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On a Volevič system of singular partial differential equations

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In this paper, we deal with a certain class of Volevič systems of linear partial differential equations with some singularities and establish an existence and uniqueness theorem for analytic solutions, that is, an analogue of the Cauchy-Kowalewski theorem. Further, we also give a uniqueness theorem for distribution solutions with some regularity. Our results are generalizations of those in Baouendi-Goulaouic [1][2], where they discussed for higher-order single equations. Analogous results are obtained in Elschner [3][4].

§1. Assumptions and results.

First, we state the existence and uniqueness theorem for analytic solutions. Let Ω be a bounded open set in $\boldsymbol{R}^n$. For $s>0$, we set

$$\Omega_s=\bigcup_{a\in\Omega}B(a, s),$$

where $B(a, s)=\{z\in\boldsymbol{C}^n;\ |z-a|<s\}$. We denote by $A_s(\Omega)$ the Banach space of all functions holomorphic in Ω_s and continuous on $\bar{\Omega}_s$ with its norm $\|u\|_s=\sup\{|u(z)|;\ z\in\Omega_s\}$. Now, let $(t, z)\in\boldsymbol{R}\times\boldsymbol{C}^n$ and let us consider an $m\times m$ system of linear partial differential equations of the form

$$\text{(S)}\qquad t^\sigma D_t u+A(t, z, t^\rho D_z)u=f(t, z)$$

in $C^0([0, T], A_{s_0}(\Omega))$, where $D_t=\partial/\partial t$, $D_z=\partial/\partial z$, $u={}^t(u_1, \cdots, u_m)$, $f(t, z)={}^t(f_1(t, z), \cdots, f_m(t, z))$ and $A(t, z, t^\rho D_z)=(A_{ij}(t, z, t^\rho D_z))_{1\leqq i,j\leqq m}$ is an $m\times m$ matrix of differential operators. We assume the following three conditions: (1) $\sigma\geqq 1$ and $\rho>\sigma-1$, (2) the order of $A_{ij}\leqq n_i-n_j+1$ for some $(n_1, \cdots, n_m)\in\boldsymbol{N}^m$ and $A_{ij}(t, z, t^\rho D_z)$ is expressed in the form

$$A_{ij}(t, z, t^\rho D_z)=\sum_{|\alpha|\leqq n_i-n_j+1}a_{ij,\alpha}(t, z)(t^\rho D_z)^\alpha$$

for some $a_{ij,\alpha}(t, z)\in C^0([0, T], A_{s_0}(\Omega))$, and (3) the eigenvalues $\alpha_j(z)$ $(1\leqq j\leqq m)$ of $A(0, z, 0)$ satisfy $\operatorname{Re}\alpha_j(z)\geqq c$ on Ω_{s_0} for some $c>0$. Then, we have the fol-

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lowing result which is an analogue of the Cauchy-Kowalewski theorem.

THEOREM 1. *For any* s $(0<s<s_0)$, *there is a positive number* ε $(0<\varepsilon<T)$ *which satisfies the following: for any* $f(t)$ $(=f(t, z))\in C^0([0, T], A_{s_0}(\Omega))$ *there exists a unique solution* $u(t)\in C^0([0, \varepsilon], A_s(\Omega))\cap C^1((0, \varepsilon], A_s(\Omega))$ *of* (S) *satisfying* $t^\sigma u_t'(t)\in C^0([0, \varepsilon], A_s(\Omega))$.

Secondly, we state the uniqueness theorem for distribution solutions. Assume that Ω contains the origin of $\boldsymbol{R}^n$. Let $(t, x)\in\boldsymbol{R}\times\boldsymbol{R}^n$ and let us consider the $m\times m$ system

$$(\mathrm{S})_{\boldsymbol{R}} \qquad t^\sigma D_t u+A(t, x, t^\rho D_x)u=0$$

in $C^0([0, T], \mathcal{D}'(\Omega))$, where $A(t, x, t^\rho D_x)$ is the restriction of $A(t, z, t^\rho D_z)$ on $[0, T]\times\Omega$ and $\mathcal{D}'(\Omega)$ is the space of all distributions on Ω. Then, combining the dual version of Theorem 1 for analytic functionals with arguments developed in Treves [**7**] and Baouendi-Goulaouic [**1**], we obtain the following result which is an analogue of Holmgren's uniqueness theorem.

