1}^\infty$ is a Cauchy sequence in $C_\infty(R^d)$ and $G_\lambda T$ must belong to $C_\infty(R^d)$. Making use of (4.7) and (5.4) again, we find a constant c_2 and

$$\sup_{R^d}|G_\lambda T|\leqq c_2(\|T\|_{\mathcal{F}^{-s}(R^d)}+\|T\|_{\mathcal{F}^{-2}(R^d)})\,. \tag{5.5}$$

Q. E. D.

Following [**12**; Proposition 2.1] we shall prove

PROPOSITION 5.3. *$G_\lambda(C_0(R^d))$ is a dense subspace of $C_\infty(R^d)$ for $\lambda>0$.*[6]

PROOF. For $u\in C_0^\infty(R^d)$ define the operator L by

(6) $C_0(R^d)$ =the space of continuous functions with compact supports.

$$(5.6)\qquad Lu=\sum_{p,q=1}^{d}\frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial u}{\partial x_q}\right)+\sum_{p,q=1}^{d-1}\frac{\partial}{\partial x_p}\left(a_{pq}(\cdot,0)\frac{\partial u}{\partial x_q}(\cdot,0)\times\delta_{\{0\}}^{(d)}\right).$$

Since a_{pq} are bounded, L maps $C_0^\infty(R^d)$ into $\mathcal{F}^{-s}(R^d)\cap\mathcal{F}^{-2}(R^d)$ with $s>d$. Hence $u=G_\lambda(\lambda-L)u\in G_\lambda(\mathcal{F}^{-s}(R^d)\cap\mathcal{F}^{-2}(R^d))$, from which $G_\lambda(\mathcal{F}^{-s}(R^d)\cap\mathcal{F}^{-2}(R^d))$ is dense in $C_\infty(R^d)$. Combining this with Proposition 3.8, Remark 3.10 and Proposition 5.2 we get the conclusion. Q. E. D.

Now we shall show

THEOREM 5.4. *For the form $\mathcal{E}$ defined by* (5.1), (5.2) *and* (5.3) *there exists a unique diffusion process $X=[x_t, \zeta, P_x]$ $(x\in R^d)$ such that*

$$G_\lambda f(x)=E_x\left[\int_0^\zeta e^{-\lambda t}f(x_t)dt\right]\qquad f\in C_\infty(R^d),\quad \lambda>0.$$

PROOF. It is easy to see that G_λ satisfies the resolvent equation: $G_\lambda-G_\mu+(\lambda-\mu)G_\lambda G_\mu=0$ $(\lambda,\mu>0)$. Since $(\mathcal{F}_0^2(R^d),\mathcal{E})$ is a Dirichlet space relative to $L^2(R^d)$ (see Proposition 4.1), λG_λ is sub-Markov. By Proposition 5.3 we can appeal to the Hille-Yoshida theorem to obtain a nonnegative, strongly continuous and sub-Markov semigroup $\{T_t;\ t>0\}$ on $C_\infty(R^d)$ such that

$$G_\lambda f=\int_0^\infty e^{-\lambda t}T_t f dt\qquad for\ f\in C_\infty(R^d),\quad \lambda>0.$$

Therefore there exists uniquely an associated Hunt process $X=[x_t,\zeta,P_x]$ $(x\in R^d)$. It remains to verify the continuity of sample paths. For this end we need

LEMMA 5.5. *Assume $s>d$. Let B be an open ball and $T\in\mathcal{F}^{-s}(R^d)\cap\mathcal{F}^{-2}(R^d)$ be represented as* (3.6) *by means of $f_p\in L^s(R^d)\cap L^2(R^d)$ $(0\leqq p\leqq d)$ and $g_p\in L^s(R^{d-1})\cap L^2(R^{d-1})$ $(0\leqq p\leqq d-1)$ such that $f_p=0$ on B and $g_p=0$ on B_0. Then*

$$G_\lambda T(x)=E_x[e^{-\lambda\sigma_B}G_\lambda T(x_{\sigma_B})]\qquad x\in B,$$

where $\sigma_B=\inf\{t;\ t>0,\ x_t\notin B\}$.

