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On the decay of local energy for wave equations with time-dependent potentials

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§1. Introduction.

In this paper, we consider the following wave equation in 3-dimensional space $\boldsymbol{R}^3$:

$$u_{tt}-\Delta u+V(x, t)u=0, \qquad t>0, \quad x\in \boldsymbol{R}^3.$$

If the potential $V(x, t)=V(x)$ is time-independent and of compact support, then it is well-known that the local energy of solutions to the equation above decays exponentially as $t\rightarrow\infty$ for initial data with compact support. ([5], [10] etc.) The aim of the present paper is to prove a similar result in case of time-dependent potentials whose supports in x remain in a bounded region uniformly in t.

The problem to be investigated here is closely related to an exterior boundary-value problem with a moving obstacle. Roughly speaking, for zero Dirichlet boundary-value problems, the exponential decay of local energy has been verified by use of the energy method under the assumptions that the moving obstacle is star-shaped for each fixed t and that the lateral boundary is time-like. ([2], [3] etc.) Our method employed here is also based on the energy method and especially the principle of limiting absorption plays an important role. This idea is due to Strauss [7] where this principle was applied to exterior problems with stationary (not necessarily star-shaped) obstacles.

For another works on the decay of local energy for wave equations with time-dependent coefficients, see also [1] and [8] etc. In [8], the author considered the wave equation with potential $V(x, t)$ whose support in x expands with time t at a speed less than the sound speed. However, the assumptions imposed on $V(x, t)$ there were more restrictive and complicated than those in the present paper.

1.1. ASSUMPTIONS AND RESULTS. We shall formulate the results obtained here more precisely with several assumptions. We consider the equation

$$u_{tt}-\Delta u+V(x, t)u=0, \qquad t>0, \quad x\in \boldsymbol{R}^3, \tag{1.1}$$

$$(1.2) \qquad u(x, 0)=f(x), \quad u_t(x, 0)=g(x),$$

where it is assumed that the initial data $f(x)$ and $g(x)$ have compact support and are of finite energy ($f\in H^1(R^3)$, $g\in L^2(R^3)$, $H^1(R^3)$ being the Sobolev space of order one). We make the following assumptions on $V(x, t)$:

(A. 1) $V(x, t)$ is a non-negative and bounded function of class C^1 with bounded derivatives;

(A. 2) There exists a constant M such that the support of $V(x, t)$ in x is contained in $\{x : |x|\leqq M\}$;

(A. 3) $V_t(x, t)$ $(=\partial_t V(x, t))=O(t^{-\alpha})$ as $t\to\infty$ for some α, $0<\alpha\leqq 1$, uniformly in x.

Throughout the whole discussion, we use the constants M and α with the meanings ascribed above.

We further define the local energy of solutions to equation (1.1) in $B_K=\{x : |x|<K\}$, $0<K\leqq\infty$, at time T as follows:

$$(1.3) \qquad E(u, K, T)=\frac{1}{2}\int_{B_K}\{|u_t(T)|^2+|\nabla u(T)|^2+V(x, T)|u(T)|^2\}\,dx\,.$$

In particular, when $K=\infty$, $E(u, \infty, T)$ denotes the total energy.

With the above assumptions and notations, the main result can be stated in the following form:

THEOREM 1.1. *Assume that* $V(x, t)$ *satisfies* (A. 1)$\sim$(A. 3). *Let* $u=u(x, t)$ *be the solution to equation* (1.1) *with initial data* (1.2) *satisfying the assumptions above. Then, for any* K, $K<\infty$, *there exist constants* C_K *and* θ, $\theta>0$, *such that*

$$E(u, K, T)\leqq C_K \exp(-\theta T)E(u, \infty, 0)$$

for any T, *where the constants* C_K *and* θ *depend on the bound of supports of initial data.*

The above result can be slightly generalized to a potential $V(x, t)$ of the following form.

(A. 4) $V(x, t)$ takes the form $V(x, t)=U(x-p(t), t)$ with $U(x, t)$ satisfying (A. 1) $\sim$(A. 3), where $p(t)=(p_1(t), p_2(t), p_3(t))$ satisfies $\partial_t p_j(t)=O(t^{-\beta})$ and $(\partial_t)^2 p_j(t)=O(t^{-1-\beta})$, $1\leqq j\leqq 3$, as $t\to\infty$ for some β, $\beta>0$.

For a potential of the above form, we obtain the following result.

THEOREM 1.2. *Assume that* $V(x, t)$ *satisfies* (A. 4). *Let* $u=u(x, t)$ *be the solution to equation* (1.1) *with initial data* (1.2) *satisfying the same assumptions as in* Theorem 1.1. *Let* $B_K(T)$ *be the ball with radius* K, $K<\infty$, *and center at* $p(T)$. *Let* $E(u, B_K(T), T)$ *denote the local energy of the solution* u *in* $B_K(T)$ *at time* T. *Then,*

$$E(u,\ B_K(T),\ T) \leqq C_K \exp(-\theta T) E(u,\ \infty,\ 0)\,.$$

Theorem 1.2 is verified basically in the same way as in the proof of Theorem 1.1, so we omit the detailed proof. We make a change of the variables $y=x-p(s)$ in proving the principle of limiting absorption (Proposition 2.1) and the other part of proof is done by a way similar to that in [**9**].

