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# The dimension of the space of cusp forms on the Siegel upper half plane of degree two related to a quaternion unitary group

By Tsuneo ARAKAWA

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## § 0. Introduction.

**0.1.** The main purpose of this paper is to give an explicit calculation of the dimension of the spaces of cusp forms on the Siegel upper half plane of degree two with respect to some arithmetic discontinuous groups having zero-dimensional cusps. Such groups are defined from $A$-hermitian forms of degree two, where $A$ is an indefinite division quaternion algebra over the rational number field $\boldsymbol{Q}$. The same result was obtained by H. Yamaguchi [18] by quite a different method; while Yamaguchi uses the Hirzebruch-Riemann-Roch theorem, our calculation is based on the Selberg trace formula.

Let $G$ be the $A$-unitary group of degree two. Since $A$ is indefinite, this determines a linear algebraic group $G$ over $\boldsymbol{Q}$ up to $\boldsymbol{Q}$-isomorphisms. Denote by $a\rightarrow a'$ $(a\in A)$ the canonical involution of $A$ and by $\mathfrak{O}$ a maximal order of $A$, and let $M_2(A)$ denote the total matrix algebra of degree two over $A$. As an explicit presentation of $G$, we define the group of $\boldsymbol{Q}$-rational points as

$$G_{\boldsymbol{Q}}=\left\{S\in M_2(A)\,\middle|\,S\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}S'=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\},$$

where $S'=\begin{pmatrix}a' & c'\\ b' & d'\end{pmatrix}$ for $S=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M_2(A)$. Let $N$ be a natural number. We consider the arithmetic discontinuous group $\Gamma(N)$ of $G_{\boldsymbol{Q}}$ such that

$$\Gamma(N)=\left\{S=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G_{\boldsymbol{Q}}\,\middle|\,a-1,\ b,\ c,\ d-1\in N\mathfrak{O}\right\}.$$

In particular, we set $\Gamma=\Gamma(1)$. Since the group $G_{\boldsymbol{R}}$ of $\boldsymbol{R}$-rational points of $G$ is conjugate in $GL(4, \boldsymbol{R})$ to the real symplectic group $Sp(2, \boldsymbol{R})$ of degree two (size 4), so we may identify the arithmetic groups $\Gamma$, $\Gamma(N)$ with the discontinuous subgroups of $Sp(2, \boldsymbol{R})$ by a fixed isomorphism of $G_{\boldsymbol{R}}$ to $Sp(2, \boldsymbol{R})$. Denote by $\mathfrak{H}_2$ the Siegel upper half plane of degree two: $\mathfrak{H}_2=\{Z\in M_2(\boldsymbol{C})\mid {}^tZ=Z,\ \mathrm{Im}(Z)>0\}$. Then, $Sp(2, \boldsymbol{R})$ operates on $\mathfrak{H}_2$ by

$$\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}: Z \longrightarrow \gamma\langle Z\rangle=(AZ+B)(CZ+D)^{-1} \qquad (\gamma \in Sp(2, \boldsymbol{R})).$$

Put $J(\gamma, Z)=\det(CZ+D)$. For any natural number $k$, let $\mathfrak{S}_k(\Gamma(N))$ be the $\boldsymbol{C}$-vector space of cusp forms of weight $k$ with respect to the group $\Gamma(N)$. Namely, $\mathfrak{S}_k(\Gamma(N))$ is the space of holomorphic functions $f(Z)$ on $\mathfrak{H}_2$ satisfying

(i) $\qquad f(\gamma\langle Z\rangle)=J(\gamma, Z)^k f(Z) \qquad$ for all $\gamma \in \Gamma(N)$,

(ii) $\qquad \det\{\mathrm{Im}(Z)\}^{k/2}|f(Z)| \qquad$ is bounded on $\mathfrak{H}_2$.

In this paper we shall calculate the dimension of the space $\mathfrak{S}_k(\Gamma(N))$ over $\boldsymbol{C}$ explicitly. The result is the following:

THEOREM. *Suppose $k \geqq 5$, $N \geqq 3$. Then,*

$$\dim_{\boldsymbol{C}} \mathfrak{S}_k(\Gamma(N))=2^{-7}3^{-3}5^{-1}[\Gamma:\Gamma(N)](k-1)(k-3/2)(k-2)\prod_{p\mid d(A)}(p-1)(p^2+1)$$
$$+2^{-4}3^{-1}[\Gamma:\Gamma(N)]N^{-3}\prod_{p\mid d(A)}(p-1),$$

*where $d(A)$ is the product of prime numbers which ramify in $A$ over $\boldsymbol{Q}$ and the group index $[\Gamma:\Gamma(N)]$ is given as*

$$[\Gamma:\Gamma(N)]=N^{10}\prod_{\substack{p\mid N \\ p\nmid d(A)}}\Big(1-\frac{1}{p^2}\Big)\Big(1-\frac{1}{p^4}\Big)\prod_{\substack{p\mid N \\ p\mid d(A)}}\Big(1-\frac{1}{p^2}\Big)\Big(1+\frac{1}{p}\Big).$$

**0.2.** T. Yamazaki [**19**], Y. Morita [**6**], and U. Christian [**2**], [**3**] calculated explicitly the dimension of the space of cusp forms with respect to the principal congruence subgroup of the Siegel modular group $Sp(2, \boldsymbol{Z})$ of level $N$ ($N \geqq 3$). Yamazaki applied the Hirzebruch-Riemann-Roch theorem, while Morita, Christian obtained the same result by using the Selberg trace formula ([**7**], Exposé 10 of [**8**]). Especially if we take $A=M_2(\boldsymbol{Q})$, $\mathfrak{O}=M_2(\boldsymbol{Z})$, then the group $\Gamma(N)$ coincides with the principal congruence subgroup of $Sp(2, \boldsymbol{Z})$ of level $N$ up to conjugations in $Sp(2, \boldsymbol{R})$. So our dimension formula gives an analogy of their result for the case where $A$ is an indefinite division quaternion algebra over $\boldsymbol{Q}$.

Our method of the calculation is essentially based on the results of Morita [**6**]. In § 1, we shall study zeta functions associated with some quadratic forms and calculate their special values after T. Shintani [**11**]. The purpose of § 2 is to obtain some properties of the group $\Gamma$ and a fundamental domain of $\Gamma$ in $\mathfrak{H}_2$. In § 3, we shall calculate $\dim_{\boldsymbol{C}} \mathfrak{S}_k(\Gamma(N))$ using the results of the previous two sections. In calculating the contribution to the dimension formula of the unipotent elements, we shall use Shintani's method (§ 3 of [**11**]). In the latter of § 3, we shall show after Morita [**6**] that the contribution to the dimension formula of the non-unipotent elements vanishes.

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**0.3.** Notation. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$, respectively, the ring of rational integers, the rational number field, the real number field, and the complex number field. For any square matrix $A$ with entries in $\boldsymbol{C}$, we denote by ${}^tA$, $\mathrm{tr}(A)$, and $\det(A)$, respectively, the transposed matrix of $A$, the trace of $A$, and the determinant of $A$. Let $E_n$ denote the unit matrix of size $n$. For any real symmetric matrices $Y_1$, $Y_2$ of the same size, we write $Y_1 > Y_2$ if $Y_1 - Y_2$ is positive definite. For any field $F$, $F^\times$ denotes the group of invertible elements in $F$. Let $(a, b)$ denote the greatest common divisor of integers $a$ and $b$. Further, we denote by $\Gamma(s)$ and $\zeta(s)$ the gamma function and the Riemann zeta function, respectively.

## §1. Zeta functions associated with an indefinite division quaternion algebra over $\boldsymbol{Q}$.

**1.1.** Preliminaries. Let $A$ be an indefinite division quaternion algebra over $\boldsymbol{Q}$. We denote by $a \to a'$ the canonical involution of $A$ and put $N(a) = aa'$, $\mathrm{tr}(a) = a + a'$. Let $\mathfrak{O}$ be a maximal order in $A$. Set, for any basis $\{u_i\}$ of $\mathfrak{O}$ over $\boldsymbol{Z}$,

$$d(A) = |\det(\mathrm{tr}(u_i u_j))|^{1/2}.$$

This number is independent of the choice of $\mathfrak{O}$ and $\{u_i\}$. Denote by $\mathfrak{O}^\times$ (resp. $\mathfrak{O}^1$) the unit group (resp. the unit group with norm 1) of $\mathfrak{O}$.

Now we fix a maximal order $\mathfrak{O}$ through out the present paper. Set

$$V_{\boldsymbol{Q}} = \{a \in A \mid \mathrm{tr}(a) = 0\}, \qquad L = \{a \in \mathfrak{O} \mid \mathrm{tr}(a) = 0\}.$$

Let $L^*$ be the dual lattice of $L$ in $V_{\boldsymbol{Q}}$ with respect to the bilinear form $\mathrm{tr}(xy)$:

$$L^* = \{b \in V_{\boldsymbol{Q}} \mid \mathrm{tr}(ab) \in \boldsymbol{Z} \quad \text{for all } a \in L\}.$$

First the following lemma is easily verified.

