J. Math. Soc. Japan
Vol. 34, No. 1, 1982

Closed *-derivations on compact groups

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(Received May 22, 1980)

§1. Introduction and preliminaries.

Unbounded derivations have recently become one of the most important branches of the theory of C^*-algebras. Several authors obtained general results concerning the relation between closed *-derivations and strongly continuous one-parameter groups of *-automorphisms (cf. [**3**], [**4**], [**10**]).

S. Sakai ([**11**, Proposition 1.17]) proved that a non-zero closed derivation δ in $C(T)$ (T: the unit circle) commuting with the rotation group $\{\theta_t\}_{t\in R}$ of T is a scalar multiple of the infinitesimal generator of $\{\theta_t\}_{t\in R}$.

In this paper we present a similar result for closed *-derivations commuting with the left translation group on arbitrary compact groups.

A linear map δ in a C^*-algebra A is said to be a derivation if it satisfies the following condition:

(i) the domain $D(\delta)$ of δ is a dense subalgebra of A and $\delta(fg)=\delta(f)g+f\delta(g)$ $(f, g\in D(\delta))$. A derivation δ is said to be a *-derivation if it satisfies:

(ii) $f\in D(\delta)\Rightarrow f^*\in D(\delta)$ and $\delta(f^*)=\delta(f)^*$.

Throughout this paper, G will denote a compact group. Let $C(G)$ be the C^*-algebra of all complex-valued continuous functions on G. Suppose that $\{g_t\}_{t\in R}$ is a continuous one-parameter subgroup of G. We define $\{\tau_t\}_{t\in R}$ by the equation $\tau_t(f)(x)=f(xg_t)$ $(f\in C(G), x\in G, t\in R)$. Then $\{\tau_t\}_{t\in R}$ is a strongly continuous one-parameter group of *-automorphisms of $C(G)$. Let δ be the infinitesimal generator of $\{\tau_t\}_{t\in R}$. Then it is well-known that δ is a closed *-derivation in $C(G)$ with domain $D(\delta)$ which is a dense *-subalgebra of $C(G)$. Further we define the left translation group $\{L_u\}_{u\in G}$ by the equation $L_u(f)(x)=f(u^{-1}x)$ $(f\in C(G), u, x\in G)$. Then it is clear that $L_u\tau_t=\tau_tL_u$, $L_u(D(\delta))=D(\delta)$ and $L_u\delta=\delta L_u$ $(u\in G, t\in R)$.

Our goal in this paper is to prove that the converse is true, that is, we have the following theorem.

THEOREM. *Let G be a compact group. Suppose that δ is a closed *-derivation in $C(G)$ commuting with the left translation group $\{L_u\}_{u\in G}$, that is, $L_u(D(\delta))=D(\delta)$, $L_u\delta=\delta L_u$ $(u\in G)$. Then there exists a continuous one-parameter subgroup $\{g_t\}_{t\in R}$ of G such that δ is the infinitesimal generator of the strongly continuous* *one-parameter group of *-automorphisms* $\{\tau_t\}_{t\in R}$ *of* $C(G)$ *defined by* $\tau_t(f)(x)=f(xg_t)$ $(f\in C(G),\ x\in G,\ t\in R)$.

In this theorem we have restricted our attention now to the left translation group of G. Naturally, we have the analogous result in case of the right translation group of G.

Our method for proving the theorem is based on the Tannaka's duality theorem for arbitrary compact groups and the representative algebras which generalize the algebra of all trigonometric polynomials on the unit circle.

The author has learned from Prof. S. Sakai that F. Goodman has recently obtained results in a more general setting. We believe, however, that our method as well as the result has its own interest for further discussion.

§ 2. Domains of the closed linear operators commuting with the left translation group.

In this section we characterize the domain of a closed linear operator in $C(G)$ commuting with the left translation group. In the proof of [**11**, Proposition 1.17] the subalgebra $R(T)$ of all trigonometric polynomials in $C(T)$ (T: the unit circle) is very useful. At first we consider the subalgebra $R(G)$ in $C(G)$ which corresponds to $R(T)$.