1.2. NOTATIONS. We list up basic notations used in this paper. For a domain G, we denote by $H^m(G)$ $(H^0(G)=L^2(G))$ the usual Sobolev space of order m over G, m being not necessarily an integer. When G is the ball with radius K and center at the origin, we denote it by B_K and the norm in $H^m(B_K)$ by $\|\ \|_{m,K}$. In particular, when $K=\infty$, we drop the subscript K to denote the norm $\|\ \|_{m,K}$; $\|\ \|_m=\|\ \|_{m,\infty}$.

We use frequently the subscripts j, $1 \leqq j \leqq 3$, and t to denote x_j-derivatives and t-derivatives of functions, respectively; $u_j=(\partial/\partial x_j)u$, $u_t=(\partial/\partial t)u$. We further use the symbol C to denote various (unessential) positive constants. When we specify the dependence of such a constant on a parameter, say k, we denote it by C_k or $C(k)$.

1.3. FUNDAMENTAL IDENTITY. We conclude this section by stating the fundamental identity due to Morawetz (Appendix 3, [**4**]).

LEMMA 1.1. *Let $u=u(x, t)$ and $V=V(x, t)$ be real-valued C^2-function and C^1-function, respectively. Then,*

$$\begin{aligned}(1.4)\qquad (u_{tt}-\Delta u+Vu)\{(r^2+t^2)u_t+2t(x\cdot\nabla u)+2tu\} \\ =X_t(t,\ u)+\nabla\cdot Y(u)+Z(u)\,, \qquad r=|x|\,,\end{aligned}$$

where

$$\begin{aligned}X(t,\ u)&=\frac{1}{2}(r^2+t^2)(|\nabla u|^2+u_t^2+Vu^2)+2tu_t(u+(x\cdot\nabla u)) \\ &\quad +r^{-2}(r^2+t^2)\Big(\frac{1}{2}u^2+(x\cdot\nabla u)u\Big)\,,\end{aligned}$$

$$\begin{aligned}Y^j(u)&=x_j\Big\{t(|\nabla u|^2+Vu^2-u_t^2)-\frac{1}{2}r^{-2}((r^2+t^2)u^2)_t\Big\} \\ &\quad -u_j\{(r^2+t^2)u_t+2t(x\cdot\nabla u)+2tu\}\,, \qquad Y=(Y^j)_{j=1,3}\,,\end{aligned}$$

$$Z(u)=-\Big\{2tV+t(x\cdot\nabla V)+\frac{1}{2}(r^2+t^2)V_t\Big\}u^2\,,$$

$x=(x_1,\ x_2,\ x_3)$ being a position vector. Furthermore, $X(t,\ u)\geqq 0$ and satisfies the estimate

$$(1.5)\qquad X(t,\ u) \geqq \frac{1}{8}t^2\{|\nabla u|^2+u_t^2+Vu^2+\nabla\cdot(r^{-2}xu^2)\}$$

for $r<\frac{1}{2}t$.

§ 2. The principle of limiting absorption.

Throughout this section, we always assume that the potential $V(x, t)$ satisfies (A. 1)$\sim$(A. 3). We write $V(x ; s)=V(x, s)$ for fixed s, $s>1$, and consider the following equation:

$$(2.1) \qquad -\Delta W+V(x ; s)W-(\lambda+i\kappa)^2W=G, \qquad \lambda\in \boldsymbol{R}^1, \quad 0<\kappa<1.$$

Here we assume that $G=G(x)$ is an L^2-function and that

$$(2.2) \qquad \operatorname{supp} G\subset B_R$$

for some R, $R>M$. Obviously, equation (2.1) has a unique L^2-solution for $\kappa>0$.

PROPOSITION 2.1. *Let* $W=W(x ; \zeta, s)$, $\zeta=\lambda+i\kappa$, *be the* L^2*-solution to equation* (2.1) *with* $G(x)$ *satisfying* (2.2). *Then, for any* K, $K<\infty$, *there exists a constant* C_K *independent of* λ, κ *and* s *such that*

$$(2.3) \qquad \|W\|_{0, K}\leqq C_K(1+|\lambda|)^{-1}\|G\|_0 .$$

Estimate (2.3) is well-known as a key estimate in the proof of the principle of limiting absorption, but for the sake of completeness, we shall prove this result. The proof is divided into two steps. The first step is to verify (2.3) for the high frequencies ($|\lambda|>\Lambda$, Λ being large enough) and the second one is to verify it for the low frequencies ($|\lambda|\leqq\Lambda$).

LEMMA 2.1. *There exists a constant* Λ *independent of* s *such that for* λ, $|\lambda|>\Lambda$

$$\|W\|_{0, K}\leqq C_K|\lambda|^{-1}\|G\|_0$$

with C_K *independent of* λ, κ *and* s.