PROOF. By Proposition 3.11 there is a sequence $\{\varphi_n\}_{n=1}^\infty\subset C_0^\infty(R^d)$ such that $\lim_{n\to\infty}\|T-\varphi_n\|_{\mathcal{F}^{-s}(R^d)}=\lim_{n\to\infty}\|T-\varphi_n\|_{\mathcal{F}^{-2}(R^d)}=0$ and $\varphi_n=0$ on B $(n\geqq 1)$. Hence by Proposition 5.2 $G_\lambda\varphi_n$ converges uniformly to $G_\lambda T$ as $n\to\infty$. On the other hand Dynkin's formula gives

$$G_\lambda\varphi_n(x)=E_x[e^{-\lambda\sigma_B}G_\lambda\varphi_n(x_{\sigma_B})]\qquad x\in B.$$

Making $n\to\infty$, we have

$$G_\lambda T(x)=E_x[e^{-\lambda\sigma_B}G_\lambda T(x_{\sigma_B})]\qquad x\in B,$$

as desired. Q. E. D.

Let us prove the continuity of the sample paths. We follow Kanda's argument [**11**]. Let B be an open ball and Ω be any open set such that $\bar{B}\cap\Omega=\emptyset$. For a nonnegative $u\in C_0^\infty(R^d)$ satisfying $u=0$ in B and $u>0$ in Ω. Put $T=(\lambda-L)u$ with L defined by (5.6), which implies $u=G_\lambda T$. Since T satisfies the assumption of Lemma 5.5, $u(x)=E_x[e^{-\lambda\sigma_B}u(x_{\sigma_B})]=0$ $(x\in B)$. This gives $P_x(x_{\sigma_B}\in\Omega)=0$ and since Ω is arbitrary, we have $P_x(x_{\sigma_B}\in\partial B)=1$. Thus the proof is complete. Q. E. D.

By virtue of (5.5) and the fact: $(G_\lambda\varphi, \psi)_{L^2(R^d)}=(\varphi, G_\lambda\psi)_{L^2(R^d)}$, $\varphi, \psi\in C_0^\infty(R^d)$, we can use the results by Blumenthal and Getoor [**3**; Theorem VI. 1.4] to get the *resolvent density* $g_\lambda(x, y)$, $\lambda>0$ such that for each $\lambda>0$,

(5.7) $g_\lambda(x, y)$ is nonnegative and jointly measurable,

(5.8) $g_\lambda(\cdot, y)$ and $g_\lambda(x, \cdot)$ are λ-excessive,

(5.9) $G_\lambda f(x)=\int g_\lambda(x, y)f(y)dy=\int g_\lambda(y, x)f(y)dy.$

Moreover by [**6**; Theorem 4] we obtain the symmetric transition probability density $p(t, x, y)$:

$$P_x(x_t\in dy,\ t<\zeta)=p(t, x, y)dy\,, \quad p(t, x, y)=p(t, y, x)$$

for each $t>0$ and $x, y\in R^d$.

Before we go to the proof of Theorem 1, we see

LEMMA 5.6. *Given a Borel measurable symmetric bounded a_{pq} satisfying the assumptions* (2.4) *and* (2.5), *there is an extension satisfying those assumptions with $\tilde{\gamma}$ and R^d instead of γ and V respectively.*

PROOF. Let $\{A_i\}_{i=1}^N$ be an open covering of ∂V and $\{\alpha_i\}_{i=1}^N$ be C^∞-partition of unity subordinate to $\{A_i\}_{i=1}^N$: $\alpha_i\in C_0^\infty(A_i)$, $0\leqq\alpha_i\leqq1$ and $\sum_{i=1}^N\alpha_i=1$ on a neighborhood D of ∂V. Then there are one-to-one transformations Φ_i, $1\leqq i\leqq N$ and open sets U_i, $1\leqq i\leqq N$ such that for each i Φ_i and its inverse Ψ_i are smooth transformations, $\Phi_i(A_i)=U_i$, $\Phi_i(A_i\cap V)=\{X=(X_1, \cdots, X_d)\in U_i;\ X_1>0\}$ and if $A_{i0}\neq\emptyset$, then $\Phi_i(\{(x_1, \cdots, x_d)\in A_i;\ x_d=0\})=\{(X_1, \cdots, X_d)\in U_i;\ X_d=0\}$. Put

$$b_{pq}^i(X)=\begin{cases} a_{pq}\circ\Psi_i(X) & \text{if } X\in\Phi_i(A_i\cap V)\,,\\ \lim\limits_{Y_1\downarrow0} a_{pq}\circ\Psi_i(Y_1, X_2, \cdots, X_d) & \text{if } X=(X_1, \cdots, X_d)\\ & \quad\in\{(X_1, \cdots, X_d)\in U_i;\ X_1=0\}\,,\\ \sum\limits_{j=1}^{[d/2]+1}\eta_j a_{pq}\circ\Psi_i(-jX_1, X_2, \cdots, X_d) & \text{if } X=(X_1, \cdots, X_d)\in\tilde{U}_i\,. \end{cases}$$

Here $\tilde{U}_i=\{(X_1, \cdots, X_d)\in\Phi_i(A_i\cap D);\ (-jX_1, X_2, \cdots, X_d)\in\Phi_i(A_i\cap V),\ 1\leqq j\leqq[d/2]+1\}$ and η_j, $1\leqq j\leqq[d/2]+1$ are unique solutions of $\sum_{j=1}^{[d/2]+1}(-j)^k\eta_j=1$, $0\leqq k\leqq[d/2]$.