If $\{U, H\}$ is a continuous irreducible unitary representation, then H is necessarily finite-dimensional, and so U is expressed as an element of $C(G)\otimes M_n(C)$ for some $n\in N$, where $M_n(C)$ is the C^*-algebra of all $n\times n$ complex matrices. We will denote by $\hat{G}$ the dual object of G, that is, the set of all equivalence classes of irreducible unitary representations of G. Further, for every $\lambda\in\hat{G}$ we choose a representative element $\{U^\lambda, H^\lambda\}$ and put $N_\lambda=\{x(\in G)\mapsto(U^\lambda(x)\xi\,|\,\eta):\ \xi,\ \eta\in H^\lambda\}$. Then the set N_λ is independent of the choice of the representative element $\{U^\lambda, H^\lambda\}$. Further the set N_λ is a finite-dimensional linear subspace of $C(G)$ stable under the left and right translations of G. A function f in $C(G)$ is called a representative function (or a trigonometric polynomial) on G if it is a finite linear combination of functions in $\bigcup_{\lambda\in\hat{G}} N_\lambda$. We define $R(G)$ as the set of all representative functions on G. $R(G)$ is called the representative algebra of G. We remark that $R(G)$ is a dense *-subalgebra of $C(G)$. Further, if we choose an orthonormal basis $\{e_i^\lambda\}_{i=1}^{d(\lambda)}$ of H^λ and define the functions $\{f_{ij}^{(\lambda)}\}_{\lambda\in\hat{G},1\le i,j\le d(\lambda)}$ by the equation $f_{ij}^{(\lambda)}=(U^\lambda(x)e_j^\lambda\,|\,e_i^\lambda)$ $(x\in G)$, then we have

$$\int_G f_{ij}^{(\lambda)}(t)\overline{f_{pq}^{(\mu)}(t)}\,d\nu(t)=\begin{cases}\dfrac{1}{d(\lambda)} & \text{if } \lambda=\mu,\ i=p \text{ and } j=q\,,\\[2ex] 0 & \text{otherwise}\,,\end{cases}\tag{2.1}$$

and $\{\sqrt{d(\lambda)}f_{ij}^{(\lambda)}\}_{\lambda\in\hat{G},\,i,j=1,2,\cdots,d(\lambda)}$ is a complete orthonormal system in $L^2(G,\nu)$

where ν is the normalized Haar measure on G (cf. [7], [14], etc.).

PROPOSITION 2.1. *Let G be a compact group. Suppose that T is a closed linear operator in $C(G)$ with dense domain $D(T)$ and that T commutes with the left translation group $\{L_u\}_{u\in G}$, that is, $L_u(D(T))=D(T)$ and $L_uT=TL_u$ $(u\in G)$. Then the domain $D(T)$ of T contains the representative algebra $R(G)$ of G. Further, for every $\lambda\in\hat{G}$, the space N_λ is stable under T, more precisely, if $U=\{f_{ij}\}_{i,j=1,2,\cdots,n}$ $(\in C(G)\otimes M_n(C))$ is an irreducible unitary representation of G, then there exists a matrix $\Lambda=\{\lambda_{ij}\}_{i,j=1,2,\cdots,n}$ $(\in M_n(C))$ such that*

$$T(f_{ij})=\sum_{k=1}^{n}\lambda_{kj}f_{ik} \qquad (i,\ j=1,\ 2,\ \cdots,\ n)\,. \tag{2.2}$$

PROOF. To prove the proposition, it suffices to show that for every irreducible unitary representation $U=\{f_{ij}\}_{i,j=1,2,\cdots,n}$, the coordinate functions f_{ij} belong to $D(T)$ and that there exists a matrix Λ satisfying the relation (2.2). Let $U=\{f_{ij}\}_{i,j=1,2,\cdots,n}$ be such a representation of G. For every f, $\phi\in C(G)$, we define the convolution $F(\phi:f)$ by the equation

$$F(\phi:f)=\int_G\phi(t)L_t(f)d\nu(t)\,. \tag{2.3}$$

The right-hand side of (2.3) is the Bochner integral of a continuous map of G into the Banach space $C(G)$. Hence we have $F(\phi:f)\in C(G)$.

Now we consider in case $\phi=f_{ij}$ and $f=f_{pq}$ $(i,\ j,\ p,\ q=1,\ 2,\ \cdots,\ n)$. By the orthogonality relation

$$\int_G f_{\alpha\beta}(t)f_{\gamma\tau}(t)d\nu(t)=\frac{1}{n}\delta_{\alpha\gamma}\delta_{\beta\tau} \qquad (\alpha,\ \beta,\ \gamma,\ \tau=1,\ 2,\ \cdots,\ n) \tag{2.4}$$

and by the equation

$$L_t(f_{\alpha\beta})(s)=f_{\alpha\beta}(t^{-1}s)=\sum_{\gamma=1}^{n}f_{\alpha\gamma}(t^{-1})f_{\gamma\beta}(s) \quad (\alpha,\ \beta=1,\ 2,\ \cdots,\ n,\ \ s,\ t\in G)\,, \tag{2.5}$$

we obtain the following relation

$$F(f_{ij}:f_{pq})=\delta_{jp}f_{iq} \qquad (i,\ j,\ p,\ q=1,\ 2,\ \cdots,\ n) \tag{2.6}$$