PROOF. We use the summation convention in the proof of this lemma and note that integrations with no domains attached are taken over the whole space. Let $h(x)=(1+|x|^2)^{-(1+\delta)/2}$, $\delta>0$, and let $\chi(x)=(\chi^{(1)}(x), \chi^{(2)}(x), \chi^{(3)}(x))$ be a real-valued smooth vector field such that the matrix with components $\{\chi_j^{(k)}+\chi_k^{(j)}\}_{j, k=1, 3}$ is positive definite and that $\chi^{(j)}(x)=(1-|x|^{-\delta})x_j/|x|$ for $|x|$ large enough. To prove this lemma, we use two identities which are obtained by multiplying (2.1) by $h(x)\overline{W}$ and by $\chi^{(j)}\overline{W}_j+\frac{1}{2}\chi_j^{(j)}\overline{W}$. First, we multiply both sides of (2.1) by $h(x)\overline{W}$, integrate the resulting identity over the whole space and take the real part. Then, we have, integrating by parts and using the boundedness of V, that

$$(2.4) \qquad |\lambda|^2\int h(x)|W|^2dx\leqq C_1\int h(x)|\nabla W|^2dx+C_2\int|G|^2dx$$

for $|\lambda|$ large enough, where the constants C_j, $j=1, 2$, are independent of λ, κ and s. Secondly, we multiply both sides of (2.1) by $\chi^{(j)}\overline{W}_j+\frac{1}{2}\chi_j^{(j)}\overline{W}$, integrate over the whole space and take the real part. Integrating by parts yields

$$\int\Big\{\mathrm{Re}\,\chi_k^{(j)}W_k\overline{W}_j-\frac{1}{2}\chi^{(j)}V_j(x\,;\,s)|W|^2-\frac{1}{4}\chi_{jkk}^{(j)}|W|^2\Big\}dx$$
$$=\mathrm{Re}\Big(G,\,\chi^{(j)}W_j+\frac{1}{2}\chi_j^{(j)}W\Big)-2\lambda\kappa\,\mathrm{Im}(W,\,\chi^{(j)}W_j)\,,$$

where (,) denotes the scalar product in $L^2(R^3)$. By the definition of $\chi(x)$, a short calculation shows that $\mathrm{Re}\,\chi_k^{(j)}W_k\overline{W}_j\geqq \mathrm{Ch}(x)|\nabla W|^2$. Furthermore, by the boundedness of $\chi(x)$ and of its derivatives and by the fact that $\chi_{jkk}^{(j)}=O(|x|^{-(3+\delta)})$, we have

$$\int h(x)|\nabla W|^2dx\leqq C_1\int(|G|^2+h(x)|W|^2)dx+C_2|\lambda|\kappa\|W\|_0\|\nabla W\|_0\,. \tag{2.5}$$

On the other hand, as is easily shown, $2|\lambda|\kappa\|W\|_0^2\leqq|(G,\,W)|$ and $\|\nabla W\|_0^2\leqq |\lambda|^2\|W\|_0^2+|(G,\,W)|$. Therefore, $|\lambda|\kappa\|W\|_0\|\nabla W\|_0\leqq C\{|\lambda|\,|(G,\,W)|+\|G\|_0^2\}$ for $|\lambda|$ large enough. Hence, it follows from (2.5) that

$$\int h(x)|\nabla W|^2dx\leqq(\eta\lambda^2+C)\int h(x)|W|^2dx+C(\eta)\|G\|_0^2 \tag{2.6}$$

for any η small enough. Thus, combining (2.6) with (2.4), we see that there exists a constant Λ independent of s for which the desired estimate holds.

LEMMA 2.2. *Let Λ be as in* Lemma 2.1. *Then, there exists a constant C_K independent of λ, κ and s such that for $|\lambda|\leqq\Lambda$*

$$\|W\|_{0,K}\leqq C_K\|G\|_0\,.$$

PROOF. The proof is done by contradiction. A similar proof can be found in Wilcox [11]. Assume that there exist three sequences $\{s_n\}$, $\{\zeta_n\}$, $\zeta_n=\lambda_n+i\kappa_n$, and $\{G_n\}$, $\mathrm{supp}\,G_n\subset B_R$, such that G_n converges to 0 in the L^2-norm as $n\to\infty$ and that $\|W_n(\zeta_n,\,s_n)\|_{0,K}=1$, where $W_n(\zeta_n,\,s_n)$ is the solution to equation (2.1) with $G=G_n$. We may assume that $s_n\to\infty$ and $\zeta_n\to\lambda_0$ for some λ_0, $|\lambda_0|\leqq\Lambda$, as $n\to\infty$ and that $K>R$ for R in (2.2). (The case in which the sequence $\{s_n\}$ is convergent to some finite value is dealt with similarly.)