LEMMA 1. *Let $\{v_1, v_2, v_3\}$ be any basis of $L$ over $\boldsymbol{Z}$. Then*

$$\det(\mathrm{tr}(v_i v_j)) = -2d(A)^2.$$

Let $K$ be a quadratic number field and let $\mathfrak{o}$ be an order of $K$. Then there uniquely exists a positive rational integer $f(\mathfrak{o})$ which satisfies $\mathfrak{o} = \boldsymbol{Z} + f(\mathfrak{o})\mathfrak{o}_K$, $\mathfrak{o}_K$ being the maximal order of $K$. The number $f(\mathfrak{o})$ is called the conductor of $\mathfrak{o}$. Denote by $d(\mathfrak{o})$ the discriminant of $\mathfrak{o}$. We call $\mathfrak{a}$ a proper $\mathfrak{o}$-ideal if $\mathfrak{a}$ is a $\boldsymbol{Z}$-lattice in $K$ of $\boldsymbol{Q}$-rank 2 and $\mathfrak{o} = \{\mu \in K \mid \mu\mathfrak{a} \subset \mathfrak{a}\}$. We can classify all the proper $\mathfrak{o}$-ideals with respect to multiplication by the elements of $K^\times$. Denote by $h(\mathfrak{o})$

the number of classes of proper $\mathfrak{o}$-ideals. Moreover, let $E(\mathfrak{o})$ (resp. $E_1(\mathfrak{o})$) be the unit group (resp. the unit group with norm 1) of $\mathfrak{o}$. If $d(\mathfrak{o})<0$, then the unit group $E(\mathfrak{o})$ is a finite group and let $w(\mathfrak{o})$ be the order of $E(\mathfrak{o})$. If $d(\mathfrak{o})>0$, let $\varepsilon(\mathfrak{o})$ be the fundamental unit of the group $E_1(\mathfrak{o})$ satisfying $\varepsilon(\mathfrak{o})>1$.

Now we quote a theorem due to Eichler (see Satz 6 of [**4**]).

THEOREM (Eichler). (i) *Let $K$ be a quadratic number field. Then there exists a $\boldsymbol{Q}$-isomorphism of $K$ into $A$, if and only if the following condition holds:*

$$(1.1) \qquad \left(\frac{K}{p}\right)\neq 1 \quad \textit{for every prime number } p\,|\,d(A),$$

*where $\left(\frac{K}{p}\right)$ is the Legendre symbol.*

(ii) *Let $K$ be a quadratic number field satisfying the condition* (1.1), *and $\mathfrak{o}$ be an order of $K$. Then there exists a $\boldsymbol{Q}$-isomorphism $\varphi$ of $K$ into $A$ satisfying $\varphi(\mathfrak{o})=\varphi(K)\cap\mathfrak{O}$, if and only if the conductor $f(\mathfrak{o})$ is coprime to $d(A)$.*

Now we take $K$ and $\mathfrak{o}$ as in (ii) of the theorem. For $x, y\in V_{\boldsymbol{Q}}$, we say that $x$ is $\mathfrak{O}^1$-conjugate to $y$ if there exists an $\varepsilon\in\mathfrak{O}^1$ satisfying $y=\varepsilon x\varepsilon^{-1}$. For $x\in V_{\boldsymbol{Q}}$, put $\boldsymbol{Q}(x)=\boldsymbol{Q}+\boldsymbol{Q}x$ and $\mathfrak{O}_x=\boldsymbol{Q}(x)\cap\mathfrak{O}$. Then $\mathfrak{O}_x$ is an order of the quadratic field $\boldsymbol{Q}(x)$. For any rational number $d$ such that $\boldsymbol{Q}(\sqrt{d})=K$, let $F(\mathfrak{o}, d)$ be the set of all $x\in V_{\boldsymbol{Q}}$ satisfying the following:

(1) $N(x)=-d$,

(2) There exists a $\boldsymbol{Q}$-isomorphism $\varphi$ of $K$ into $A$ such that $\varphi(\mathfrak{o})=\mathfrak{O}_x$.

Call $n(F(\mathfrak{o}, d))$ the number of $\mathfrak{O}^1$-conjugacy classes in $F(\mathfrak{o}, d)$. The number $n(F(\mathfrak{o}, d))$ is independent of the choice of $d$ such that $\boldsymbol{Q}(\sqrt{d})=K$ and it has been calculated by Eichler [**4**]:

$$(1.2) \qquad n(F(\mathfrak{o}, d))=\frac{2h(\mathfrak{o})}{[E(\mathfrak{o}):E_1(\mathfrak{o})]}\lambda(\mathfrak{o}),$$

where

$$\lambda(\mathfrak{o})=\prod_{p\,|\,d(A)}\left\{1-\left(\frac{K}{p}\right)\right\}.$$

Here we note that the formula (1.2) has been proved precisely in the subsection 26 of Shimizu [**9**].

**1.2.** Zeta functions associated with the norm form. Put $A_\infty=A\otimes_{\boldsymbol{Q}}\boldsymbol{R}$. Then $A_\infty$ is isomorphic to the total matrix algebra $M_2(\boldsymbol{R})$ of degree 2 over $\boldsymbol{R}$. We fix an isomorphism $A_\infty\cong M_2(\boldsymbol{R})$ throughout the present paper. We consider $A$ as being contained in $M_2(\boldsymbol{R})$. Let $g\to g'$ ($g\in M_2(\boldsymbol{R})$) be the canonical involution of $M_2(\boldsymbol{R})$. Put

$$V_{\boldsymbol{R}}=\{x\in M_2(\boldsymbol{R}) \mid \operatorname{tr}(x)=0\}.$$

Then, $V_{\boldsymbol{Q}}\subset V_{\boldsymbol{R}}$, and $L$, $L^*$ are $\boldsymbol{Z}$-lattices in $V_{\boldsymbol{R}}$. Write $H$ for $GL(2, \boldsymbol{R})$ and let $\rho$ be the representation of $H$ on $V_{\boldsymbol{R}}$ defined as follows:

$$\rho(h)x=hxh' \qquad (x\in V_{\boldsymbol{R}},\ h\in H).$$

Thus, $H$ acts on $V_{\boldsymbol{R}}$. We note that $\det(x)=N(x)$ for $x\in V_{\boldsymbol{Q}}$. Denote by $\mathcal{S}(V_{\boldsymbol{R}})$ the space of rapidly decreasing functions on $V_{\boldsymbol{R}}$. For any $f\in\mathcal{S}(V_{\boldsymbol{R}})$, let $f^*$ be the Fourier transform of $f$, which can be defined by setting

$$f^*(x)=\int_{V_{\boldsymbol{R}}} f(y)\exp(2\pi\sqrt{-1}\operatorname{tr}(xy))dy$$

$$\left(dy=dy_1dy_2dy_3 \quad \text{for } y=\begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1\end{pmatrix}\in V_{\boldsymbol{R}}\right).$$

By Lemma 1 we have

LEMMA 2 (the Poisson summation formula). *For any* $f\in\mathcal{S}(V_{\boldsymbol{R}})$,

$$d(A)\sum_{a\in L} f(a)=\sum_{a\in L^*} f^*(a).$$

Set $V_i=\{x\in V_{\boldsymbol{R}} \mid (-1)^{i-1}\det(x)>0\}$ $(i=1, 2)$,

$$\Phi_i(f, s)=\int_{V_i} f(y)\,|\det(y)|^s dy \qquad (f\in\mathcal{S}(V_{\boldsymbol{R}})).$$

Obviously, $\Phi_i(f, s)$ $(i=1, 2)$ are absolutely convergent for $\operatorname{Re}(s)>0$, and they are holomorphic functions of $s$ for $\operatorname{Re}(s)>0$. The following lemma is a part of Lemma 15 of Shintani [**11**].

LEMMA 3. *The functions* $\Phi_i(f, s)$ $(i=1, 2,\ f\in\mathcal{S}(V_{\boldsymbol{R}}))$ *have analytic continuations to meromorphic functions on the whole complex plane which satisfy the following functional equations:*

$$\begin{pmatrix}\Phi_1(f^*, s-3/2) \\ \Phi_2(f^*, s-3/2)\end{pmatrix}=2^{1-2s}\pi^{1/2-2s}\Gamma(s)\Gamma(s-1/2)\begin{pmatrix}\cos\pi s & 1 \\ 0 & \sin\pi s\end{pmatrix}\begin{pmatrix}\Phi_1(f, -s) \\ \Phi_2(f, -s)\end{pmatrix}.$$

Set $H_+=\{h\in H \mid \det(h)>0\}$, and let $dh$ be the Haar measure on $H_+$ normalized by

$$dh=\det(h)^{-2}\prod_{i=1}^{4} dh_i \qquad \left(h=\begin{pmatrix} h_1 & h_2 \\ h_3 & h_4\end{pmatrix}\right).$$

The group $\mathfrak{O}^1$ is regarded as a discrete subgroup of $H_+$ (also of $SL(2, \boldsymbol{R})$). For every $\alpha\in V_{\boldsymbol{Q}}\cap V_i$ $(i=1, 2)$, set

$$H_\alpha=\{h\in H_+ \mid \rho(h)\alpha=\alpha\}, \qquad \Omega_\alpha=\mathfrak{O}^1\cap H_\alpha.$$

Then we see easily that $\Omega_\alpha=\{\varepsilon\in\mathfrak{O}_\alpha \mid N(\varepsilon)=1\}=E_1(\mathfrak{O}_\alpha)$. For any bounded domain $U_\alpha$ such that $U_\alpha \subset \bar{U}_\alpha \subset V_i$, let $W_\alpha = \{h \in H_+ \mid \rho(h)\alpha \in U_\alpha\}$ and $(W_\alpha)_0$ be a fundamental domain of $W_\alpha$ with respect to $\Omega_\alpha$. After Siegel [16], Shintani [11, p. 50], set

$$\mu(\alpha) = \int_{(W_\alpha)_0} dh \Big/ \int_{U_\alpha} |\det(y)|^{-3/2} dy .$$

The number $\mu(\alpha)$ is finite and independent of the choice of $U_\alpha$.