Next, we consider $F(\phi:f)$ for $\phi=f_{ij}$ $(i,\ j=1,\ 2,\ \cdots,\ n)$ and for every f in $C(G)$. We have

$$\begin{aligned}
L_sF(f_{ij}:f)&=L_s\Big(\int_G f_{ij}(t)L_t(f)d\nu(t)\Big)\\
&=\int_G f_{ij}(t)L_{st}(f)d\nu(t)=\int_G f_{ij}(s^{-1}t)L_t(f)d\nu(t)\\
&=\int_G \sum_{k=1}^{n}f_{ik}(s^{-1})f_{kj}(t)L_t(f)d\nu(t)
\end{aligned}$$

$$=\sum_{k=1}^{n} f_{ik}(s^{-1})F(f_{kj}:f), \qquad (s\in G),$$

and so

$$F(f_{ij}:f)=\sum_{k=1}^{n} F(f_{kj}:f)(e)f_{ik} \qquad (i, j=1, 2, \cdots, n).$$

Denoting the scalar product in $L^2(G, \nu)$ by $(\ |\ )_{L^2(G)}$, we have

$$F(f_{kj}:f)(e)=(f|f_{jk})_{L^2(G)}.$$

Thus we have the following equation for every $f\in C(G)$,

$$F(f_{ij}:f)=\sum_{k=1}^{n}(f|f_{jk})_{L^2(G)}f_{ik} \qquad (i, j=1, 2, \cdots, n). \tag{2.7}$$

For every $f\in D(T)$, we define $|||f|||_T$ by the equation

$$|||f|||_T=\operatorname{supp}_{t\in G}\left\|\begin{pmatrix} f(t) & T(f)(t) \\ 0 & f(t)\end{pmatrix}\right\|.$$

As T is closed, $D(T)$ is a complex Banach space with respect to the norm $|||\ |||_T$. Next we consider $F(\phi:f)$ for $f\in D(T)$ and $\phi\in C(G)$. Since the map $t\ (\in G)\mapsto \phi(t)L_t(f)\ (\in D(T))$ is continuous with respect to the norm $|||\ |||_T$ on $D(T)$, $F(\phi:f)$ belongs to $D(T)$. Since the operator T is a bounded linear operator of the Banach space $(D(T), |||\ |||_T)$ into the Banach space $(C(G), \|\ \|_{\text{unif.}})$, we have

$$T\left(\int_G \phi(t)L_t(f)d\nu(t)\right)=\int_G \phi(t)TL_t(f)d\nu(t)=\int_G \phi(t)L_tT(f)d\nu(t).$$

Hence the following equation holds for every $f\in D(T)$ and for every $\phi\in C(G)$,

$$TF(\phi:f)=F(\phi:T(f)). \tag{2.8}$$

Specially, for every $f\in D(T)$ the function $F(f_{ij}:f)$ belongs to $D(T)$ and we have the equation

$$TF(f_{ij}:f)=F(f_{ij}:T(f)) \qquad (i, j=1, 2, \cdots, n). \tag{2.9}$$

Next, using the above relation, we show that the functions $\{f_{ij}\}_{i,j=1,2,\dots,n}$ belong to $D(T)$. The orthogonality relations (2.4) imply that the functions $\{f_{ij}\}_{i,j=1,2,\dots,n}$ are linearly independent in $L^2(G, \nu)$ and so in $C(G)$. Choose α as an arbitrary number in $\{1, 2, \cdots, n\}$. We suppose that there exist linear combinations $\{g_p\}_{p=1,2,\dots,n}$ of functions $f_{\alpha 1}, f_{\alpha 2}, \cdots, f_{\alpha n}$ such that

$$g_p=\sum_{q=1}^{n} c_{pq}f_{\alpha q}\in D(T) \qquad (p=1, 2, \cdots, n) \tag{2.10}$$

and that

$$\det\left((c_{pq})_{p,q=1,2,\dots,n}\right)\neq 0\,. \tag{2.11}$$

Putting $C=(c_{pq})_{p,q=1,2,\dots,n}$ and $C^{-1}=(b_{\beta\gamma})_{\beta,\gamma=1,2,\dots,n}$, we have $\sum_{\gamma=1}^{n} b_{\beta\gamma}g_\gamma=f_{\alpha\beta}$ ($\beta=1, 2, \cdots, n$). Hence we obtain $(f_{\alpha j})_{j=1,2,\dots,n}\subseteqq D(T)$. Therefore to prove that $(f_{ij})_{i,j=1,2,\dots,n}\subseteqq D(T)$, it suffices to prove that for every $\alpha\in(1, 2, \cdots, n)$, there exist coefficients $(c_{pq})_{p,q=1,2,\dots,n}$ satisfying (2.10) and (2.11). For $h_1, h_2, \cdots, h_n$ in $C(G)$, we define $D(h_1, h_2, \cdots, h_n)$ by the equation

$$\begin{aligned} D(h_1, h_2, \cdots, h_n)&=\det\left(((h_i|f_{1j})_{L^2(G)})_{i,j=1,2,\dots,n}\right)\\ &=\sum_{\sigma\in S_n}\operatorname{sgn}\sigma(h_1|f_{1\sigma(1)})_{L^2(G)}(h_2|f_{1\sigma(2)})_{L^2(G)}\cdots(h_n|f_{1\sigma(n)})_{L^2(G)}\,. \end{aligned}$$