We write

$$-\Delta W_n+V(x\,;\,s_n)W_n-(\lambda_n+i\kappa_n)^2W_n=G_n\,. \tag{2.7}$$

Let $M<R<R_2<R_1<K$ and let $\rho(x)$, $\rho\geqq 0$, be a smooth function such that $\rho(x)=1$ in $|x|\leqq R_1$ and $\rho(x)=0$ in $|x|>K$. We multiply both sides of (2.7) by

$\rho(x)\overline{W}_n$ and integrate the resulting identity over the whole space. Then, integrating by parts yields

$$\int \rho(x)|\nabla W_n|^2 dx \leqq C \tag{2.8}$$

for C independent of n. Furthermore, the elliptic estimate applied to the domain $D=\{x: R\leqq|x|\leqq R_1\}$ shows that $\{W_n\}$ forms a bounded set in $H^2(D)$. Hence, the trace $W_n|_\Sigma$ of W_n to $\Sigma=\{x:|x|=R_2\}$ is well-defined, and $\{W_n|_\Sigma\}$ and $\{(\partial/\partial\nu)W_n\}$, $(\partial/\partial\nu)W_n$ being the exterior normal derivative on Σ, are also bounded sets in $H^{3/2}(\Sigma)$ and in $H^{1/2}(\Sigma)$, respectively. Let $\Gamma_n=\Gamma_n(x, y)=(4\pi)^{-1}|x-y|^{-1}\cdot\exp(i\zeta_n|x-y|)$ be the Green function for the operator $-\Delta-\zeta_n^2$. We apply the Green formula to W_n and Γ_n in the region $\{x:|x|\geqq R_2\}$. Then, we have

$$W_n(x)=\int_\Sigma \{(\Gamma_n(x, y)(\partial/\partial\nu)W_n(y)-W_n(y)(\partial/\partial\nu)\Gamma_n(x, y)\}\, dS_y \tag{2.9}$$

for $|x|>R_2$. This relation combined with (2.8) shows that for any bounded domain Ω, $\{W_n\}$ forms a bounded set in $H^1(\Omega)$ and hence $\{W_n\}$ becomes a precompact set in $L^2(B_K)$. Moreover, we note that $\{V(x; s_n)\}$ also forms a precompact set in $L^2(R^3)$, which follows from (A. 1) and (A. 2) at once. Thus, we see that there exist subsequences of $\{s_n\}$ and $\{\zeta_n\}$ (denoted by the same symbols) such that $W_n=W_n(\zeta_n, s_n)$ converges to some W_0, $\|W\|_{0,K}=1$, strongly in $L^2(\Omega)$ for any bounded domain Ω and that $V(x; s_n)$ also converges to some non-negative bounded function $V_0(x)$ with compact support in the L^2-norm. We also see from expression (2.9) that W_0 satisfies the out-going radiation condition at infinity. Hence, W_0 is an out-going solution of the equation

$$-\Delta W_0+V_0(x)W_0-\lambda_0^2W_0=0\,.$$

On the other hand, the Rellich uniqueness theorem shows that such a solution must be zero, which contradicts $W_0\neq 0$. Here we should note that the uniqueness theorem for $\lambda_0=0$ follows from the non-negativity of $V_0(x)$. Thus, the proof is completed.

Proposition 2.1 is readily verified by combining Lemmas 2.1 and 2.2.

§ 3. *A priori* estimate.

Throughout this section, we again assume that the potential $V(x, t)$ satisfies (A. 1)$\sim$(A. 3) and derive an *a priori* estimate (Theorem 3.1) which plays an important role in proving Theorem 1.1. The derivation of such an estimate is based on the principle of limiting absorption.

We fix s, $s\geqq 1$, and write $V(x, t; s)=V(x, t+s)$, $t\geqq 0$, and $V(x; s)=V(x, s)$ $(=V(x, 0; s))$. We consider the equation

(3.1) $$u_{tt}-\Delta u+V(x, t; s)u=0,$$

(3.2) $$u(x, 0; s)=f(x; s), \quad u_t(x, 0; s)=g(x; s).$$

Here we assume that the initial data are of finite energy and have compact support which is contained in B_R, $R>M$. Furthermore, without loss of generality, we may assume that the initial data are real-valued and hence the solution u is also real-valued.

As an auxiliary equation, we also consider the equation

(3.3) $$v_{tt}-\Delta v+V(x; s)v=0$$

with the same initial data as in (3.2). (3.3) is a wave equation with time-independent potential and, as is well-known, the local energy decays exponentially.