Now $e_{pq}\equiv\sum_{i=1}^{N}\alpha_i(b_{pq}^i\circ\Phi_i)+(1-\sum_{i=1}^{N}\alpha_i)a_{pq}$ satisfies (2.5) on V_ε, a certain neighborhood of V (cf. [1; Theorem 4.26]). Furthermore note that $e_{pq}=a_{pq}$ on V and e_{pq} satisfy (2.4) with the same constant γ on V_ε. Let us choose functions $\varphi, \psi\in C^\infty(R^d)$ such that $0\leqq\varphi, \psi\leqq 1$ and

$$\varphi=\begin{cases}1 & \text{on } V_1^c\\ 0 & \text{on } V\end{cases}, \qquad \psi=\begin{cases}1 & \text{on } V_2\\ 0 & \text{on } V_3^c\end{cases},$$

where $V\subset\bar{V}\subset V_1\subset\bar{V}_1\subset V_2\subset\bar{V}_2\subset V_3\subset\bar{V}_3\subset V_\varepsilon$. Then $\tilde{a}_{pq}\equiv\varphi\delta_{pq}+\psi e_{pq}$ belong to both of $C_u^{[d/2]}(R_+^d)$ and $C_u^{[d/2]}(R_-^d)$ $a_{pq}(\cdot, 0)$ belong to $C_u^{[d/2]}(R^{d-1})$ and $\tilde{a}_{pq}$ satisfy (2.4) with some constant $\tilde{\gamma}$. Q.E.D.

PROOF OF THEOREM 1. Let us denote by $\tilde{a}_{pq}$ an extension of a_{pq} in Lemma 5.6. Define the form $\tilde{\mathcal{E}}$ on $\mathcal{F}_0^2(R^d)\times\mathcal{F}_0^2(R^d)$ by

$$\tilde{\mathcal{E}}(u, v)=\sum_{p,q=1}^{d}\int_{R^d}\frac{\partial u}{\partial x_p}(x)\frac{\partial v}{\partial x_q}(x)\tilde{a}_{pq}(x)\nu_{pq}(dx) \qquad u, v\in\mathcal{F}_0^2(R^d).$$

Since $\tilde{\mathcal{E}}$ satisfies the assumptions of Theorem 5.4, there is a unique diffusion process $\tilde{X}=[\tilde{x}_t, \tilde{\zeta}, \tilde{P}_x]$ $(x\in R^d)$ whose resolvent $\tilde{G}_\lambda$ satisfies the equation

$$\begin{cases}\tilde{G}_\lambda f\in\mathcal{F}_0^2(R^d)\\ \tilde{\mathcal{E}}_\lambda(\tilde{G}_\lambda f, \varphi)=(f, \varphi)_{L^2(R^d)} \qquad \varphi\in\mathcal{F}_0^2(R^d),\end{cases}$$

f being in $L^2(R^d)$. Let $\tilde{\sigma}_V$ be the first hitting time of V^c: $\tilde{\sigma}_V=\inf\{t\,;\, t>0, \tilde{x}_t\in V^c\}$. Then $\tilde{X}_V=[\tilde{x}(t\wedge\tilde{\sigma}_V), \tilde{\sigma}_V, \tilde{P}_x]$ $(x\in V)$ which is called the *part of* $\tilde{X}$ *on* V satisfies the desired conditions of Theorem 1. Indeed, it follows from Fukushima's result [7] that

$$\begin{cases}u(x)\equiv\tilde{E}_x\left[\int_0^{\tilde{\sigma}_V}e^{-\lambda t}f(\tilde{x}_t)dt\right]\in\mathcal{F}_0^2(V)\\ \mathcal{E}_\lambda(u, \varphi)=(f, \varphi)_{L^2(V)} \qquad \varphi\in\mathcal{F}_0^2(V),\end{cases}$$

where $\mathcal{E}$ is the form defined by (2.2) and (2.3). Noting that the solution of (2.6) is unique, we get the conclusion. Q.E.D.