It is clear that the map D is continuous and that $D(T)\times D(T)\times\cdots\times D(T)$ is dense in $C(G)\times C(G)\times\cdots\times C(G)$. Since $D(f_{11}, f_{12}, \cdots, f_{1n})=(1/n)^n\neq 0$, there exist functions $\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n$ in $D(T)$ such that $D(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n)\neq 0$. Thus we have

$$F(f_{\alpha 1}:\tilde{h}_p)=\sum_{q=1}^{n}(\tilde{h}_p|f_{1q})_{L^2(G)}f_{\alpha q}\in D(T) \qquad (p=1, 2, \cdots, n)\,.$$

Putting $c_{pq}=(\tilde{h}_p|f_{1q})_{L^2(G)}$ ($p, q=1, 2, \cdots, n$) and $g_p=\sum_{q=1}^{n}c_{pq}f_{\alpha q}$ ($p=1, 2, \cdots, n$), we have $(g_p)_{p=1,2,\dots,n}\subseteqq D(T)$ and $\det((c_{pq})_{p,q=1,2,\dots,n})\neq 0$. Hence we obtain $(f_{ij})_{i,j=1,2,\dots,n}\subseteqq D(T)$. Moreover we have for every $i, j=1, 2, \cdots, n$,

$$T(f_{ij})=nTF(f_{i1}:f_{1j})=nF(f_{i1}:T(f_{1j}))=n\sum_{q=1}^{n}(T(f_{1j})|f_{1q})_{L^2(G)}f_{iq}\,.$$

Putting $\lambda_{qj}=(T(f_{1j})|f_{1q})_{L^2(G)}$ ($q, j=1, 2, \cdots, n$), we obtain

$$T(f_{ij})=\sum_{q=1}^{n}\lambda_{qj}f_{iq} \qquad (i, j=1, 2, \cdots, n)\,.$$

This completes the proof.

DEFINITION 2.2. Let E be a Banach space and let T be a closed densely defined linear operator on E with the domain $D(T)$ of T. A linear subspace D_0 of $D(T)$ is said to be a core for T if $\{(x, Tx): x\in D_0\}$ is dense in the graph $\{(x, Tx): x\in D(T)\}$ of T.

We recall that if T is a closed linear operator, then T is completely determined by a core for T.

Next we show that $R(G)$ is a core for every closed densely defined linear operator in $C(G)$ commuting with the left translation group of G. To prove this we need the following lemma.

LEMMA 2.3. *Let G be a compact group and let D be a dense linear subspace of $C(G)$ which is closed under the *-operation. Then there exists a family of functions $\{\phi_\gamma\}_{\gamma\in\Gamma}$ in D subordinate to a directed set Γ satisfying the following* *conditions:*

(2.12) $$\phi_\gamma \geqq 0 \qquad (\gamma \in \Gamma)$$

(2.13) $$\int_G \phi_\gamma(t) d\nu(t) = 1 \qquad (\gamma \in \Gamma)$$

(2.14) *if $N(e)$ is an arbitrary open neighborhood of the identity e of G, we have* $\int_{G \setminus N(e)} \phi_\gamma(t) d\nu(t) \to 0$.

PROOF. Let $\{N_v(e)\}_{v \in V}$ be a fundamental system of open neighborhood of e. Define an order on V by the inclusion, that is, $v' \leqq v$ if and only if $N_v(e) \subseteqq N_{v'}(e)$. Then V is a directed set. Put $\Gamma = V \times N$ and define an order on $\Gamma = V \times N$ by recognizing $(v', n') \leqq (v, n)$ equivalent to $N_v(e) \subseteqq N_{v'}(e)$ and $n' \leqq n$. Then Γ is also a directed set. For every $v \in V$ there exists a function $\psi_v \in C(G)$ such that