We use the notation $E(u, K, T; s)$ to denote the local energy of solutions to (3.1) in B_K at time T;

(3.4) $$E(u, K, T; s)=\frac{1}{2}\int_{B_K}\{|u_t(T)|^2+|\nabla u(T)|^2+V(x, T; s)|u(T)|^2\}dx.$$

LEMMA 3.1. *Let $v=v(x, t; s)$ be the solution to equation (3.3) with initial data (3.2) satisfying the assumptions stated above. Then, there exists a constant C independent of s and T such that*

$$\int_0^T \|v\|_{0,M}^2 dt \leqq CE(v, \infty, 0; s).$$

PROOF. Let $\phi(t)$, $\phi\geqq 0$, be a smooth function such that $\phi(t)=0$ for $-\infty<t<1$ and $\phi(t)=1$ for $t\geqq 2$. We set $\tilde{v}=\phi v$ for the solution v. Then, $\tilde{v}$ satisfies the equation

$$\tilde{v}_{tt}-\Delta\tilde{v}+V(x; s)\tilde{v}=h$$

with zero initial data, where $h=h(x, t; s)=2\phi_t v_t+\phi_{tt}v$. By assumption, the initial data are supported in B_R. Hence, by a domain of dependence argument, it follows that the support of h in x is contained in B_{R+2}.

We now define the functions $W=W(x; \lambda+i\kappa, s)$ and $H=H(x; \lambda+i\kappa, s)$, $0<\kappa<1$, as follows:

$$W=\int_{-\infty}^{\infty} e^{i(\lambda+i\kappa)t}\tilde{v}(x, t; s)dt,$$

$$H=\int_{-\infty}^{\infty} e^{i(\lambda+i\kappa)t}h(x, t; s)dt.$$

Then, the support of H is also contained in B_{R+2} and W satisfies the equation

(3.5) $$-\Delta W+V(x; s)W-(\lambda+i\kappa)^2W=H.$$

Since the total energy is conserved for equation (3.3), the Poincaré inequality shows that $\|H\|_0^2 \leqq CE(v, \infty, 0; s)$ with C independent of λ, κ and s.

We now apply Proposition 2.1 to equation (3.5) and obtain, using the Parseval relation, that

$$\int_{-\infty}^{\infty} e^{-2\kappa t}\|\tilde{v}\|_{0,M}^2 dt = (2\pi)^{-1}\int_{-\infty}^{\infty}\|W(x; \lambda+i\kappa, s)\|_{0,M}^2 d\lambda \leqq CE(v, \infty, 0; s)$$

for C independent of s. Hence, the integral of $\|v\|_{0,M}^2$ over the interval $(2, T)$ is majorized by $CE(v, \infty, 0; s)$. On the other hand, by the Poincaré inequality and by the conservation law of total energy, we have that the integral of $\|v\|_{0,M}^2$ over $(0, 2)$ is also majorized by $CE(v, \infty, 0; s)$ with another C independent of s. Thus, the proof is completed.

Next, we set $w = w(x, t; s) = u(x, t; s) - v(x, t; s)$, where u and v are solutions to equations (3.1) and (3.3), respectively. Then, w satisfies the equation

$$\begin{aligned} &w_{tt} - \Delta w + V(x; s)w = F, \\ &w(x, 0; s) = w_t(x, 0; s) = 0, \end{aligned} \tag{3.6}$$

where $F = F(x, t; s) = (V(x; s) - V(x, t; s))u$. Let $0 < \gamma < \dfrac{2}{3}\alpha$, α being as in (A.3). We fix T so that $1 < T < s^\gamma$ and define $\tilde{F} = \tilde{F}(x, t; s)$ as $\tilde{F} = F$ for $0 \leqq t \leqq T$ and $\tilde{F} = 0$ for $t > T$. We denote by $\tilde{w} = \tilde{w}(x, t; s)$ the solution to equation (3.6) with F replaced by $\tilde{F}$. Then,

$$\tilde{w} = w \qquad \text{for} \quad 0 \leqq t \leqq T. \tag{3.7}$$

Furthermore, for $t < 0$, we extend $\tilde{w}$ and $\tilde{F}$ as $\tilde{w} = 0$ and $\tilde{F} = 0$, respectively, and define $P = P(x; \lambda+i\kappa, s)$ and $Q = Q(x, \lambda+i\kappa, s)$, $0 < \kappa < 1$, as follows:

$$P = \int_{-\infty}^{\infty} e^{i(\lambda+i\kappa)t}\tilde{w}(x, t; s)dt,$$

$$Q = \int_{-\infty}^{\infty} e^{i(\lambda+i\kappa)t}\tilde{F}(x, t; s)dt.$$

Then, P satisfies the equation

$$-\Delta P + V(x; s)P - (\lambda+i\kappa)^2 P = Q. \tag{3.8}$$

LEMMA 3.2. *Let* $0 < \gamma < \dfrac{2}{3}\alpha$ *and let* $1 < T < s^\gamma$. *Let* Q *be as above. Then,* Q *is supported in* B_M, *and*

$$\|Q\|_0^2 \leqq Cs^{-\nu}\int_0^T \|u\|_{0,M}^2 dt, \quad \nu = 2\alpha - 3\gamma > 0,$$

for C *independent of* λ, κ *and* s.

PROOF. The first assertion follows from (A. 2) at once. The second one follows from (A. 3) by use of the Schwarz inequality.