We notice that there is the resolvent density satisfying (5.7), (5.8) and (5.9) in this case, too. Because by (4.8)

$$\|G_\lambda T\|_{L^\infty(V;\, d\nu_{11})}\leqq c\|T\|_{\mathcal{F}^{-s}(V)}$$

for some constant c and any $T\in\mathcal{F}^{-s}(V)$ with $s>d$.

PROOF OF THEOREM 2. Let $\mathcal{E}^{ij}$ be a Dirichlet form on $L^2(V)$ defined by (2.2) and (2.3) with coefficients a_{pq}^{ij}. Then there is uniquely an associated diffusion Y^{ij} on V by means of Theorem 1. Considering the time changed process, we may suppose that Y^{ij} has the speed measure $|J^{ij}(X)|\,dX$, where

$J^{ij}(X)$ is the Jacobian determinant: $J^{ij}(X)=\left|\dfrac{\partial(x_1,\cdots,x_d)}{\partial(X_1,\cdots,X_d)}\right|$ if $X=\Phi^{ij}(x)$. Notice that $c_1\leqq|J^{ij}|\leqq c_2$ on V for some positive constants $c_k=c_k(i,j)$ $(k=1,2)$. We put $X^{ij}=\Phi^{ij}(Y^{ij})$. If $U^{ij}\subset\Omega$, X^{ij} is associated with $\mathcal{E}$ on U^{ij} because $\mathcal{E}_\lambda(u,v)=\mathcal{E}^{ij}(u\circ\Psi^{ij},v\circ\Psi^{ij})+\lambda(u\circ\Psi^{ij},v\circ\Psi^{ij})_{L^2(V,|J^{ij}|dX)}$ for $u,v\in\mathcal{D}[\mathcal{E}]$ with supports in U^{ij}. If $U^{ij}\cap\Omega^c\neq\emptyset$, $X^{ij}_{U^{ij}\cap\Omega}$, the part of X^{ij} on $U^{ij}\cap\Omega$, is associated with $\mathcal{E}$ on $U^{ij}\cap\Omega$. In fact, setting $X^{ij}=[x_t,\zeta,P_x]$ $(x\in U^{ij})$, we can appeal to the results by Fukushima [7] and find that

$$(5.10)\qquad\begin{cases} u(x)\equiv E_x\left[\int_0^\sigma e^{-\lambda t}f(x_t)dt\right]\in\mathcal{F} \\ \mathcal{E}_\lambda(u,\varphi)=(f,\varphi)_{L^2(U^{ij}\cap\Omega)} \qquad \varphi\in\mathcal{F}, \end{cases}$$

for $\lambda>0$ and $f\in L^2(U^{ij}\cap\Omega)$. Hence $\sigma=\inf\{t;\ t>0,\ x_t\notin U^{ij}\cap\Omega\}$ and $\mathcal{F}$ is the completion of $C_0^\infty(U^{ij}\cap\Omega)$ with respect to the norm $\mathcal{E}_1$. Since the solution of (5.10) is unique, $X^{ij}_{U^{ij}\cap\Omega}$ is the unique associated diffusion on $U^{ij}\cap\Omega$. By $X[U^{ij}\cap\Omega]$ we mean X^{ij} or $X^{ij}_{U^{ij}\cap\Omega}$ according to $U^{ij}\cap\Omega^c=\emptyset$ or $\neq\emptyset$.

Let Ω^i, $i\geqq1$ be bounded subdomains as in (2.7). Noting that $\mathcal{E}(u,v)=\sum_{1\leq p,q\leq d}\int\dfrac{\partial u}{\partial x_p}\dfrac{\partial v}{\partial x_q}\alpha_{pq}dx$ for $u,v\in C_0^\infty(\Omega^i)$ and α_{pq} are symmetric, bounded and uniformly elliptic on Ω^i, we can apply Kunita's results [**12**; Theorem 1] to obtain a unique associated diffusion $X[\Omega^i]$ on Ω^i.

Let us denote by $\mathcal{O}$ the collection of Ω^i, $i\geqq1$ and $U^{ij}\cap\Omega$, $i,j\geqq1$. Then $\mathcal{O}$ is an open covering of Ω and we have just showed that for each $U\in\mathcal{O}$ there is a unique diffusion $X[U]$ on U associated with $\mathcal{E}$. If $U\equiv U_1\cap U_2\neq\emptyset$ for some $U_1,U_2\in\mathcal{O}$, then by utilizing the results due to Fukushima [7] again, we get that $X[U_1]_U$ is equivalent with $X[U_2]_U$. Therefore the arguments of P. Courrege and P. Priouret [**4**; Theorems 1 and 2] guarantee the existence of a unique diffusion process X on Ω whose part on U, X_U, is equivalent with $X[U]$ for every $U\in\mathcal{O}$. Thus we have a desired diffusion process of the theorem.