THEOREM 2. *Let* $u(t)$ $(=u(t, x))\in C^0([0, T], \mathcal{D}'(\Omega))\cap C^1((0, T], \mathcal{D}'(\Omega))$ *be a solution of* $(\mathrm{S})_{\boldsymbol{R}}$. *Then, we have* $u(t, x)=0$ *in a neighbourhood of* $(0, 0)$ *in* $[0, T]\times\boldsymbol{R}^n$.

Finally, we remark the case of higher-order single equations. Let

$$(t^\sigma D_t)^m u+\sum_{\substack{j+|\alpha|\leq m\\ j<m}} a_{j,\alpha}(t, z)(t^\rho D_z)^\alpha(t^\sigma D_t)^j u=f(t, z)$$

be a single equation of order m with $\sigma\geqq 1$, $\rho>\sigma-1$ and $a_{j,\alpha}(t, z)\in C^0([0, T], A_{s_0}(\Omega))$. Then, by the usual method (for example, see Tahara [**5**]), we can rewrite the above equation into an $m\times m$ system of type (S) with $(n_1, \cdots, n_m)=(1, \cdots, m)$. Therefore, we can obtain the same results as Theorems 1 and 2 above. The case $\sigma=1$ is already proved in Baouendi-Goulaouic [**1**][**2**] and the case $\rho\geqq\sigma>1$ in Elschner [**3**]. Elschner [**4**] treats the case $\sigma\geqq 1$ and $\rho>\sigma-1$, but his function spaces are somewhat different from ours. In Tahara [**6**], analogous equations are discussed in the space of differentiable functions under some hyperbolicity.

§2. Basic estimates for a Volevič system.

Before the proofs of Theorems 1 and 2, we prepare some estimates for resolvent operators of the system of ordinary differential equations

$$t^\sigma D_t u+A(t, z, 0)u=f(t, z) \tag{2.1}$$

in $C^0([0, \varepsilon], A_s(\Omega))$, where u, $f(t, z)$ and $A(t, z, 0)$ are the same as in §1. Note that the (i, j) component of $A(t, z, 0)$ vanishes identically, if $n_i-n_j+1<0$. This fact will play an essential role in our discussion. The existence of resolvent operators of (2.1) is guaranteed by the following lemma.

LEMMA 1. *Assume that* ε (>0) *is sufficiently small. Then, for any* s $(0<s\leqq s_0)$ *and for any* $f(t)$ $(=f(t, z))\in C^0([0, \varepsilon], A_s(\Omega))$ *there exists a unique solution* $u(t)\in C^0([0, \varepsilon], A_s(\Omega))\cap C^1((0, \varepsilon], A_s(\Omega))$ *of* (2.1) *satisfying* $t^\sigma u_t'(t)\in C^0([0, \varepsilon], A_s(\Omega))$.

PROOF. Notice the following facts: (i) for any ε, s and $g(t, z)\in C^0([0, \varepsilon], A_s(\Omega))$ there exists a unique solution $v(t)\in C^0([0, \varepsilon], A_s(\Omega))\cap C^1((0, \varepsilon], A_s(\Omega))$ of the equation $t^\sigma D_t v+A(0, z, 0)v=g(t, z)$ satisfying $t^\sigma v_t'(t)\in C^0([0, \varepsilon], A_s(\Omega))$, and (ii) the unique solution $v(t, z)$ $(=v(t))$ in (i) is given by

$$v(t, z)=\int_0^\infty e^{-A(0,z,0)s}g(\phi_\sigma(t, s), z)ds\,, \tag{2.2}$$

where $\phi_\sigma(t, s)$ is the function defined by

$$\phi_\sigma(t, s)=\begin{cases} te^{-s}\,, & \text{when } \sigma=1\,, \\ t\Big(\dfrac{1}{(\sigma-1)st^{\sigma-1}+1}\Big)^{1/(\sigma-1)}\,, & \text{when } \sigma>1\,. \end{cases}$$