For any $\mathfrak{O}^1$-invariant subset $M$ in $V_Q$ (i.e., $\rho(\varepsilon)M = M$ for all $\varepsilon \in \mathfrak{O}^1$), denote by $M/\sim$ the set of $\mathfrak{O}^1$-conjugacy classes in $M$. Now define the zeta functions $\xi_i(s)$, $\xi_i^*(s)$ $(i=1, 2)$ as follows:

$$\xi_i(s) = \sum_{\alpha \in L_i/\sim} \mu(\alpha) |N(\alpha)|^{-s}, \quad \xi_i^*(s) = \sum_{\alpha \in L_i^*/\sim} \mu(\alpha) |N(\alpha)|^{-s} \qquad (i=1, 2),$$

where $L_i = L \cap V_i$, $L_i^* = L^* \cap V_i$. For $\alpha \in V_Q$ $(\alpha \neq 0)$, let $\varphi$ be an isomorphism of $\boldsymbol{Q}(\sqrt{-N(\alpha)})$ to $\boldsymbol{Q}(\alpha)$, and let $\mathfrak{o}$ be the order of $\boldsymbol{Q}(\sqrt{-N(\alpha)})$ satisfying $\varphi(\mathfrak{o}) = \mathfrak{O}_\alpha$. By an easy calculation, we have

$$\mu(\alpha) = \begin{cases} \dfrac{\pi}{2w(\mathfrak{o})} & \text{for } \alpha \in V_Q \cap V_1, \\ \dfrac{1}{4} \log \varepsilon(\mathfrak{o}) & \text{for } \alpha \in V_Q \cap V_2. \end{cases} \tag{1.3}$$

Thus, using the theorem of Eichler in the subsection 1.1 and the formulae (1.2), (1.3), it is not difficult to see that

$$\xi_1(s) = \pi\zeta(2s) \left\{ \sum\nolimits_1 \frac{h(\mathfrak{o})\lambda(\mathfrak{o})}{w(\mathfrak{o})|d(\mathfrak{o})|^s} + 4^s \sum\nolimits_2 \frac{h(\mathfrak{o})\lambda(\mathfrak{o})}{w(\mathfrak{o})|d(\mathfrak{o})|^s} \right\},$$

where the summation $\sum_1$ (resp. $\sum_2$) indicates that $\mathfrak{o}$ runs through all orders of all imaginary quadratic number fields which satisfy $d(\mathfrak{o}) \equiv 1 \bmod 4$ and $(f(\mathfrak{o}), d(A)) = 1$ (resp. $d(\mathfrak{o}) \equiv 0 \bmod 4$ and $(f(\mathfrak{o}), d(A)) = 1$). For any order $\mathfrak{o}$ of any real quadratic number field, set $h^*(\mathfrak{o}) = 2h(\mathfrak{o})([E(\mathfrak{o}) : E_1(\mathfrak{o})])^{-1}$. Similarly, we have

$$\xi_2(s) = \frac{\zeta(2s)}{4} \left\{ \sum\nolimits_3 \frac{h^*(\mathfrak{o})\lambda(\mathfrak{o}) \log \varepsilon(\mathfrak{o})}{|d(\mathfrak{o})|^s} + 4^s \sum\nolimits_4 \frac{h^*(\mathfrak{o})\lambda(\mathfrak{o}) \log \varepsilon(\mathfrak{o})}{|d(\mathfrak{o})|^s} \right\},$$

where in the summation $\sum_3$ (resp. $\sum_4$), $\mathfrak{o}$ runs through all orders of all real quadratic number fields which satisfy $d(\mathfrak{o}) \equiv 1 \bmod 4$ and $(f(\mathfrak{o}), d(A)) = 1$ (resp. $d(\mathfrak{o}) \equiv 0 \bmod 4$ and $(f(\mathfrak{o}), d(A)) = 1$). If we note that all Dirichlet series in (1.32) of [11] are convergent for $\operatorname{Re}(s) > 3/2$, we see easily from the above representations of $\xi_i(s)$ $(i=1, 2)$ that our zeta functions $\xi_i(s)$, $\xi_i^*(s)$ $(i=1, 2)$ are absolutely convergent for $\operatorname{Re}(s) > 3/2$.

Put for simplicity $v(A) = \dfrac{\pi^2}{6} \prod_{p \mid d(A)} (p-1)$.

PROPOSITION 1. (i) *The zeta functions* $\xi_i(s)$, $\xi_i^*(s)$ $(i=1, 2)$ *have analytic continuations to meromorphic functions on the whole complex plane. They satisfy the following functional equations:*

$$\begin{pmatrix}\xi_1(3/2-s)\\ \xi_2(3/2-s)\end{pmatrix}=d(A)^{-1}2^{1-2s}\pi^{1/2-2s}\Gamma(s)\Gamma(s-1/2)\begin{pmatrix}\cos\pi s & 0\\ 1/2 & \sin\pi s\end{pmatrix}\begin{pmatrix}\xi_1^*(s)\\ \xi_2^*(s)\end{pmatrix}.$$

(ii) *They are holomorphic for* $s\neq 3/2$. *Moreover, they have simple poles at* $s=3/2$ *and the residues are given in the following table. The values of* $\xi_1(s)$, $\xi_1^*(s)$ *at* $s=0$ *are also given in the table:*

| | *the residue at* $s=3/2$ | *the value at* $s=0$ |
|---|---|---|
| $\xi_1(s)$ | $v(A)/d(A)$ | $v(A)/4\pi$ |
| $\xi_2(s)$ | $v(A)/2d(A)$ | ? |
| $\xi_1^*(s)$ | $2d(A)v(A)$ | $v(A)/4\pi$ |
| $\xi_2^*(s)$ | $d(A)v(A)$ | ? |

We can prove Proposition 1 by a usual argument using Lemma 3 (cf. Theorem 5 of [11]). Here we shall only prove the table in (ii).

PROOF OF THE TABLE. For the lattice $M$ ($M=L$ or $L^*$), put

$$Z(f, M, s)=\int_{H_+/D^1}\chi(h)^s\sum_{\alpha\in M'}f(\rho(h)\alpha)dh,$$

where $f\in\mathcal{S}(V_R)$, $M'=M-\{0\}$, and $\chi(h)=\det(h)^2$. Then it is easy to see that the integrals $Z(f, L, s)$, $Z(f, L^*, s)$ are absolutely convergent for $\mathrm{Re}(s)>3/2$ and that

$$Z(f, L, s)=\frac{1}{2}\xi_1(s)\Phi_1(f, s-3/2)+\xi_2(s)\Phi_2(f, s-3/2),$$

$$Z(f, L^*, s)=\frac{1}{2}\xi_1^*(s)\Phi_1(f, s-3/2)+\xi_2^*(s)\Phi_2(f, s-3/2).$$

Let $\mathfrak{H}_1$ be the upper half plane and let $dz$ $(z\in\mathfrak{H}_1)$ be the invariant measure defined by $dz=y^{-2}dxdy$ $(z=x+\sqrt{-1}y)$. Then it is easily checked that

$$\int_{H_+/D^1,\,\chi(h)\leq 1}\chi(h)^s dh=\frac{1}{2s}\frac{\pi}{2}\int_{D^1\backslash\mathfrak{H}_1}dz \tag{1.4}$$

$$=\frac{1}{2s}v(A)\qquad(\mathrm{Re}(s)>0).$$

For $f\in\mathcal{S}(V_R)$, put

$$Z_+(f, M, s)=\int_{H_+/D^1,\,\chi(h)\geq 1}\chi(h)^s\sum_{\alpha\in M'}f(\rho(h)\alpha)dh\qquad(M=L,\text{ or }L^*).$$

It is not difficult to see from Lemma 2 and (1.4) that

$$Z(f, L, s)=Z_+(f, L, s)+d(A)^{-1}Z_+(f^*, L^*, 3/2-s) \tag{1.5}$$
$$-\frac{1}{2s}v(A)f(0)+\frac{1}{2s-3}d(A)^{-1}v(A)f^*(0) \qquad (\mathrm{Re}(s)>3/2).$$

Since the integrals $Z_+(f, L, s)$, $Z_+(f, L^*, s)$ ($f\in\mathcal{S}(V_{\boldsymbol{R}})$) are easily seen to be entire functions of $s$, $Z(f, L, s)$ has an analytic continuation to a meromorphic function on the whole complex plane. We can take an $f\in\mathcal{S}(V_{\boldsymbol{R}})$ with compact support such that its support is contained in $V_1$ and that $f^*(0)\neq 0$. Then we have

$$Z(f, L, s)=\frac{1}{2}\xi_1(s)\Phi_1(f, s-3/2).$$

By the identity (1.5), we see that

$$\lim_{s\to 3/2}(s-3/2)\xi_1(s)=d(A)^{-1}v(A).$$

Similarly, we can calculate the residues of the functions $\xi_1^*(s)$, $\xi_2(s)$, $\xi_2^*(s)$ at $s=3/2$. Finally, we note that the value of $\xi_1(s)$ (resp. $\xi_1^*(s)$) at $s=0$ can be easily calculated from the residue of $\xi_1^*(s)$ (resp. $\xi_1(s)$) at $s=3/2$ due to the functional equations of (i). q. e. d.