Q. E. D.

§ 6. Examples.

1° First we see an example corresponding to the case that all S^i are noncompact in Theorem 2. Put $S^i=\{(x_1,\cdots,x_d)\in R^d;\ x_d=f^i(x_1,\cdots,x_{d-1})\}$, where each f^i belongs to $C^{[d/2]+1}(R^{d-1})$, and assume (2.7) with $\Omega=R^d$. We shall introduce the form $\mathcal{E}$:

$$(6.1)\qquad\begin{cases} \mathcal{D}[\mathcal{E}]=C_0^\infty(R^d), \\ \mathcal{E}(u,v)=\sum_{p,q=1}^d\int_{R^d}\dfrac{\partial u}{\partial x_p}\dfrac{\partial v}{\partial x_q}a_{pq}dx+\sum_{i=1}^\infty\sum_{p,q=1}^d\int_{S^i}\dfrac{\partial u}{\partial x_p}\dfrac{\partial v}{\partial x_q}b^i_{pq}d\sigma^i. \end{cases}$$

Here a_{pq} and b^i_{pq} $(1\leqq p,\ q\leqq d,\ i\geqq 1)$ are Borel measurable, symmetric and satisfies the following: For every compact subset K there are constants $\gamma=\gamma(K)$ and $\lambda=\lambda(K, i)$ such that

$$\gamma^{-1}\sum_{p=1}^{d}\xi_p^2\leqq\sum_{p,q=1}^{d}\xi_p\xi_q a_{pq}(x)\leqq\gamma\sum_{p=1}^{d}\xi_p^2 \qquad x\in K,$$

$$\lambda^{-1}\sum_{p=1}^{d-1}\xi_p^2\leqq\sum_{p,q=1}^{d-1}\xi_p\xi_q b^i_{pq}(x)\leqq\lambda\sum_{p=1}^{d-1}\xi_p^2 \qquad x\in S^i\cap K,$$

for every $\xi\in R^d$. Moreover $a_{pq}\in C_u^{[d/2]}(U^{ij}-S^i)$ and $b^i_{pq}\in C_u^{[d/2]}(S^i\cap U^{ij})$ for all p, q, i, j, where $\{U^{ij}\}_{j\geqq 1}$ is an open covering of S^i as in (2.7). $b^i_{pd}=b^i_{dp}=\sum_{q=1}^{d-1}b^i_{pq}\dfrac{\partial f^i}{\partial x_q}$ if $1\leqq p\leqq d-1$. $b^i_{dd}=\sum_{p,q=1}^{d-1}b^i_{pq}\dfrac{\partial f^i}{\partial x_p}\dfrac{\partial f^i}{\partial x_q}$. $d\sigma^i$ stands for the surface element of S^i.

Defining the transformation Φ^i: $x=(x_1,\cdots,x_d)\mapsto X=(X_1,\cdots,X_d)$ by $x_1=X_1,\cdots,x_{d-1}=X_{d-1}$, $x_d=X_d+f^i(X_1,\cdots,X_{d-1})$ and writing its inverse by Ψ^i, we get

$$\sum_{p,q=1}^{d}\int_{S^i}\frac{\partial u}{\partial x_p}\frac{\partial v}{\partial x_q}b^i_{pq}d\sigma^i \tag{6.2}$$
$$=\sum_{p,q=1}^{d-1}\int_{R^{d-1}}\frac{\partial(u\circ\Psi^i)}{\partial X_p}(X',0)\frac{\partial(v\circ\Psi^i)}{\partial X_q}(X',0)b^i_{pq}\circ\Psi^i(X',0)J^i(X')dX',$$

where $X'=(X_1,\cdots,X_{d-1})$ and $J^i(X')=\left\{1+\sum_{p=1}^{d-1}\left(\dfrac{\partial f^i}{\partial X_p}\right)^2\right\}^{1/2}$. Following [**6**; Theorem 11] we now prove

PROPOSITION 6.1. *The form $\mathcal{E}$ defined by* (6.1) *is closable on* $L^2(R^d)$.

PROOF. Assume that $u_n\in C_0^\infty(R^d)$, $\mathcal{E}(u_n-u_m, u_n-u_m)\to 0$ as $n, m\to\infty$ and $(u_n, u_n)_{L^2(R^d)}\to 0$ as $n\to\infty$. It suffices to show $\mathcal{E}(u_n, u_n)\to 0$ as $n\to\infty$.