(2.15) $$\psi_v \geqq 0$$

(2.16) $$\int_G \psi_v(s) d\nu(s) = 1$$

(2.17) $$\psi_v(G \setminus N_v(e)) = \{0\}.$$

In fact, by Urysohn's lemma, we can get a function $\psi'_v \in C(G)$ such that $0 \leqq \psi'_v$, $\psi'_v(e) = 1$ and $\psi'_v(G \setminus N_v(e)) = \{0\}$. Then we have $\int_G \psi'_v(t) d\nu(t) > 0$. We obtain a function ψ_v satisfying conditions (2.15)-(2.17) by multiplying ψ'_v by a suitable positive real number. Since D is a dense *-subspace of $C(G)$, for every $v \in V$ and for every $\varepsilon > 0$ there exists a function $\phi'_{v,\varepsilon} \in D$ such that $\bar{\phi}'_{v,\varepsilon} = \phi'_{v,\varepsilon}$ and that $\|\phi'_{v,\varepsilon} - (\psi_v + \varepsilon)\|_{\text{unif.}} < \varepsilon/2$. Then we have the following inequalities; $0 \leqq \phi'_{v,\varepsilon}$, $1 - \varepsilon/2 < \int_G \phi'_{v,\varepsilon}(t) d\nu(t) < 1 + 3\varepsilon/2$ and $0 \leqq \int_{G \setminus N_v(e)} \phi'_{v,\varepsilon}(t) d\nu(t) \leqq 3\varepsilon/2$. If we take ε as $0 < \varepsilon < 1/3$, and put

$$\phi_{v,\varepsilon} = \frac{1}{\int_G \phi'_{v,\varepsilon}(s) d\nu(s)} \phi'_{v,\varepsilon},$$

then we have the following properties:

(2.18) $$\phi_{v,\varepsilon} \in D,$$

(2.19) $$0 \leqq \phi_{v,\varepsilon},$$

(2.20) $$\int_G \phi_{v,\varepsilon}(t) d\nu(t) = 1,$$

(2.21) $$\int_{G \setminus N_v(e)} \phi_{v,\varepsilon}(t) d\nu(t) < 2\varepsilon.$$

Setting $\phi_{(v,n)}=\phi_{v,1/n}$, we obtain a family $\{\phi_\gamma\}_{\gamma\in\Gamma}$ satisfying conditions (2.12)-(2.14). This completes the proof.

PROPOSITION 2.4. *Let G be a compact group. Suppose that T is a closed densely defined linear operator in $C(G)$ commuting with the left translation group $\{L_u\}_{u\in G}$ of G. Then $R(G)$ is a core for T.*

PROOF. We use the notations as in the proof of Proposition 2.1. By equation (2.7), we have for every $\phi\in R(G)$ and for every $f\in C(G)$

$$F(\phi:f)\in R(G). \tag{2.22}$$

Since $R(G)$ is a dense *-subalgebra of $C(G)$, we can choose a net $\{\phi_\gamma\}_{\gamma\in\Gamma}$ in $R(G)$ satisfying the conditions (2.12)-(2.14). Now we show that for every $f\in C(G)$ $F(\phi_\gamma:f)$ converges to f in the sense of the uniform norm on $C(G)$. In fact, for every $\gamma\in\Gamma$, we have

$$\begin{aligned}\|F(\phi_\gamma:f)-f\|&=\sup_{s\in G}\left|\int_G\phi_\gamma(t)L_t(f)(s)d\nu(t)-f(s)\right|\\&=\sup_{s\in G}\left|\int_G\phi_\gamma(t)\{f(t^{-1}s)-f(s)\}\,d\nu(t)\right|.\end{aligned}$$

For every $\varepsilon>0$, there exists an open neighborhood $N(e)$ of the identity e of G such that $|f(t^{-1}s)-f(s)|<\varepsilon$ $(t\in N(e),\ s\in G)$. Then we have

$$\sup_{s\in G}\left|\int_{N(e)}\phi_\gamma(t)\{f(t^{-1}s)-f(s)\}\,d\nu(t)\right|<\varepsilon\int_{N(e)}\phi_\gamma(t)d\nu(t)<\varepsilon.$$

On the other hand, condition (2.14) implies

$$\begin{aligned}&\sup_{s\in G}\left|\int_{G\setminus N(e)}\phi_\gamma(t)\{f(t^{-1}s)-f(s)\}\,d\nu(t)\right|\\&\leqq 2\|f\|_{\text{unif.}}\int_{G\setminus N(e)}\phi_\gamma(t)d\nu(t)\longrightarrow 0.\end{aligned}$$

Thus we have for every $f\in C(G)$

$$\|F(\phi_\gamma:f)-f\|_{\text{unif.}}\longrightarrow 0. \tag{2.23}$$

By (2.8) and (2.23), for every $f\in D(T)$ we have

$$\|TF(\phi_\gamma:f)-Tf\|_{\text{unif.}}=\|F(\phi_\gamma:Tf)-Tf\|_{\text{unif.}}\longrightarrow 0.$$

Consequently, by (2.22) we have the proposition. This completes the proof.

§ 3. Proof of Theorem.

In this final section we complete the proof of the theorem by using Tanaka's duality theorem.