REMARK. The zeta functions $\xi_i(s)$, $\xi_i^*(s)$ ($i=1, 2$) are closely related to the zeta functions associated with indefinite quadratic forms studied by Siegel [**13**], [**14**]. Proposition 1 could be induced from the results of those.

## § 2. Arithmetic discontinuous subgroups of $Sp(2, \boldsymbol{R})$.

**2.1.** $A$-unitary group. We use the terminology in Shimura [**10**] in this subsection.

Let $A$ be as in § 1. Let $W$ be the product of two copies of $A$: $W=A\times A$ ($A$-space), and let $f$ be a non-degenerate quaternion hermitian form on $W$. Then, an $A$-unitary group $G$ with respect to $f$ is defined to be the group consisting of all $A$-automorphisms $\sigma$ of $W$ such that $f(x\sigma, y\sigma)=f(x, y)$. Since $A$ is indefinite, the $A$-unitary group $G$ is uniquely determined up to conjugations by $\boldsymbol{Q}$-regular endomorphisms of $W$ independent of the choice of $f$ (see Proposition 2.1 of [**10**]).

So we take a quaternion hermitian form $f$ as follows:

$$f(x, y)=x_1y_2'+x_2y_1' \qquad (x=(x_1, x_2),\ y=(y_1, y_2)\in W). \tag{2.1}$$

Now we define a linear algebraic group $G$ defined over $\boldsymbol{Q}$ as the $A$-unitary group with respect to the quaternion hermitian form (2.1) on $W$. Let $G_{\boldsymbol{Q}}$ denote the group of $\boldsymbol{Q}$-rational points of $G$. For $\sigma\in G_{\boldsymbol{Q}}$, we may write $x\sigma=$

$(x_1a+x_2b,\ x_1c+x_2d)$ $(x=(x_1, x_2)\in W)$ with some $a, b, c, d\in A$. Thus we may identify $\sigma$ with a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $M_2(A)$. Then, from the form (2.1) of $f$, we have

$$G_Q=\left\{S\in M_2(A)\,\middle|\, S\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}S'=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\},$$

where $S'=\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$ for $S=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(A)$. For any element $S=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(A)$, $S$ belongs to $G_Q$, if and only if

$$ad'+bc'=1, \qquad \mathrm{tr}\,(ab')=0, \qquad \mathrm{tr}\,(cd')=0.$$

Put $M=\mathfrak{O}\times\mathfrak{O}$. Then, $M$ is a maximal lattice with respect to $f$ which belongs to the principal genus with the order $\mathfrak{O}$ (see 2.3, 4.4 of [**10**]). Let $\Gamma$ be the group of all $\sigma\in G_Q$ satisfying $M\sigma=M$. Then we have

$$\Gamma=\left\{S=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G_Q\,\middle|\, a, b, c, d\in\mathfrak{O}\right\}. \tag{2.2}$$

We call a pair $(c, d)$ $(c, d\in\mathfrak{O})$ an $\mathfrak{O}$-coprime symmetric pair, if $\mathrm{tr}\,(cd')=0$ and there exist some $x, y\in\mathfrak{O}$ such that $cx+dy=1$. Two $\mathfrak{O}$-coprime symmetric pairs $(c, d)$, $(c_1, d_1)$ are said to be associated if there exists some $\varepsilon\in\mathfrak{O}^\times$ such that $c_1=\varepsilon c$, $d_1=\varepsilon d$. It is easy to show the following two lemmas. Especially, Lemma 5 is reduced to the fact that the class number of $A$ is one.

LEMMA 4. *If $(c, d)$ is an $\mathfrak{O}$-coprime symmetric pair, then there exist some $a, b\in\mathfrak{O}$ satisfying $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$.*

LEMMA 5. *The associated classes of $\mathfrak{O}$-coprime symmetric pairs $(c, d)$ satisfying $c\neq 0$ correspond one to one onto the space $V_Q$ by the correspondence $(c, d)\to c^{-1}d$.*

For any $S\in M_2(A)$, $S$ may be considered as an element of $M_4(\boldsymbol{R})$. The following proposition will be used in § 3.

PROPOSITION 2. *Let $S$ be an element of $\Gamma$. If all the eigenvalues of $S$ are one (i.e., $S$ is a unipotent element of $\Gamma$), then there exist some $\gamma\in\Gamma$ and $\alpha\in L$ such that*

$$S=\gamma^{-1}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\gamma.$$

PROOF. Put $S=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c=0$, the assertion is easily verified. We assume $c\neq 0$. Since $S$ is a unipotent element of $G_Q$, it is not difficult to see that

$$c+c'=0, \qquad a'+d=2. \tag{2.3}$$

Take any $x\in A$ $(x\neq 0)$, and put $y=x(1-a)c^{-1}$. Then we see easily from the relation (2.3) that $xa+yc=x$, $xb+yd=y$. Since $x^{-1}y\in V_{Q}$, by Lemma 5 there exists an $\mathfrak{O}$-coprime symmetric pair $(x_1, y_1)$ such that $x=\beta x_1$, $y=\beta y_1$ with some $\beta\in A$. By Lemma 4, we can take some $\gamma\in\Gamma$ of the form $\gamma=\begin{pmatrix} * & * \\ x_1 & y_1 \end{pmatrix}$. Thus we have $(0, 1)\gamma S=(0, 1)\gamma$. Hence, we obtain

$$\gamma S\gamma^{-1}=\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \quad \text{with some } \alpha\in L. \qquad \text{q.e.d.}$$

**2.2.** Congruence subgroup. For any natural number $N$, define the principal congruence subgroup $\Gamma(N)$ of $\Gamma$ of level $N$ as follows:

$$\Gamma(N)=\left\{S=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma \,\middle|\, a-1,\ b,\ c,\ d-1\in N\mathfrak{O}\right\}. \tag{2.4}$$

Then the principal congruence subgroup $\Gamma(N)$ is a normal subgroup of $\Gamma$ of finite index. The following lemma is essentially well-known (see for example Proposition 1 of [6]).

LEMMA 6. *Suppose $N\geqq 3$. Let $S$ be an element of the principal congruence subgroup $\Gamma(N)$. Then some power of $S$ is unipotent, if and only if it is unipotent. In particular, $\Gamma(N)$ is torsion free.*

**2.3.** Fundamental domain. Let $G_{\boldsymbol{R}}$ denote the group of $\boldsymbol{R}$-rational points of $G$. Then the group $G_{\boldsymbol{R}}$ is considered to be the subgroup of $GL(4, \boldsymbol{R})$;

$$G_{\boldsymbol{R}}=\left\{g\in M_4(\boldsymbol{R}) \,\middle|\, g\begin{pmatrix} 0 & E_2 \\ E_2 & 0 \end{pmatrix}g'=\begin{pmatrix} 0 & E_2 \\ E_2 & 0 \end{pmatrix}\right\},$$

where $g'=\begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}$ for $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D\in M_2(\boldsymbol{R})$. Put

$$J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\in M_2(\boldsymbol{R}) \quad \text{and} \quad K=\begin{pmatrix} E_2 & 0 \\ 0 & J \end{pmatrix}\in M_4(\boldsymbol{R}).$$

Then we have $Sp(2, \boldsymbol{R})=KG_{\boldsymbol{R}}K^{-1}$. Let $N$ be any natural number and define discontinuous subgroups $\Gamma^*$, $\Gamma^*(N)$ of $Sp(2, \boldsymbol{R})$ as follows:

$$\Gamma^*=K\Gamma K^{-1}, \qquad \Gamma^*(N)=K\Gamma(N)K^{-1}. \tag{2.5}$$

In the introduction we identified the groups $\Gamma$, $\Gamma(N)$ with $\Gamma^*$, $\Gamma^*(N)$ respectively.

Let $\mathfrak{H}_2$ be the Siegel upper half plane of degree two. For any $X\in M_2(\boldsymbol{C})$, we use the symbol $\mathrm{Abs}(X)$ for the absolute value of $\det(X)$. After § 2 of Siegel [15], we construct a fundamental domain of the group $\Gamma^*$ in $\mathfrak{H}_2$. The following lemma is easily verified in the same manner as in Braun [1].

LEMMA 7. *Suppose $\mu$ is a real number satisfying $\mu>3$. Let $Z=X+\sqrt{-1}Y$ be any element of $\mathfrak{H}_2$. Then there exist some positive constants $C_1$, $C_2$, and $\lambda$ depending only on $\mu$ satisfying the following inequalities:*

(i) $$\sum_{\alpha\in L}\mathrm{Abs}(Z-\alpha J)^{-\mu}<C_1\det(Y)^{-\mu+3/2}\exp(\lambda\,\mathrm{tr}(Y^{-1})),$$

(ii) $$\sum_{(c,d)}{}'\mathrm{Abs}(cZ-dJ)^{-\mu}<C_2\det(Y)^{-\mu+3/2}\exp(\lambda\,\mathrm{tr}(Y^{-1})),$$

*where the summation indicates that* $(c, d)$ *runs through all associated classes of* $\mathfrak{O}$*-coprime symmetric pairs satisfying* $c\neq 0$.