First we observe the following: $\sum_{1\leqq p\leqq d}\int_{R^d}\left\{\dfrac{\partial}{\partial x_p}(\alpha u_n-\alpha u_m)\right\}^2dx\to 0$ as $n, m\to\infty$ for every $\alpha\in C_0^\infty(R^d)$, so that by closability Dirichlet integrals $\sum_{1\leqq p\leqq d}\int_K\left(\dfrac{\partial u_n}{\partial x_p}\right)^2dx\to 0$ as $n\to\infty$ for every compact set K.

Set $E_1=\{x\in R^d;\ |x|<1\}$ and $E_l=\{x\in R^d;\ |x|<l\}-E_{l-1}$ $(l\geqq 2)$. Since for each l

$$0\leqq\sum_{p,q=1}^{d}\int_{E_l}\frac{\partial u_n}{\partial x_p}\frac{\partial u_n}{\partial x_q}a_{pq}dx\leqq\gamma(E_l)\sum_{p=1}^{d}\int_{E_l}\left(\frac{\partial u_n}{\partial x_p}\right)^2dx\to 0 \quad \text{as} \quad n\to\infty,$$

we have

$$0\leqq\sum_{p,q=1}^{d}\int_{R^d}\frac{\partial u_n}{\partial x_p}\frac{\partial u_n}{\partial x_q}a_{pq}dx=\sum_{l=1}^{\infty}\sum_{p,q=1}^{d}\int_{E_l}\frac{\partial u_n}{\partial x_p}\frac{\partial u_n}{\partial x_q}a_{pq}dx$$

$$\leqq \sum_{l=1}^{\infty} \lim_{m\to\infty} \sum_{p,q=1}^{d} \int_{E_l} \frac{\partial(u_n-u_m)}{\partial x_p} \frac{\partial(u_n-u_m)}{\partial x_q} a_{pq} dx \leqq \lim_{m\to\infty} \mathcal{E}(u_n-u_m,\ u_n-u_m),$$

which implies $\lim_{n\to\infty} \sum_{1\leq p,q\leq d} \int_{R^d} \frac{\partial u_n}{\partial x_p} \frac{\partial u_n}{\partial x_q} a_{pq} dx = 0.$

Fix i and l, and set $E=[\Phi^i(S^i\cap E_l)]_0$. Then by (6.2)

$$\sum_{p=1}^{d-1} \int_E \left\{\frac{\partial}{\partial X_p}((u_n-u_m)\circ\Psi^i)(X',0)\right\}^2 dX'$$

$$\leqq \lambda(E_l,i) \sum_{p,q=1}^{d-1} \int_E \frac{\partial}{\partial X_p}((u_n-u_m)\circ\Psi^i)(X',0) \frac{\partial}{\partial X_q}((u_n-u_m)\circ\Psi^i)(X',0)$$
$$\times b_{pq}^i\circ\Psi^i(X',0) J^i(X') dX'$$

$$=\lambda(E_l,i) \sum_{p,q=1}^{d} \int_{S^i\cap E_l} \frac{\partial(u_n-u_m)}{\partial x_p} \frac{\partial(u_n-u_m)}{\partial x_q} b_{pq}^i d\sigma^i \to 0 \quad \text{as} \quad n,\ m\to\infty.$$

On the other hand by the Sobolev imbedding theorem (cf. [**1**; Theorem 5.4])

$$\int_E \{u_n\circ\Psi^i(X',0)\}^2 dX' \leqq c_1\left\{\sum_{p=1}^{d} \int_{\Phi^i(E_l)} \left(\frac{\partial}{\partial X_p}(u_n\circ\Psi^i)\right)^2 dX + \int_{\Phi^i(E_l)} (u_n\circ\Psi^i)^2 dX\right\}$$

$$\leqq c_2\left\{\sum_{p=1}^{d} \int_{E_l} \left(\frac{\partial u_n}{\partial x_p}\right)^2 dx + \int_{E_l} u_n^2 dx\right\} \to 0 \quad \text{as} \quad n\to\infty,$$

with some constants c_i $(i=1, 2)$. Therefore the completeness of the Sobolev space $W^1(E)$ gives

$$\sum_{p,q=1}^{d} \int_{S^i\cap E_l} \frac{\partial u_n}{\partial x_p} \frac{\partial u_n}{\partial x_q} b_{pq}^i d\sigma^i \leqq \lambda(E_l,i) \max_{X'\in\bar{E}} J(X') \sum_{p=1}^{d-1} \int_E \left\{\frac{\partial}{\partial X_p}(u_n\circ\Psi^i)(X',0)\right\}^2 dX'$$
$$\to 0 \quad \text{as} \quad n\to\infty,$$