At first we consider the following general fact. Let Ω be a compact Hausdorff space and let δ be a *-derivation in $C(\Omega)$ whose domain $D(\delta)$ contains 1. For every natural number n, we define δ_n by

$$\delta_n\begin{pmatrix} f_{11} & f_{12}\cdots f_{1n} \\ f_{21} & f_{22}\cdots f_{2n} \\ & \cdots \\ f_{n1} & f_{n2}\cdots f_{nn}\end{pmatrix}=\begin{pmatrix} \delta(f_{11}) & \delta(f_{12})\cdots\delta(f_{1n}) \\ \delta(f_{21}) & \delta(f_{22})\cdots\delta(f_{2n}) \\ & \cdots \\ \delta(f_{n1}) & \delta(f_{n2})\cdots\delta(f_{nn})\end{pmatrix}$$

for $f_{ij}\in D(\delta)$ $(i, j=1, 2, \cdots, n)$. Then $D(\delta_n)$ is a dense *-subalgebra of $C(\Omega)\otimes M_n(C)$. If δ is closed, then δ_n is also closed. The family $\{\delta_n\}_{n\in N}$ has the following fundamental properties:

(3.1) if $A\in M_n(C)$, then $1_\Omega\otimes A\in D(\delta_n)$ and $\delta(1_\Omega\otimes A)=0$; and if $A\in GL(n:C)$ and $F=(f_{ij})_{i,j=1,2,\cdots,n}\in D(\delta_n)$, then $(1_\Omega\otimes A)^{-1}F(1_\Omega\otimes A)\in D(\delta_n)$ and $\delta_n((1_\Omega\otimes A)^{-1}F(1_\Omega\otimes A)=(1_\Omega\otimes A)^{-1}\delta_n(F)(1_\Omega\otimes A)$;

(3.2) if $k\in N$ and $F_1\in D(\delta_{n_1})$, $F_2\in D(\delta_{n_2})$, $\cdots$, $F_k\in D(\delta_{n_k})$, then $F_1\oplus F_2\oplus\cdots\oplus F_k\in D(\delta_{n_1+n_2+\cdots+n_k})$ and $\delta_{n_1+n_2+\cdots+n_k}(F_1\oplus F_2\oplus\cdots\oplus F_k)$ $=\delta_{n_1}(F_1)\oplus\delta_{n_2}(F_2)\oplus\cdots\oplus\delta_{n_k}(F_k)$;

(3.3) if $F=(f_{ij})_{i,j=1,2,\cdots,n}\in D(\delta_n)$. then $\bar{F}=(\overline{f_{ij}})_{i,j=1,2,\cdots,n}\in D(\delta_n)$ and $\delta_n(\bar{F})=\overline{\delta_n(F)}$;

(3.4) if n, $m\in N$ and $F\in D(\delta_n)$, $\Phi\in D(\delta_m)$, then $F\otimes\Phi\in D(\delta_{nm})$ and $\delta_{nm}(F\otimes\Phi)=\delta_n(F)\otimes\Phi+F\otimes\delta_m(\Phi)$.

Now we return to the proof of the theorem. Let G be a compact group. Suppose that $\delta: D(\delta)\mapsto C(G)$ is a closed *-derivation commuting with the left translation group $\{L_u\}_{u\in G}$, that is, $L_u(D(\delta))=D(\delta)$ and $L_u\delta=\delta L_u$ $(u\in G)$. To prove the theorem, by Propositions 2.1 and 2.4, it suffices to show that there exists a continuous one-parameter subgroup $\{g_t\}_{t\in R}$ of G having the following properties: if we set $\tau_t(f)(x)=f(xg_t)$ $(f\in C(G), x\in G, t\in R)$, and if we define $\tilde{\delta}$ as the infinitesimal generator of the strongly continuous one-parameter group $\{\tau_t\}_{t\in R}$ of *-automorphisms of $C(G)$, we have $\delta(f)=\tilde{\delta}(f)$ for every $f\in R(G)$.

We denote by $\bar{G}$ the set of all continuous matricial representations of G. For $D\in\bar{G}$, we denote by $d(D)$ the degree of D. If $U\in\bar{G}$ is an irreducible unitary representation with $d(U)=n$, then by Proposition 2.1, U belongs to $D(\delta_n)$ and there exists a matrix $\Lambda(U)\in M_n(C)$ such that $\delta_n(U)=U\Lambda(U)$. Further, if $V\in\bar{G}$ is a unitary representation with $d(V)=n$, then V is decomposed into a direct sum of irreducible unitary representations $U_1, U_2, \cdots, U_k$ of G, that is, $V(x)=U_1(x)\oplus U_2(x)\oplus\cdots\oplus U_k(x)$ $(x\in G)$ (cf. [7], [14], etc.). Then we have $V\in D(\delta_n)$ and

$$\begin{aligned}\delta_n(V)&=\delta_{d(U_1)}(U_1)\oplus\delta_{d(U_2)}(U_2)\oplus\cdots\oplus\delta_{d(U_k)}(U_k)\\&=(U_1\oplus U_2\oplus\cdots\oplus U_k)(\Lambda(U_1)\oplus\Lambda(U_2)\oplus\cdots\oplus\Lambda(U_k))\\&=V(\Lambda(U_1)\oplus\Lambda(U_2)\oplus\cdots\oplus\Lambda(U_k)).\end{aligned}$$