Denote by $\mathfrak{P}$ the set of all positive definite real symmetric matrices of size 2. The unit group $\mathfrak{O}^\times$ of $\mathfrak{O}$ operates on $\mathfrak{P}$ by

$$Y\longrightarrow \varepsilon Y{}^t\varepsilon \qquad (Y\in\mathfrak{P},\ \varepsilon\in\mathfrak{O}^\times). \tag{2.6}$$

Let $\mathfrak{R}$ be a fundamental domain of $\mathfrak{O}^\times$ in $\mathfrak{P}$ under the operation (2.6). Put $\mathfrak{P}^1=\{Y\in\mathfrak{P}\mid\det(Y)=1\}$. Then the set $\mathfrak{R}\cap\mathfrak{P}^1$ is compact. Moreover, we may take $\mathfrak{R}$ such that $\mathfrak{R}$ is a connected convex cone and that its boundaries lie on a finite collection of hyperplanes.

Let $\{v_1, v_2, v_3\}$ be a basis of $L$ over $\boldsymbol{Z}$, and put

$$V_{\boldsymbol{R}}/L=\{x=\textstyle\sum x_iv_i\in V_{\boldsymbol{R}}\mid -1/2\leqq x_i\leqq 1/2\quad (i=1, 2, 3)\}.$$

Now define the subset $\mathfrak{F}$ of $\mathfrak{H}_2$ as

$$\mathfrak{F}=\left\{Z=X+\sqrt{-1}Y\in\mathfrak{H}_2\ \middle|\ \begin{array}{l}\text{(i) Abs}(cZ+dJ)\geqq 1\ \text{ for all }\mathfrak{O}\text{-coprime}\\ \qquad\text{symmetric pairs }(c, d)\text{ with }c\neq 0\\ \text{(ii) }Y\in\mathfrak{R},\quad XJ\in V_{\boldsymbol{R}}/L\end{array}\right\}. \tag{2.7}$$

Denote by $dZ$ the invariant measure on $\mathfrak{H}_2$ normalized by

$$dZ=\det(Y)^{-3}dx_1dx_{12}dx_2dy_1dy_{12}dy_2\ \left(X=\begin{pmatrix}x_1 & x_{12}\\ x_{12} & x_2\end{pmatrix},\ \ Y=\begin{pmatrix}y_1 & y_{12}\\ y_{12} & y_2\end{pmatrix}\right).$$

PROPOSITION 3. (i) *The set* $\mathfrak{F}$ *is a fundamental domain of the group* $\Gamma^*$ *in* $\mathfrak{H}_2$.

(ii) $\mathfrak{F}$ *is connected and the boundaries of* $\mathfrak{F}$ *consist of a finite number of algebraic surfaces.*

(iii) *For any positive number* $\mu$, *put*

$$\mathfrak{F}(\mu)=\{Z=X+\sqrt{-1}Y\in\mathfrak{H}_2\mid Y\in\mathfrak{R},\ \det(Y)>\mu,\ XJ\in V_{\boldsymbol{R}}/L\}.$$

*Then there exists some positive constant c independent of Z such that* $\mathfrak{F}$ *is contained in* $\mathfrak{F}(c)$.

(iv) *The volume* $\int_{\mathfrak{F}}dZ$ *is finite.*

Since Proposition 3 is easily proved similarly as in § 2 of [15], we omit the proof (cf. § 9 of [5]).

## § 3. The dimension of the space of cusp forms.

**3.1.** The dimension formula of Godement. Let the groups $\Gamma$, $\Gamma(N)$, $\Gamma^*$, and $\Gamma^*(N)$ be the same as in (2.2), (2.4), and (2.5). For any element $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $Sp(2, \boldsymbol{R})$ and $Z\in\mathfrak{H}_2$, put

$$\gamma\langle Z\rangle=(AZ+B)(CZ+D)^{-1}, \qquad J(\gamma, Z)=\det(CZ+D).$$

For any positive integer $k$, denote by $\mathfrak{S}_k(\Gamma^*(N))$ the complex vector space of cusp forms of weight $k$ with respect to the group $\Gamma^*(N)$. It is known that the dimension of the space $\mathfrak{S}_k(\Gamma^*(N))$ over $\boldsymbol{C}$ is finite. Moreover, we note that its finiteness is easily verified similarly as in § 13 of [5].

In Exposé 10 of [8], Godement expressed the dimension of the space of cusp forms with respect to a discontinuous subgroup of the real symplectic group as an integral of an infinite series. In our case, the theorem of Godement is stated as follows:

THEOREM (Godement). *Suppose* $N\geqq 3$, $k>4$. *Let* $\mathfrak{F}(N)$ *be a fundamental domain of* $\Gamma^*(N)$ *in* $\mathfrak{H}_2$. *Put*

$$H_\gamma(Z)=\det\left(\frac{\gamma\langle Z\rangle-\bar{Z}}{2\sqrt{-1}}\right)^{-k} J(\gamma, Z)^{-k}\det(Y)^k$$

$$(\gamma\in Sp(2, \boldsymbol{R}),\ Z=X+\sqrt{-1}Y\in\mathfrak{H}_2),$$

$$a(k)=2^{-5}\pi^{-3}(k-1)(k-3/2)(k-2).$$

*Then*

(3.1) $$\Big|\sum_{\gamma\in\Gamma^*(N)} H_\gamma(Z)\Big| \quad \text{is bounded on } \mathfrak{H}_2,$$

(3.2) $$\dim_C\mathfrak{S}_k(\Gamma^*(N))=a(k)\int_{\mathfrak{F}(N)}\sum_{\gamma\in\Gamma^*(N)} H_\gamma(Z)dZ.$$

Let $\mathfrak{F}$ be the fundamental domain of the modular group $\Gamma^*$ in $\mathfrak{H}_2$ defined in (2.7). Since the subgroup $\Gamma^*(N)$ is a normal subgroup of $\Gamma^*$, the kernel function

$$\sum_{\gamma\in\Gamma^*(N)} H_\gamma(Z)$$

is $\Gamma^*$-invariant due to the relation $H_\gamma(\delta\langle Z\rangle)=H_{\delta^{-1}\gamma\delta}(Z)$ ($\gamma, \delta\in Sp(2, \boldsymbol{R})$). Hence, by the dimension formula (3.2), we have

(3.3) $$\dim_C\mathfrak{S}_k(\Gamma^*(N))=a(k)([\Gamma:\Gamma(N)]/2)\int_{\mathfrak{F}}\sum_{\gamma\in\Gamma^*(N)} H_\gamma(Z)dZ.$$

**3.2.** The contribution of unipotent elements. In this subsection we shall calculate the contribution of unipotent elements to the dimension formula (3.3) by the method introduced by Shintani [**11**, § 3].

Let $\Pi_N$ be the subset of $\Gamma^*(N)$ consisting of all unipotent elements of $\Gamma^*(N)$ different from the identity element. Then we see from Proposition 2 that

$$\Pi_N=\bigcup_{\gamma\in\Gamma^*_\infty\backslash\Gamma^*}\bigcup_{\alpha\in L'}\gamma^{-1}\begin{pmatrix}1 & N\alpha J\\ 0 & 1\end{pmatrix}\gamma \quad \text{(disjoint union)}, \tag{3.4}$$

where $\Gamma^*_\infty=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma^*\,\middle|\,C=0\right\}$, and $L'=L-\{0\}$. Put

$$I_1=\int_{\mathfrak{F}}\Big|\sum_{\gamma\in\Pi_N}H_\gamma(Z)\Big|\,dZ. \tag{3.5}$$

Let $D$ be a fundamental domain of $\Gamma^*_\infty$ in $\mathfrak{H}_2$. Then, by the decomposition (3.4) of $\Pi_N$, we have $I_1\leqq I_2$, where

$$I_2=\int_D\Big|\sum_{\alpha\in L'}\det\Big(Y-\frac{N\alpha J}{2\sqrt{-1}}\Big)^{-k}\det(Y)^k\Big|\,dZ.$$

By an easy calculation, we have

$$I_2=2^4N^{-3}\pi^{-1}d(A)\int_{H_+/\mathfrak{D}^1}\det(h)^3\Big|\sum_{\alpha\in L'}\det\Big(J+\frac{h\alpha h'}{\sqrt{-1}}\Big)^{-k}\Big|\,dh.$$

Now we consider the integral

$$I(s)=\int_{H_+/\mathfrak{D}^1}\chi(h)^s\sum_{\alpha\in L'}\det\Big(J+\frac{\rho(h)\alpha}{\sqrt{-1}}\Big)^{-k}dh$$

as a function of $s$. Define a function $\varphi(x)$ on $V_R$ by

$$\varphi(x)=\begin{cases}\det(x)^{k-3/2}\exp(-2\pi\,\mathrm{tr}(xJ)) & \text{for } xJ>0,\\ 0 & \text{otherwise}.\end{cases}$$

Then, its Fourier transform is given by

$$\varphi^*(x)=\gamma(k)^{-1}\det\Big(J-\frac{x}{\sqrt{-1}}\Big)^{-k},$$

where

$$\gamma(k)=(2\pi)^{2k}\pi^{-1/2}(\Gamma(k)\Gamma(k-1/2))^{-1} \quad \text{(see Hilfssatz 37 of [12])}.$$

By Lemma 2, we obtain

$$\sum_{\alpha\in L}\det\Big(J+\frac{\rho(h)\alpha}{\sqrt{-1}}\Big)^{-k}=\frac{\gamma(k)}{2d(A)}\chi(h)^{-3/2}\sum_{\alpha\in L^*}\varphi(\rho(h^*)\alpha), \tag{3.6}$$

where $h^*=\det(h)^{-1}h$ for $h\in H_+$.