from which we derive that

$$0 \leqq \sum_{i=1}^{\infty} \sum_{p,q=1}^{d} \int_{S^i} \frac{\partial u_n}{\partial x_p} \frac{\partial u_n}{\partial x_q} b_{pq}^i d\sigma^i$$

$$\leqq \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \lim_{m\to\infty} \sum_{p,q=1}^{d} \int_{S^i\cap E_l} \frac{\partial(u_n-u_m)}{\partial x_p} \frac{\partial(u_n-u_m)}{\partial x_q} b_{pq}^i d\sigma^i$$

$$\leqq \lim_{m\to\infty} \mathcal{E}(u_n-u_m,\ u_n-u_m).$$

Letting n tend to infinity, we obtain the conclusion. Q. E. D.

We denote by $\bar{\mathcal{E}}$ the smallest closed extension of $\mathcal{E}$. It is obvious that $\bar{\mathcal{E}}$ satisfies the conditions in Theorem 2. Thus we are led to

THEOREM 6.2. *For the smallest closed extension $\bar{\mathcal{E}}$ of $\mathcal{E}$ defined by* (6.1) *there is a unique diffusion process X on R^d and its resolvent G_λ satisfies*

$$\begin{cases} G_\lambda f \in \mathcal{D}[\bar{\mathcal{E}}] \\ \bar{\mathcal{E}}_\lambda(G_\lambda f, \varphi) = (f, \varphi)_{L^2(R^d)} \qquad \varphi \in \mathcal{D}[\bar{\mathcal{E}}], \end{cases}$$

for $f \in L^2(R^d)$ *and* $\lambda > 0$.

According to the terminologies in [10], this diffusion possesses the speed measure dx, the system of energy measures $\{a_{pq}dx + \sum_{i \geqq 1} b^i_{pq} d\sigma^i;\ 1 \leqq p,\ q \leqq d\}$ and the killing measure $k(dx) \equiv 0$.

2° We next consider the case when each S^i is a sphere centered at the origin in Theorem 2. Let $S^i = \{x \in R^d;\ |x| = \rho^i\}$ $(i = 1, 2, \cdots)$ and assume (2.7) with $\Omega = R^d$. For each p, q, i $(1 \leqq p,\ q \leqq d,\ i \geqq 1)$ a_{pq} belongs to $C_u^{[d/2]}(N^i - S^i)$, where $N^i = \bigcup_{j \geqq 1} U^{ij}$ and $\{U^{ij}\}_{j \geqq 1}$ is a finite open covering of S^i as in (2.7). b^i_{pq} belongs to $C_u^{[d/2]}(S^i)$ for every $1 \leqq p,\ q \leqq d-1$, $i \geqq 1$. For convenience we introduce polar coordinates $(r, \theta) \equiv (r, \theta_1, \cdots, \theta_{d-1})$ defined by: $x_1 = r\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1}$, $x_2 = r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1}$, $x_3 = r\cos\theta_2 \cdots \sin\theta_{d-1}$, $\cdots$, $x_d = r\cos\theta_{d-1}$, $0 \leqq r < +\infty$, $-\pi \leqq \theta_1 \leqq \pi$, $0 \leqq \theta_2 \leqq \pi$, $\cdots$, $0 \leqq \theta_{d-1} \leqq \pi$. a_{pq} and b^i_{pq} are Borel measurable, symmetric and satisfies the following: For each i there are positive constants $\gamma = \gamma(i)$ and $\lambda = \lambda(i)$ such that

$$\gamma^{-1} \sum_{p=1}^d \xi_p^2 \mu_p(r, \theta) \leqq \sum_{p,q=1}^d \xi_p \xi_q a_{pq}(r, \theta) \leqq \gamma \sum_{p=1}^d \xi_p^2 \mu_p(r, \theta) \qquad r < \rho^i,$$

$$\lambda^{-1} \sum_{p=1}^{d-1} \xi_p^2 \mu_p(1, \theta) \leqq \sum_{p,q=1}^{d-1} \xi_p \xi_q b^i_{pq}(\rho^i, \theta) \leqq \lambda \sum_{p=1}^{d-1} \xi_p^2 \mu_p(1, \theta),$$

for every $\xi \in R^d$, where

$$\mu_p(r, \theta_1, \cdots, \theta_{d-1}) = \begin{cases} (r\sin\theta_{p+1} \cdots \sin\theta_{d-1})^{-2} & \text{if} \quad 1 \leqq p \leqq d-2, \\ r^{-2} & \text{if} \quad p = d-1, \\ 1 & \text{if} \quad p = d. \end{cases}$$