LEMMA 3.3. *Let γ, T and ν be as in* Lemma 3.2. *Let $w=w(x, t; s)$ be the solution to equation* (3.6). *Then, there exists a constant C independent of s and T such that*

$$\int_0^T \|w\|_{0,M}^2 dt \leqq C s^{-\nu} \int_0^T \|u\|_{0,M}^2 dt .$$

PROOF. We apply Proposition 2.1 to equation (3.8). Then, the Parseval relation combined with (3.7) and Lemma 3.2 yields the desired estimate.

As an immediate consequence of Lemmas 3.1 and 3.3, we obtain the following result.

LEMMA 3.4. *Let γ be as in* Lemma 3.2. *Let $u=u(x, t; s)$ be the solution to equation* (3.1) *with initial data* (3.2). *If s is taken large enough, $s\geqq s_1$, then there exists a constant C independent of s, $s\geqq s_1$, and T, $1<T<s^\gamma$, such that*

$$\int_0^T \|u\|_{0,M}^2 dt \leqq CE(u, \infty, 0; s) .$$

Now, we can state the main estimate in this section.

THEOREM 3.1. *Assume that the potential $V(x, t)$ satisfies* (A. 1)$\sim$(A. 3). *Let $0<\gamma<\frac{2}{3}\alpha$. Let $u=u(x, t; s)$ be the solution to equation* (3.1) *with initial data* (3.2) *whose supports are contained in B_R, $R>M$. If s is taken large enough, $s\geqq s_0$, then, for $1<T<s^\gamma$,*

$$E\left(u, \frac{1}{2}T, T; s\right) \leqq CT^{-1}E(u, \infty, 0; s),$$

where C is independent of s, $s\geqq s_0$, and T.

PROOF. We may assume that the solution u is a real-valued C^2-function. Indeed, all solutions of finite energy can be obtained as a limit of such solutions in the energy norm. For the proof, we use identity (1.4) with $V=V(x, t; s)$. We integrate (1.4) over $\boldsymbol{R}^3\times(0, T)$ and obtain

$$\int X(T, u)dx = \int X(0, u)dx - \int_0^T \int Z(u)dxdt . \tag{3.9}$$

For the first term on the right side, we use the Poincaré inequality to obtain that $\int X(0, u)dx \leqq CE(u, \infty, 0; s)$. If s is taken large enough, it then follows from (A. 3) and Lemma 3.4 that the second term is majorized by $CTE(u, \infty, 0; s)$ for $1<T<s^\gamma$ with C independent of s.

On the other hand, for the left side, we use (1.5) to obtain

$$\int X(T,\ u)dx \geqq \frac{1}{8}T^2 E\left(u,\ \frac{1}{2}T,\ T\ ;\ s\right),$$

which completes the proof.

§ 4. Proof of Theorem 1.1.

In this section, we shall prove Theorem 1.1 by use of the method due to Morawetz [6].

We fix T so large that $T > s_0$, s_0 being as in Theorem 3.1, and define the sequence $\{S_k\}_{k=0}^{\infty}$ by $S_k = kT$. How large T must be chosen is determined by (4.3) below. We may assume that the initial data in (1.2) are supported in B_M by taking M large enough, if necessary.

LEMMA 4.1. *Let $u=u(x, t)$ be the solution to equation* (1.1) *with initial data* (1.2) *which are supported in B_M. Then, u may be written as $u=R_0+F_0$, where $F_0=F_0(x, t)$ is the free space solution, $\square F_0=0$, with the same initial data as u and $F_0=0$ in $|x|<t-M$, while $R_0=R_0(x, t)$ is a solution to* (1.1) *for $t>2M$ and has support in B_{3M} at $t=2M$. Furthermore, according to the notation* (1.3), *we have*

$$E(R_0,\ \infty,\ t) \leqq C(t)E(u,\ \infty,\ 0)\,. \tag{4.1}$$

PROOF. It is clear from Huygens' principle that $F_0=0$ in $|x|<t-M$ and hence $F_0=0$ in $|x|<M$ for $t>2M$, which implies that F_0 is a solution of equation (1.1) for $t>2M$. Therefore, R_0 is also a solution for $t>2M$. Furthermore, by a domain of dependence argument, we know that R_0 has support of at most $|x|\leqq 3M$ at $t=2M$. Finally we shall prove (4.1). We write $E(R_0,\ \infty,\ t)\leqq 2(E(u,\ \infty,\ t)+E(F_0,\ \infty,\ t))$. By the inequality

$$\int |x|^{-2}|F_0(x,\ t)|^2 dx \leqq 4\int |\nabla F_0(x,\ t)|^2 dx$$

and by the equation $\square F_0=0$, we have $(d/dt)E(F_0,\ \infty,\ t)\leqq CE(F_0,\ \infty,\ t)$ and hence $E(F_0,\ \infty,\ t)\leqq C(t)E(u,\ \infty,\ 0)$. Similarly, we obtain the same estimate for $E(u,\ \infty,\ t)$. This proves (4.1) and the proof is complete.