PROPOSITION 4. (i) *For* $0<\mathrm{Re}(s)<k-1/2$, *we have*

$$\int_{H+/\mathfrak{O}^1}\left|\chi(h)^s\sum_{\alpha\in L'}\det\left(J+\frac{\rho(h)\alpha}{\sqrt{-1}}\right)^{-k}\right|dh<+\infty\,.$$

(ii) *The function* $I(s)$ *is holomorphic for* $0<\mathrm{Re}(s)<k-1/2$ *and it satisfies the relation:*

$$I(s)=\frac{(2\pi)^{2s}}{4d(A)}\frac{\Gamma(k-s)\Gamma(k-s-1/2)}{\Gamma(k)\Gamma(k-1/2)}\xi_1{}^*(3/2-s)\,.$$

PROOF. As in Proposition 1, we have

$$I(s)=\frac{1}{2}\gamma(k)\xi_1(s)\Phi_1(\varphi^*,\ s-3/2)+\gamma(k)\xi_2(s)\Phi_2(\varphi^*,\ s-3/2)\,.$$

By Lemma 19 of [11], the functions $\Phi_i(\varphi^*, s-3/2)$ $(i=1, 2)$ are absolutely convergent if $1/2<\mathrm{Re}(s)<k-1/2$. Therefore, the integral $I(s)$ is absolutely convergent for $3/2<\mathrm{Re}(s)<k-1/2$. Further, we see easily from (3.6) that, for $0<\mathrm{Re}(s)<k-1/2$,

$$I(s)=\gamma(k)Z_+(\varphi^*,\ L,\ s)+\frac{\gamma(k)}{2d(A)}Z_+(\varphi,\ L^*,\ 3/2-s)-\frac{1}{2s}v(A) \tag{3.7}$$

(see Proposition 1). Hence, the assertion (i) easily follows from (3.7). Moreover, the right hand of (3.7) gives an analytic continuation of $I(s)$ for $\mathrm{Re}(s)<k-1/2$. If $\mathrm{Re}(s)<0$, we have

$$I(s)=\frac{\gamma(k)}{2d(A)}Z(\varphi,\ L^*,\ 3/2-s)\,.$$

Thus we easily obtain the relation in (ii). q. e. d.

By the bounded convergence theorem and the Fubini theorem, we see from Proposition 4 that the integral $I_1$ in (3.5) converges and that

$$\int_{\mathfrak{F}}\sum_{\gamma\in\Pi_N}H_\gamma(Z)dZ=2^4\pi^{-1}N^{-3}d(A)I(3/2)\,. \tag{3.8}$$

Thus, by Proposition 1 and the equality (3.8), we obtain the following:

PROPOSITION 5. *Suppose* $N\geqq 3$, $k>4$.

(i) $\int_{\mathfrak{F}}|\sum_{\gamma\in\Pi_N}H_\gamma(Z)|\,dZ<+\infty\,.$

(ii) *The contribution of unipotent elements* ($\neq$ *the identity element*) *of the group* $\Gamma^*(N)$ *to the dimension formula* (3.3) *is given by*

$$a(k)[\Gamma:\Gamma(N)]/2\int_{\mathfrak{F}}\prod_{\gamma\in\Pi_N}H_\gamma(Z)dZ=[\Gamma:\Gamma(N)]N^{-3}3^{-1}2^{-4}\prod_{p\mid d(A)}(p-1)\,.$$

(iii) *The contribution of the identity element of the group* $\Gamma^*(N)$ *to the dimension formula* (3.3) *is given by*

$$a(k)[\Gamma:\Gamma(N)]/2\int_{\mathfrak{F}} dZ.$$

**3.3.** Estimates of infinite series. We shall estimate some infinite series in the dimension formula (cf. § 4 of [6]).

Let $\mathfrak{F}$ be the fundamental domain of the group $\Gamma^*$ in $\mathfrak{H}_2$ defined in (2.7).

PROPOSITION 6. *Suppose $k \geqq 5$. Let $Z=X+\sqrt{-1}Y$ be any element of $\mathfrak{F}$. Then there exist positive constants $C_3$, $C_4$, and $C_5$ independent of $Z$ satisfying the following inequalities:*

(i) $$\sum_{\gamma\in\Gamma^*-\Gamma^*_\infty} |H_\gamma(Z)| < C_3 \det(Y)^{3/2} \sum_{(c,d)}{}' \operatorname{Abs}(cZ-dJ)^{-k+3/2},$$

(ii) $$\sum_{\gamma\in\Gamma^*-\Gamma^*_\infty} |H_\gamma(Z)| < C_4,$$

(iii) $$\sum_{\gamma\in\Gamma^*_\infty} |H_\gamma(Z)| < C_5 \det(Y)^{3/2}.$$

PROOF. For every $\delta=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma^*$, we have

$$\sum_{\gamma\in\Gamma^*_\infty\cdot\delta} |H_\gamma(Z)| = |J(\gamma, Z)|^{-k}\det(Y)^k \sum_{\varepsilon\in\mathfrak{O}^\times}\sum_{\alpha\in L} \operatorname{Abs}\{\varepsilon(\delta\langle Z\rangle)^t\varepsilon-\bar{Z}-\alpha J\}^{-k}.$$

Put $Z_1=\delta\langle Z\rangle$ and $Z_1=X_1+\sqrt{-1}Y_1$. Moreover, set $Y=tV$, $Y_1=t_1V_1$ with $t$, $t_1>0$ and $V$, $V_1\in\mathfrak{P}^1$. Since $Y={}^t(C\bar{Z}+D)Y_1(CZ+D)$ and $Z\in\mathfrak{F}$, we have $V\in\mathfrak{R}\cap\mathfrak{P}^1$ and $t\geqq t_1$. As we noted in 2.3, $\mathfrak{R}\cap\mathfrak{P}^1$ is compact. So there exists a positive constant $\mu_1$ such that $W>\mu_1E_2$ for any $W\in\mathfrak{R}\cap\mathfrak{P}^1$. By Lemma 7 and (iii) of Proposition 3, we obtain

$$\sum_{\varepsilon\in\mathfrak{O}^\times}\sum_{\alpha\in L}\operatorname{Abs}(\varepsilon Z_1{}^t\varepsilon-\bar{Z}-\alpha J)^{-k} < C_6\sum_{\varepsilon\in\mathfrak{O}^\times}\det(t_1\varepsilon V_1{}^t\varepsilon+tV)^{-k+3/2}, \tag{3.9}$$

where $C_6$ is a positive constant independent of $Z\in\mathfrak{F}$. We may assume that $V_1$ is contained in $\mathfrak{R}\cap\mathfrak{P}^1$ in the right hand of (3.9). Since $\det(E_2+T)\geqq 1+\operatorname{tr}(T)$ for any $T\in\mathfrak{P}$, we have

$$\sum_{\varepsilon\in\mathfrak{O}^\times}\det(t_1\varepsilon V_1{}^t\varepsilon+tV)^{-k+3/2} < (\mu_1 t)^{-2k+3}\sum_{\varepsilon\in\mathfrak{O}^\times}\{1+t_1t^{-1}\operatorname{tr}(\varepsilon{}^t\varepsilon)\}^{-k+3/2}.$$

Take a basis $\{u_i\}$ of $\mathfrak{O}$ over $\boldsymbol{Z}$, and put $\varepsilon=\sum_{i=1}^4\varepsilon_iu_i$ for $\varepsilon\in\mathfrak{O}^\times$. Then there exists a positive constant $\mu_2$ such that $\operatorname{tr}(\varepsilon{}^t\varepsilon)>\mu_2\sum_{i=1}^4\varepsilon_i^2$. We note that, for any rational integers $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, the number of rational integers $\varepsilon_4$ satisfying $N(\varepsilon)=\pm 1$ $(\varepsilon=\sum\varepsilon_iu_i)$ is at most four. Then it is easy to see from Lemma 5 of [6] that

$$\sum_{\varepsilon\in\mathfrak{O}^\times}\{1+t_1t^{-1}\operatorname{tr}(\varepsilon{}^t\varepsilon)\}^{-k+3/2} < 4\times\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3\in\boldsymbol{Z}}\{1+\mu_2t_1t^{-1}\sum_{i=1}^3\varepsilon_i^2\}^{-k+3/2}$$
$$< C_7(t/t_1)^{3/2},$$

where $C_7$ is a positive constant independent of $Z\in\mathfrak{H}$. Since $t/t_1=|J(\delta, Z)|$, we obtain

$$\sum_{\gamma\in\Gamma_\infty^*\cdot\delta} |H_\gamma(Z)| < \mu_1^{-2k+3} C_6 C_7 \det(Y)^{3/2} |J(\delta, Z)|^{-k+3/2}.$$

Therefore, by Lemma 7 and (iii) of Proposition 3, we easily obtain the inequalities (i), (ii), and (iii). q. e. d.