Thus, by putting $\Lambda(V)=\Lambda(U_1)\oplus\Lambda(U_2)\oplus\cdots\oplus\Lambda(U_k)$, we have $\delta_n(V)=V\Lambda(V)$. Moreover, if D is an arbitrary element in $\bar{G}$ with $d(D)=n$, then there exists a matrix $P\in GL(n:C)$ and a unitary representation $V\in\bar{G}$ such that $D(x)=P^{-1}V(x)P$ $(x\in G)$ (cf. [8], etc.). Then we have $D\in D(\delta_n)$ and $\delta_n(D)=P^{-1}\delta_n(V)P=P^{-1}V\Lambda(V)P$ $=(P^{-1}VP)(P^{-1}\Lambda(V)P)=D(P^{-1}\Lambda(V)P)$. Hence by putting $\Lambda(D)=P^{-1}\Lambda(V)P$, we have $\delta_n(D)=D\Lambda(D)$. It is clear that the map $\Lambda: D(\in\bar{G})\mapsto\Lambda(D)$ $(\in\bigcup_{n=1}^{\infty}M_n(C))$ is uniquely determined by the equation $\delta_n(D)=D\Lambda(D)$ $(D\in\bar{G})$.

Next we show that for every $t\in R$, the map D $(\in\bar{G})\mapsto\exp(t\Lambda(D))$ $(\in\bigcup_{n=1}^{\infty}GL(n:C))$ has the following properties;

(3.5) if $D\in\bar{G}$ with $d(D)=n$ and $P\in GL(n:C)$, then $\exp(t\Lambda(P^{-1}DP))=P^{-1}\exp(t\Lambda(D))P$;

(3.6) if $k\in N$ and $D_1, D_2, \cdots, D_k\in\bar{G}$, then $\exp(t\Lambda(D_1\oplus D_2\oplus\cdots\oplus D_k))$ $=\exp(t\Lambda(D_1))\oplus\exp(t\Lambda(D_2))\oplus\cdots\oplus\exp(t\Lambda(D_k))$;

(3.7) if $D\in\bar{G}$, then $\exp(t\Lambda(\bar{D}))=\overline{\exp(t\Lambda(D))}$;

(3.8) if $D_1, D_2\in\bar{G}$, then $\exp(t\Lambda(D_1\otimes D_2))=\exp(t\Lambda(D_1))\otimes\exp(t\Lambda(D_2))$.

In fact, the properties (3.1)-(3.4) of the family of *-derivations $\{\delta_n\}_{n\in N}$ ensure that the map D $(\in\bar{G})\mapsto\Lambda(D)$ $(\in\bigcup_{n=1}^{\infty}M_n(C))$ has the following properties;

(3.9) if $D\in\bar{G}$ with $d(D)=n$ and $P\in GL(n:C)$, then $\Lambda(P^{-1}DP)=P^{-1}\Lambda(D)P$;

(3.10) if $k\in N$ and $D_1, D_2, \cdots, D_k\in\bar{G}$, then $\Lambda(D_1\oplus D_2\oplus\cdots\oplus D_k)$ $=\Lambda(D_1)\oplus\Lambda(D_2)\oplus\cdots\oplus\Lambda(D_k)$;

(3.11) if $D\in\bar{G}$, then $\Lambda(\bar{D})=\overline{\Lambda(D)}$;

(3.12) if $D_i\in\bar{G}$ with $d(D_i)=n_i$ $(i=1, 2)$, then $\Lambda(D_1\otimes D_2)$ $=\Lambda(D_1)\otimes I_{n_2}+I_{n_1}\otimes\Lambda(D_2)$.

On the other hand, the above properties (3.9)-(3.12) of the map Λ ensure the properties (3.5)-(3.8) of the map D $(\in\bar{G})\mapsto\exp(t\Lambda(D))$ $(\in\bigcup_{n=1}^{\infty}GL(n, C))$. By Tannaka's duality theorem ([15]) for every $t\in R$, there uniquely exists an element g_t of G such that

(3.13) $\exp(t\Lambda(D))=D(g_t)$ for every $D\in\bar{G}$.