LEMMA 4.2. *Let $\{S_k\}_{k=0}^{\infty}$ be the sequence defined above. Let R_0 and F_0 be as above. Then, we can construct $\{R_k\}_{k=1}^{\infty}$ and $\{F_k\}_{k=1}^{\infty}$ with the following properties:*

(a) *$R_{k-1}=R_k+F_k$ for $t>S_k$;*

(b) *$F_k=F_k(x, t)$ is the free space solution with the same initial data as R_{k-1} at $t=S_k$ and $F_k=0$ in $|x|<t-S_k-M$;*

(c) *$R_k=R_k(x, t)$ is a solution of equation* (1.1) *for $t>S_K+2M$ and has support in B_{3M} at $t=S_K+2M$;*

(d) *If we use the notation* (3.4), *the total energy of R_k at time S_k+2M,*

$E(R_k, \infty, 0\,; S_k+2M)$, *is estimated as*

$$E(R_k, \infty, 0\,; S_k+2M)$$
$$\leqq 2\{E(R_{k-1}, 3M, T\,; S_{k-1}+2M)+E(R_{k-1}, 5M, T-2M\,; S_{k-1}+2M)\}\,;$$

(e) $E(R_k, \infty, 0\,; S_k+2M) \leqq C(k, T) E(R_{k-1}, \infty, 0\,; S_{k-1}+2M)$ *for some* $C(k, T)$ *depending on* k *and* T.

PROOF. We prove this lemma by induction on k. First we consider the case $k=1$. Let F_1 be the free space solution with the same initial data as R_0 at $t=S_1$ $(=T)$. We continue F_1 as R_0 for $t<S_1$. Then, $\square F_1=0$ in the domain exterior to $\{(x, t): |x| \leqq M, 0<t<S_1\}$. We apply Huygens' principle to F_1 in this domain. Let (x, t) be a point with $|x|<t-S_1-M$. Then, the backward cone with vertex (x, t), $\{(y, s): |y-x|=t-s\}$, does not intersect $\{(x, t): |x| \leqq M, 0<t<S_1\}$ and intersect the plane $t=2M$ outside the sphere $|x|=T-M$. On the other hand, if T is taken so large that $T>4M$, then $R_0=0$ there, since R_0 has support in B_{3M} by Lemma 4.1. Thus, from Huygens' principle, we may conclude that $F_1=0$ in $|x|<t-S_1-M$. Therefore, $F_1=0$ in $|x|<M$ for $t>S_1+2M$. This implies that R_1 is a solution of equation (1.1) for $t>S_1+2M$, since R_0 is a solution for $t>2M$. We also see by a domain of dependence argument that R_1 has support of at most $|x| \leqq 3M$ at $t=S_1+2M$. Thus, we can construct R_1 and F_1 with the required properties (a)～(c). We shall prove (d) and (e) for these R_1 and F_1. Property (e) follows from (d) by using the same argument as (4.1) was proved. For the proof of (d), we write

$$E(R_1, \infty, 0\,; S_1+2M)=E(R_1, 3M, 0\,; S_1+2M)$$
$$\leqq 2\{E(R_0, 3M, T\,; 2M)+E(F_1, 3M, 0\,; S_1+2M)\}\,.$$

Here we introduce the new notation $E_0(u, K, t\,; s)$ to denote

$$E_0(u, K, t\,; s)=\frac{1}{2}\int_{B_K} \{|u_t(x, t\,; s)|^2+|\nabla u(x, t\,; s)|^2\}\, dx\,.$$

Since $F_1=0$ in $|x|<M$ at $t=S_1+2M$, it follows that

$$E(F_1, 3M, 0\,; S_1+2M)=E_0(F_1, 3M, 0\,; S_1+2M)\,.$$

Hence, by the usual energy method applied to the free space equation, we see that

$$E(F_1, 3M, 0\,; S_1+2M) \leqq E_0(R_0, 5M, 0\,; S_1) \leqq E(R_0, 5M, T-2M\,; 2M)\,.$$

This proves (d) for $k=1$. For the construction of R_k and F_k with general k, we follow the same method as above. Thus, we can construct R_k and F_k with the required properties (a)～(e) inductively.

Now, we shall determine how large T must be taken. Let γ be as before and let $k(T)$ be an integer such that for any k, $k>k(T)$, $T<S_{k-1}^{\gamma}$ $(=(k-1)^{\gamma}T^{\gamma})$. We want to apply Theorem 3.1 to the terms on the right side of (d) in Lemma 4.2. To do this, we have to check that all the assumptions in Theorem 3.1 are satisfied. First, by (c) in Lemma 4.2, R_{k-1} has support in B_{3M} and hence the condition on initial data in Theorem 3.1 is satisfied. Furthermore, by the definition of $k(T)$, $T-2M<T<(S_{k-1}+2M)^{\gamma}$ for $k>k(T)$. Thus, if T is taken so large that $T>\max(12M, s_0)$, s_0 being as in Theorem 3.1, then $S_k>s_0$ and $5M<\frac{1}{2}(T-2M)$. Hence, Theorem 3.1 shows that for $k>k(T)$

$$E(R_k, \infty, 0; S_k+2M)\leqq C_M T^{-1}E(R_{k-1}, \infty, 0; S_{k-1}+2M) \tag{4.2}$$

with C_M independent of k, $k>k(T)$. We now fix T as follows:

$$T>\max(12M, s_0, C_M) \tag{4.3}$$

so that

$$\theta=-\log C_M T^{-1}>0\,. \tag{4.3'}$$

Furthermore, we set $k_0=k(T)$ for T fixed above.