**3.4.** A classification of conjugacy classes. We shall classify conjugacy classes of the group $\Gamma^*(N)$ ($N\geqq 3$) in $Sp(2, \boldsymbol{R})$ similarly as in Morita [**6**].

For any $\delta\in Sp(2, \boldsymbol{R})$, denote by $C_{\boldsymbol{R}}(\delta)$ the centralizer of $\delta$ in $Sp(2, \boldsymbol{R})$. For any $\gamma\in\Gamma^*$, denote by $C(\gamma)$ the centralizer of $\gamma$ in $\Gamma^*$. For the sake of convenience, set

$$A(a, b)=\begin{pmatrix} a & 0 & & \\ 0 & b & & \\ & & a^{-1} & 0 \\ & & 0 & b^{-1} \end{pmatrix} \quad (a\in\boldsymbol{R}^\times,\ b\in\boldsymbol{R}^\times),$$

$$B(\lambda, a)=\begin{pmatrix} \cos\lambda & 0 & \sin\lambda & 0 \\ 0 & a & 0 & 0 \\ -\sin\lambda & 0 & \cos\lambda & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \quad (\lambda\in\boldsymbol{R},\ a\in\boldsymbol{R}^\times),$$

$$T(b_1, b_{12}, b_2)=\begin{pmatrix} 1 & 0 & b_1 & b_{12} \\ 0 & 1 & b_{12} & b_2 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \quad (b_1, b_{12}, b_2\in\boldsymbol{R}),$$

$$K(\lambda)=\begin{pmatrix} \cos\lambda & \sin\lambda & & \\ -\sin\lambda & \cos\lambda & & \\ & & \cos\lambda & \sin\lambda \\ & & -\sin\lambda & \cos\lambda \end{pmatrix} \quad (\lambda\in\boldsymbol{R}).$$

The following proposition is essentially due to § 1, § 2 of [**6**] (cf. Christian [**2**, p. 131, p. 132]).

PROPOSITION 7. *Suppose* $N\geqq 3$. *Let* $\gamma$ *be an element of* $\Gamma^*(N)$. *Then* $\gamma$ *is conjugate in* $Sp(2, \boldsymbol{R})$ *to one of the following representatives from* (i) *to* (vi). *For each representative* $\delta$ *of the cases from* (i) *to* (v), *the centralizer* $C_{\boldsymbol{R}}(\delta)$ *of* $\delta$ *in* $Sp(2, \boldsymbol{R})$ *is also given in the following:*

(i) $\delta=B(\lambda, a)$ $(\sin\lambda\neq 0,\ a\neq\pm 1)$,

$C_{R}(\delta)=\{B(\alpha, \beta) \mid \alpha\in \boldsymbol{R},\ \beta\in \boldsymbol{R}^{\times}\}$;

(ii) $\delta=A(a_1, a_2)$ $(a_1^2,\ a_2^2,\ a_1a_2\neq 1)$,

$C_{R}(\delta)=\{A(\alpha, \beta) \mid \alpha,\ \beta\in \boldsymbol{R}^{\times}\}\cdots$ if $a_1\neq a_2$,

$C_{R}(\delta)=\left\{\begin{pmatrix} V & 0 \\ 0 & {}^tV^{-1}\end{pmatrix} \middle| V\in GL(2, \boldsymbol{R})\right\}\cdots$ if $a_1=a_2$;

(iii) $\delta=T(0, 0, b)A(a, 1)$ $(a\neq\pm 1)$,

$C_{R}(\delta)=\{T(0, 0, \beta)A(\alpha, 1) \mid \alpha\in \boldsymbol{R}^{\times},\ \beta\in \boldsymbol{R}\}\cdots$ if $b\neq 0$,

$$C_{R}(\delta)=\left\{\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \beta_2 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & \beta_3 & 0 & \beta_4 \end{pmatrix} \middle| \alpha\in \boldsymbol{R}^{\times},\ \begin{pmatrix}\beta_1 & \beta_2 \\ \beta_3 & \beta_4\end{pmatrix}\in SL(2, \boldsymbol{R})\right\}\cdots \quad \text{if } b=0;$$

(iv) $\delta=A(\mu, \mu)K(\lambda)$ $(\mu\neq\pm 1,\ \sin\lambda\neq 0)$,

$C_{R}(\delta)=\{A(\alpha, \alpha)K(\beta) \mid \alpha\in \boldsymbol{R}^{\times},\ \beta\in \boldsymbol{R}\}$;

(v) $\delta=T(0, b, 0)A(a, a^{-1})$ $(a\neq\pm 1,\ b\neq 0)$,

$C_{R}(\delta)=\{T(0, \beta, 0)A(\alpha, \alpha^{-1}) \mid \alpha\in \boldsymbol{R}^{\times},\ \beta\in \boldsymbol{R}\}$;

(vi) $\delta=T(b_1, b_{12}, b_2)$.

We note that every unipotent element of $\Gamma^*(N)$ is conjugate in $\Gamma^*$ to some element of the form (vi) by Proposition 2. If $\gamma$ is a non-unipotent element of $\Gamma^*(N)$ $(N\geqq 3)$, we can prove that $\gamma$ is conjugate in $Sp(2, \boldsymbol{R})$ to one of the representatives from (i) to (v) similarly as in Theorem 1 of [6]. For each representative $\delta$ from (i) to (v), the centralizer of $\delta$ in $Sp(2, \boldsymbol{R})$ has been determined by Proposition 2, 3, 4, 5, 6 of [6]. So, we omit the precise proof of Proposition 7.

The following two lemmas are easily verified (cf. Proposition 8, 10 of [6]).

LEMMA 8. *Let $\gamma$ be an element of $\Gamma^*(N)\cap\Gamma^*_\infty$ with $N\geqq 3$. Then, $\gamma$ is conjugate in $Sp(2, \boldsymbol{R})$ to one of the following representatives:*

(1) $T(0, b, 0)A(a, a^{-1})$ *with* $a\neq\pm 1$,

(2) $T(b_1, b_{12}, b_2)$.

LEMMA 9. *Let $\gamma_1, \gamma_2$ be elements of $\Gamma^*_\infty$ which satisfy the relation $\gamma_1=\varepsilon\gamma_2\varepsilon^{-1}$ with some $\varepsilon\in\Gamma^*$. Suppose $\gamma_1, \gamma_2$ are conjugate in $Sp(2, \boldsymbol{R})$ to $T(0, b, 0)A(a, a^{-1})$ with $a\neq\pm 1$, $b\neq 0$. Then, $\varepsilon$ is contained in $\Gamma^*_\infty$.*

**3.5.** The contribution of non-unipotent elements. By the same method used in §6 of [6] (cf. [2, p. 150~p. 154]), we shall prove that the contribution to the dimension formula (3.3) of all non-unipotent elements of $\Gamma^*(N)$ $(N\geqq 3)$ vanishes.

The following is the analogy of Theorem 4 of [6] and is similarly proved. So we omit the proof.

PROPOSITION 8. *Suppose* $k\geqq 5$, $N\geqq 3$. *Then we have*

$$\int_{\mathfrak{F}}\sum_{\gamma}|H_{\gamma}(Z)|dZ<+\infty,$$

*where* $\gamma$ *runs over all elements of the group* $\Gamma^{*}(N)$ *that are conjugate in* $Sp(2, \boldsymbol{R})$ *to*

(1) $B(\lambda, a)$ $(\sin\lambda\neq 0,\ a\neq\pm 1)$ *or*

(2) $A(a_1, a_2)\neq E_4$ *or*

(3) $T(0, 0, b)A(a, 1)$ $(a\neq\pm 1)$ *or*

(4) $A(\mu, \mu)K(\lambda)$ $(\mu\neq\pm 1,\ \sin\lambda\neq 0)$.

PROPOSITION 9. *Suppose* $k\geqq 5$, $N\geqq 3$. *Then the contribution to the dimension formula* (3.3) *of the elements of* $\Gamma^{*}(N)$ *that are conjugate in* $Sp(2, \boldsymbol{R})$ *to*

(1) $B(\lambda, a)$ $(\sin\lambda\neq 0,\ a\neq\pm 1)$ *or*

(2) $A(a_1, a_2)\neq E_4$ *or*

(3) $T(0, 0, b)A(a, 1)$ $(a\neq\pm 1)$ *or*

(4) $A(\mu, \mu)K(\lambda)$ $(\mu\neq\pm 1,\ \sin\lambda\neq 0)$

*is zero. Hence, in the dimension formula* (3.3), *we may disregard all terms such that* $\gamma$ *is conjugate in* $Sp(2, \boldsymbol{R})$ *to one of the above types of elements.*

PROOF. For $\gamma\in\Gamma^{*}$, let $\mathfrak{F}_{\gamma}$ be a fundamental domain of the centralizer $C(\gamma)$ of $\gamma$ in $\mathfrak{H}_2$. Due to the relation $H_{\gamma}(\delta\langle Z\rangle)=H_{\delta^{-1}\gamma\delta}(Z)$ $(\gamma, \delta\in Sp(2, \boldsymbol{R}))$ and Proposition 8, it is sufficient to show that

$$\int_{\mathfrak{F}_{\gamma}}H_{\gamma}(Z)dZ=0 \tag{3.10}$$

for any $\gamma\in\Gamma^{*}(N)$ which is conjugate in $Sp(2, \boldsymbol{R})$ to one of the above four types of elements. The relation (3.10) can be proved in the same manner as in Theorem 5 of [6]. q. e. d.