For every $\lambda\in\hat{G}$, we choose a representative element $\{U^{\lambda}, H^{\lambda}\}$. Setting $H=\bigoplus_{\lambda\in\hat{G}} H^{\lambda}$ and $J(x)=\bigoplus_{\lambda\in\hat{G}} U^{\lambda}(x)$ $(x\in G)$. J is continuous for the weak operator topology on the operator algebra $B(H)$. For every $x\in G$ such that $x\neq e$, there exists an element $\lambda\in\hat{G}$ such that $U^{\lambda}(x)\neq I_{H}\lambda$ (cf. [7] etc.). Thus J is a faithful representation. For every $\lambda\in\hat{G}$, by Proposition 2.1, there exists a linear operator $\Lambda(U^{\lambda})\in B(H^{\lambda})$ such that $\delta_{d(U^{\lambda})}(U^{\lambda})=U^{\lambda}\Lambda(U^{\lambda})$. Since $\Lambda(U^{\lambda})^{*}+\Lambda(U^{\lambda})=\delta_{d(U^{\lambda})}(U^{\lambda *}U^{\lambda})=\delta_{d(U^{\lambda})}(1_{d(U^{\lambda})})=0$, $\Lambda(U^{\lambda})$ is skew-hermitian. On the other hand, by (3.13) we have $\exp(t\Lambda(U^{\lambda}))=U^{\lambda}(g_{t})$ for every $\lambda\in G$ and for every $t\in R$. We define a linear subspace D_{0} of H by $D_{0}=\{\xi=\bigoplus_{\lambda\in L}\xi_{\lambda}:\xi_{\lambda}\in H^{\lambda}$ $(\lambda\in L)$, L is a finite subset of $\hat{G}\}$. Further, we define a one-parameter family of linear operators of D_{0} into D_{0} by $\tilde{U}_{t}(\xi)=\tilde{U}_{t}(\bigoplus_{\lambda\in L}\xi_{\lambda})=\bigoplus_{\lambda\in L}\exp(t\Lambda(U^{\lambda}))\xi_{\lambda}$ for every $\xi=\bigoplus_{\lambda\in L}\xi_{\lambda}\in D_{0}$. Then we have the following relations $\tilde{U}_{0}=\mathrm{id}_{D_{0}}$, $\tilde{U}_{s+t}=\tilde{U}_{s}\tilde{U}_{t}$ $(s, t\in R)$ and $(\tilde{U}_{t}\xi|\tilde{U}_{t}\eta)_{H}=(\xi|\eta)_{H}$ $(\xi, \eta\in D_{0}, t\in R)$. Thus there exists a one-parameter group $\{U_{t}\}_{t\in R}$ of unitary operators on H such that $U_{t}\xi=\tilde{U}_{t}\xi$ for every $\xi\in D_{0}$ and for every $t\in R$. Moreover $\{U_{t}\}_{t\in R}$ is continuous for the weak operator topology. On the other hand, H_{λ} is stable under $\{U_{t}\}_{t\in R}$ and the restriction $U_{t|H_{\lambda}}$ is equal to $U^{\lambda}(g_{t})$ $(\lambda\in\hat{G}, t\in R)$, and this implies that $J(g_{t})=U_{t}$ $(t\in R)$. Since $\{U_{t}\}_{t\in R}$ is a one-parameter group and J is faithful, $\{g_{t}\}_{t\in R}$ is a one-parameter subgroup of G. Further since G is compact, G is homeomorphic to the image $J(G)$ of J. Therefore the weak continuity of $\{U_{t}\}_{t\in R}$ implies the continuity of $\{g_{t}\}_{t\in R}$.

Now we define $\{\tau_{t}\}_{t\in R}$ by the equation $\tau_{t}(f)(x)=f(xg_{t})$ $(f\in C(G), x\in G, t\in R)$. It is clear that $\{\tau_{t}\}_{t\in R}$ is a strongly continuous one-parameter group of *-automorphisms of $C(G)$ commuting with $\{L_{u}\}_{u\in G}$. Then the infinitesimal generator $\tilde{\delta}$ of $\{\tau_{t}\}_{t\in R}$ commutes with $\{L_{u}\}_{u\in G}$. By Proposition 2.1, $R(G)$ is contained in $D(\delta)\cap D(\tilde{\delta})$. For every $\lambda\in\hat{G}$, we have

$$\begin{aligned}\tilde{\delta}_{d(U^{\lambda})}(U^{\lambda})(x)&=\lim_{t\to 0}\frac{1}{t}\{U^{\lambda}(xg_{t})-U^{\lambda}(x)\}\\&=U^{\lambda}(x)\lim_{t\to 0}\frac{1}{t}\{U^{\lambda}(g_{t})-I_{H^{\lambda}}\}\\&=U^{\lambda}(x)\Lambda(U^{\lambda})=\delta_{d(U^{\lambda})}(U^{\lambda})(x)\qquad(x\in G).\end{aligned}$$

Therefore we obtain $\delta(f)=\tilde{\delta}(f)$ for every $f\in R(G)$. Hence by Proposition 2.4, we have $\delta=\tilde{\delta}$. This completes the proof.

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Added in Proof.

After having finished this paper, independently, the author received from Prof. F. M. Goodman his recent preprint, "Translation invariant closed *-derivations", in which we found the results cited in the introduction. The author is grateful to Prof. S. Sakai for pointing him about Goodman's results and to Prof. F. M. Goodman for sending him the preprint.