We are now given the following form:

$$(6.3) \qquad \begin{cases} \mathcal{D}[\mathcal{E}] = C_0^\infty(R^d) \\ \mathcal{E}(u, v) = \sum_{p,q=1}^d \int u_p(r, \theta) v_q(r, \theta) a_{pq}(r, \theta) J(\theta) r^{d-1} dr d\theta \\ \qquad\qquad + \sum_{i=1}^\infty \sum_{p,q=1}^{d-1} (\rho^i)^{d-1} \int u_p(\rho^i, \theta) v_q(\rho^i, \theta) b^i_{pq}(\rho^i, \theta) J(\theta) d\theta \end{cases}$$

where u_p means $\dfrac{\partial u}{\partial \theta_p}$ if $1 \leqq p \leqq d-1$, or $\dfrac{\partial u}{\partial r}$ if $p = d$, and $J(\theta) = \sin\theta_2 \sin^2\theta_3 \cdots \sin^{d-2}\theta_{d-1}$. By the same method as 1° we can prove the closability of $\mathcal{E}$ in this case, too. It is easy to see that the smallest closed extension $\bar{\mathcal{E}}$ satisfies the assumptions of Theorem 2. Thus we arrive at

THEOREM 6.3. *There is a unique diffusion process X on R^d which is associated with the smallest closed extension $\bar{\mathcal{E}}$ of* (6.3):

$$\begin{cases} G_\lambda f \in \mathcal{D}[\bar{\mathcal{E}}] \\ \bar{\mathcal{E}}_\lambda(G_\lambda f, \varphi) = (f, \varphi)_{L^2(R^d)} \qquad \varphi \in C_0^\infty(R^d), \end{cases}$$

for $f \in L^2(R^d)$ and $\lambda > 0$, G_λ being the resolvent of X.

Finally we note the following: Let $\mathring{\mathcal{E}}$ be the form defined by (6.3) with a_{pq} and b^i_{pq} replaced by $\delta_{pq}\mu_p/2$ and $(\delta_{pq}\mu_p/2)|_{r=\rho^i}$ respectively. Namely

$$\text{(6.4)} \qquad \begin{cases} \mathcal{D}[\mathring{\mathcal{E}}] = C_0^\infty(R^d) \\ \mathring{\mathcal{E}}(u, v) = \dfrac{1}{2} \displaystyle\sum_{p=1}^{d} \int u_p(r, \theta) v_p(r, \theta) \mu_p(r, \theta) J(\theta) r^{d-1} dr d\theta \\ \qquad + \dfrac{1}{2} \displaystyle\sum_{i=1}^{\infty} \sum_{p=1}^{d-1} (\rho^i)^{d-1} \int u_p(\rho^i, \theta) v_p(\rho^i, \theta) \mu_p(\rho^i, \theta) J(\theta) d\theta. \end{cases}$$

Then $\mathring{\mathcal{E}}$ corresponds to the case that $d-1$-dimensional spherical Brownian motions on $\{|x| = \rho^i\}$, $i \geqq 1$ are superposed on d-dimensional Brownian motion on R^d. In fact, the spherical Brownian motion on $S^{d-1} = \{x \in R^d; |x| = 1\}$ is the diffusion with generator $\frac{1}{2}\Delta^{d-1}$, Δ^{d-1} being the spherical Laplace operator:

$$\Delta^1 = \frac{\partial^2}{\partial \theta_1^2}, \quad \Delta^p = (\sin\theta_p)^{1-p} \frac{\partial}{\partial \theta_p} (\sin\theta_p)^{p-1} \frac{\partial}{\partial \theta_p} + (\sin\theta_p)^{-2} \Delta^{p-1} \qquad (p \geqq 2).$$

For every $u, v \in C^\infty(S^{d-1})$ vanishing at $\theta_1 = -\pi, \pi$, $\theta_p = 0, \pi$ $(2 \leqq p \leqq d-1)$

$$-\int_{S^{d-1}} (\Delta^{d-1} u) v J d\theta = \sum_{p=1}^{d-1} \int_{S^{d-1}} u_p v_p \mu_p(1, \cdot) J d\theta.$$

Since

$$\sum_{p=1}^{d} \int u_p v_p \mu_p J r^{d-1} dr d\theta = \sum_{p=1}^{d} \int \left(\frac{\partial (u \circ \Phi)}{\partial x_p} \right)^2 dx \qquad u \in \mathcal{D}[\mathring{\mathcal{E}}],$$

we obtain the desired conclusion.

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