For any positive number $s$, put

$$\mathfrak{F}_s=\left\{Z=X+\sqrt{-1}Y\in\mathfrak{F}\,\middle|\,\det(Y)>\exp\left(\frac{1}{s}\right)\right\},\qquad B_s=\bigcup_{\varepsilon\in\Gamma_{\infty}^{*}}\varepsilon\langle\mathfrak{F}_s\rangle.$$

The following proposition is an easy corollary of Proposition 6.

PROPOSITION 10. *Suppose* $k\geqq 5$. *Then we have*

(i) $\displaystyle\lim_{s\to+0}\int_{\mathfrak{F}_s}\sum_{\gamma\in\Gamma^{*}-\Gamma_{\infty}^{*}}|H_{\gamma}(Z)|dZ\longrightarrow 0,$

(ii) $\displaystyle\int_{\mathfrak{F}-\mathfrak{F}_s} \sum_{\gamma\in\Gamma^*_\infty} |H_\gamma(Z)|\, dZ < +\infty\,.$

PROPOSITION 11. *Suppose* $k\geqq 5$, $N\geqq 3$. *The contribution to the dimension formula* (3.3) *of the elements of* $\Gamma^*(N)$ *that are conjugate in* $Sp(2, \boldsymbol{R})$ *to* $T(0, b, 0)A(a, a^{-1})$ $(a\neq\pm 1,\ b\neq 0)$ *is zero.*

PROOF. Now let $\gamma$ be an element of $\Gamma^*(N)$ that is conjugate in $Sp(2, \boldsymbol{R})$ to $T(0, b, 0)A(a, a^{-1})$ $(a\neq\pm 1,\ b\neq 0)$. If $\gamma$ is not conjugate in $\Gamma^*$ to any element of $\Gamma^*_\infty$, we can prove the assertion more simply as in Proposition 9, so we omit the proof in such a case. Let $C_N$ be the subset of $\Gamma^*(N)$ consisting of all elements that are conjugate in $Sp(2, \boldsymbol{R})$ to $T(0, b, 0)A(a, a^{-1})$ $(a\neq\pm 1,\ b\neq 0)$ and are conjugate in $\Gamma^*$ to some elements of $\Gamma^*_\infty$. Further, denote by $C_\infty$ the set of all $\Gamma^*_\infty$-conjugacy classes in $C_N\cap\Gamma^*_\infty$. We may assume $\gamma\in C_N\cap\Gamma^*_\infty$. Then, $\varepsilon\gamma\varepsilon^{-1}$ $(\varepsilon\in\Gamma^*)$ belongs to $\Gamma^*_\infty$ if and only if $\varepsilon\in\Gamma^*_\infty$ by Lemma 9. Thus, the centralizer $C(\gamma)$ of $\gamma$ in $\Gamma^*$ is contained in $\Gamma^*_\infty$. Hence, $B_s$ is stable by the operation of $C(\gamma)$. Let $\mathfrak{F}_{\gamma,s}$ be a fundamental domain of $C(\gamma)$ in $\mathfrak{H}_2-B_s$. By the property (3.1) and Proposition 5, 8, we have

$$\int_{\mathfrak{F}} \Big|\sum_{\gamma\in C_N} H_\gamma(Z)\Big|\, dZ < +\infty\,.$$

Therefore, we see easily from Proposition 6, 10 that

$$\int_{\mathfrak{F}} \sum_{\gamma\in C_N} H_\gamma(Z)\,dZ = \lim_{s\to+0} \sum_{\gamma\in C_\infty} \int_{\mathfrak{F}_{\gamma,s}} H_\gamma(Z)\,dZ\,.$$

By the same arguments of [6, 6.2 in § 6], we easily obtain

$$\int_{\mathfrak{F}_{\gamma,s}} H_\gamma(Z)\,dZ = 0 \qquad (\gamma\in C_N\cap\Gamma^*_\infty)\,.$$

Hence we have proved Proposition 11.

**3.6.** The index of the principal congruence subgroup to the full modular group and the volume of the fundamental domain. Let the notation be as in § 2. For any natural number $N$, put

$$\Gamma(\mathfrak{O}/N\mathfrak{O}) = \left\{ S=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \bmod N\mathfrak{O} \;\middle|\; a, b, c, d\in\mathfrak{O},\ S\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}S' \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \bmod N\mathfrak{O} \right\}.$$

Then, $\Gamma(\mathfrak{O}/N\mathfrak{O})$ is a finite group, and denote by $\sharp(\Gamma(\mathfrak{O}/N\mathfrak{O}))$ the order of the group $\Gamma(\mathfrak{O}/N\mathfrak{O})$. Let $N=\prod p^e$ be the decomposition of $N$ into the product of distinct prime numbers. Since we can easily prove that the canonical isomorphism of $\Gamma$ to $\Gamma(\mathfrak{O}/N\mathfrak{O})$ defined by $S\to S \bmod N\mathfrak{O}$ is surjective, we have

$$[\Gamma : \Gamma(N)] = \prod_{p|N} \sharp(\Gamma(\mathfrak{O}/p^e\mathfrak{O}))\,.$$

Thus, by an easy calculation, we have

$$[\Gamma:\Gamma(N)]=N^{10}\prod_{\substack{p\mid N\\ p\nmid d(A)}}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p_4}\right)\prod_{\substack{p\mid N\\ p\mid d(A)}}\left(1-\frac{1}{p^2}\right)\left(1+\frac{1}{p}\right). \tag{3.11}$$

It is known that the Tamagawa number of the group $G$ is one (see Theorem 4.4.1 of [17]). From this fact, the volume of the fundamental domain of $\Gamma^*$ in $\mathfrak{H}_2$ is calculated. The result is the following:

$$\int_{\mathfrak{F}} dZ=2^{-1}3^{-3}5^{-1}\pi^3\prod_{p\mid d(A)}(p-1)(p^2+1). \tag{3.12}$$

By Proposition 5, 7, 9, 11, and the equalities (3.11), (3.12), we obtain the theorem in the introduction.

## References


[1] H. Braun, Konvergenz verallgemeinerter Eisensteinscher Reihen, Math. Z., **44** (1939), 387-397.

[2] U. Christian, Berechnung des Ranges der Schar der Spitzenformen zur Modulgruppe zweiten Grades und Stufe $q>2$, J. Reine Angew. Math., **277** (1975), 130-154.

[3] U. Christian. Zur Berechnung des Ranges der Schar der Spitzenformen zur Modulgruppe zweiten Grades und Stufe $q>2$, J. Reine Angew. Math., **296** (1977), 108-118.

[4] M. Eichler, Zur Zahlentheorie der Quaternionen Algebren, J. Reine Angew. Math., **195** (1955), 127-151.

[5] H. Maass, Siegel's Modular Forms and Dirichlet Series, Lecture Notes in Math., No. **216**, Springer, 1971.

[6] Y. Morita, An explicit formula for the dimension of spaces of Siegel modular forms of degree two, J. Fac. Sci. Univ. Tokyo, **21** (1974), 167-248.

[7] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., **20** (1956), 47-87.

[8] Séminaire H. Cartan, Fonctions automorphes, 1957-1958.

[9] H. Shimizu, On discontinuous groups operating on the product of the upper half planes, Ann. of Math., **77** (1963), 33-71.

[10] G. Shimura, Arithmetic of alternating forms and quaternion hermitian forms, J. Math. Soc. Japan, **15** (1963), 33-65.

[11] T. Shintani, On zeta-functions associated with the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo, **22** (1975), 25-65.

[12] C.L. Siegel, Über die analytische Theorie der quadratischen Formen, Ann. of Math., **36** (1935), 527-606.

[13] C.L. Siegel, Über die Zetafunktionen indefiniter quadratischer Formen, Math. Z., **43** (1938), 682-708.

[14] C.L. Siegel, Über die Zetafunktionen indefiniter quadratischer Formen II, Math. Z., **44** (1939), 398-426.

[15] C.L. Siegel, Einführung in die Theorie der Modulfunktionen $n$-ten Grades, Math. Ann., **116** (1939), 617-657.

[16] C. L. Siegel, The average measure of quadratic forms with given determinant and signature, Ann. of Math., **45** (1944), 667-685.
[17] A. Weil, Adeles and algebraic groups, Lecture notes, Institute for Advanced Study, Princeton, 1961.
[18] H. Yamaguchi, The parabolic contribution to the dimension of the space of cusp forms on Siegel space of degree two, to appear.
[19] T. Yamazaki, On Siegel modular forms of degree two, Amer. J. Math., **98** (1976), 39-53.

Tsuneo ARAKAWA
Department of Mathematics
Rikkyo University
Nishi-Ikebukuro
Tokyo 171
Japan