Now, we shall prove Theorem 1.1 with the aid of Lemma 4.2.

PROOF OF THEOREM 1.1. According to Lemma 4.2, we may write

$$u=\sum_{j=0}^{n}F_j+R_n \qquad \text{for} \quad t>S_n\,,$$

where $F_j=0$ in $|x|<t-S_j-M$, $j=0, 1, \cdots, n$, while R_n is a solution of equation (1.1) for $t>S_n+2M$. We may assume that $K<M$. Let $S_{n+2}>t>S_n+2M$. Then, $u=R_n=R_{n-1}$ in B_K at time t and hence we may write

$$E(u, K, t)=E(R_{n-1}, K, t-S_{n-1}-2M; S_{n-1}+2M)\,.$$

We may assume that n, $n>k_0$, is taken so large that

$$T<t-S_{n-1}-2M<3T<(S_{n-1}+2M)^{\gamma}\,.$$

Hence, by Theorem 3.1,

$$E(u, K, t)\leqq C_K E(R_{n-1}, \infty, 0; S_{n-1}+2M)\,. \tag{4.4}$$

We apply (4.2) to the right side of (4.4) k-times repeatedly until $n-k-1=k_0$ and obtain

$$E(u, K, t)\leqq C_K \exp(-\theta k)E(R_{k_0}, \infty, 0; S_{k_0}+2M)\,,$$

where θ is defined by (4.3′). Furthermore, we see from Lemma 4.1 and (e) in Lemma 4.2 that $E(R_{k_0}, \infty, 0; S_{k_0}+2M)\leqq C(k_0)E(u, \infty, 0)$. Thus, if $S_{n+2}>t>S_n$

$+2M$ and if $n-k-1\geqq k_0$, then

$$E(u, K, t)\leqq C_K \exp(-\theta k)E(u, \infty, 0)$$

for C_K independent of k. For given t (large enough), we take a maximal integer k so as to satisfy the two conditions above. Then, as is easily seen, $k\geqq C(T)t$. This completes the proof.

As an immediate consequence of Theorem 1.1, we obtain the following result.

THEOREM 4.1. *Under the same assumptions as in* Theorem 1.1, *the total energy* $E(u, \infty, t)$ *is bounded in* t;

$$E(u, \infty, t)\leqq CE(u, \infty, 0),$$

where C *depends on the bound of support of initial data.*

§ 5. Concluding remark.

Theorem 1.1 can be generalized to a wider class of potentials by use of a Lorentz transformation. We consider the potential $V(x, t)$ which takes the form

$$V(x, t)=U(x-(a/b)t, t) \tag{5.1}$$

with $U(x, t)$ satisfying (A. 1)$\sim$(A. 3), where $a=(a_1, a_2, a_3)$ is a constant vector such that

$$|a|<b, \quad |a|^2=(a_1^2+a_2^2+a_3^2). \tag{5.2}$$

We define the moving ball $B_K(t)$ as $B_K(t)=\{x\,;\,|x-(a/b)t|<K\}$, $K<\infty$, and denote by $E(u, B_K(t), t)$ the local energy in $B_K(t)$ at time t, which is given by (1.3) with $V=U(x-(a/b)t, t)$. Then, we have the following result.

THEOREM 5.1. *Assume that the potential* $V(x, t)$ *takes the form* (5.1) *with* (5.2). *Let* $u=u(x, t)$ *be a solution to* (1.1) *with* V *of the form* (5.1). *Furthermore, assume that the initial data* (1.2) *are of compact support and of finite energy. Then,* $E(u, B_K(t), t)$ *decays exponentially as* $t\to\infty$.

For the proof, we use a Lorentz transformation. Let l be the time-like line parameterized as $x=(a/b)t$. We make a Lorentz transformation $L: (x, t) \to(y, \tau)$ which preserves the sense of time and which maps l into the τ-axis in the (y, τ) coordinates. Let $v(y, \tau)$ be the representation of $u(x, t)$ in terms of the (y, τ) coordinates. Similarly, we denote by $W(y, \tau)$ the representation of the potential $V(x, t)$. Then, v satisfies the equation (1.1) with $V=W(y, \tau)$. Furthermore, we see from (5.1) that $W(y, \tau)$ satisfies (A. 1)$\sim$(A. 3) in the (y, τ) coordinates. Hence, the local energy of v decays exponentially as $\tau\to\infty$. Using this fact, we can verify Theorem 5.1 in a way similar to that in Cooper and Strauss [3], so we omit the detailed proof of this theorem.

Finally, we note that Theorem 1.2 is also extended to a wider class of potentials by use of a Lorentz transformation.

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