Using these facts, we can solve (2.1) by the method of successive approximations. For given $f(t)\in C^0([0, \varepsilon], A_s(\Omega))$, we define $u^{(p)}\in C^0([0, \varepsilon], A_s(\Omega))\cap C^1((0, \varepsilon], A_s(\Omega))$ by the solution of

$$t^\sigma D_t u^{(p)}+A(0, z, 0)u^{(p)}=(A(0, z, 0)-A(t, z, 0))u^{(p-1)}+f \tag{2.3}$$

inductively on $p\geqq 0$, where $u^{(-1)}=0$. Put $v^{(p)}=u^{(p)}-u^{(p-1)}$. Since $\sup\{|e^{-A(0,z,0)t}|\,;\ z\in\Omega_{s_0}\}\leqq Me^{-(c/2)t}$ holds for some $M>0$ (by (3) in § 1), we obtain the estimate

$$\|v^{(p)}(t)\|_s\leqq CM^{p+1}\Big(\frac{2}{c}\Big)^{p+1}\delta(t)^p \tag{2.4}$$

for any $p\geqq 0$, where $C=\sup\{|f(\tau, z)|\,;\ 0\leqq\tau\leqq\varepsilon,\ z\in\Omega_{s_0}\}$ and $\delta(t)=\sup\{|A(0, z, 0)-A(\tau, z, 0)|\,;\ 0\leqq\tau\leqq t,\ z\in\Omega_{s_0}\}$. If we choose ε (>0) such that $\delta(\varepsilon)<c/2M$, it follows from (2.4) that the series $\sum_p v^{(p)}$ is convergent in $C^0([0, \varepsilon], A_s(\Omega))$ and therefore the sequence $\{u^{(p)}\}$ converges to a function u in $C^0([0, \varepsilon], A_s(\Omega))$. Applying (2.2) to (2.3) and making $p\to\infty$, we can see that u is a desired solution in $C^0([0, \varepsilon], A_s(\Omega))\cap C^1((0, \varepsilon], A_s(\Omega))$. Thus, the existence of solutions is obtained. The uniqueness may be proved in the same way. Q.E.D.

By Lemma 1, we can define the resolvent matrix R of (2.1) by $u=R[f]$, where $f\in C^0([0, \varepsilon], A_s(\Omega))$ and u is the unique solution of (2.1) given in Lemma 1. In other words, R is an $m\times m$ matrix of operators in $C^0([0, \varepsilon], A_s(\Omega))$ which satisfies the following: (i) $R[f]\in C^0([0, \varepsilon], A_s(\Omega))\cap C^1((0, \varepsilon], A_s(\Omega))$ and $t^\sigma(R[f])_t'\in C^0([0, \varepsilon], A_s(\Omega))$ for $f\in C^0([0, \varepsilon], A_s(\Omega))$, (ii) $(t^\sigma D_t+A(t, z, 0))R[f]=f$ for $f\in C^0([0, \varepsilon], A_s(\Omega))$, and (iii) $R[(t^\sigma D_t+A(t, z, 0))u]=u$, if $u\in C^0([0, \varepsilon], A_s(\Omega))\cap$

$C^1((0, \varepsilon], A_s(\Omega))$ satisfies $t^\sigma u_t' \in C^0([0, \varepsilon], A_s(\Omega))$. We denote by R^{ij} the (i, j) component of R.

We now introduce the following notation

$$\mathcal{H}[g](t)=\int_0^\infty e^{-(c/2)s} g(\phi_\sigma(t, s))ds\,,$$

where c is the constant in (3) of § 1. Note that the formula

$$\mathcal{H}^k[g](t)=\frac{1}{(k-1)!}\int_0^\infty s^{k-1}e^{-(c/2)s}g(\phi_\sigma(t, s))ds \tag{2.5}$$

holds for any $k \geqq 1$. This is verified by the change of variables and the fact $\phi_\sigma(\phi_\sigma(t, s_1), s_2)=\phi_\sigma(t, s_1+s_2)$.

Under these notations, we obtain the following a priori estimate which is the main result of this section.

LEMMA 2. *Assume that ε (>0) is sufficiently small. Then, there is a positive constant C such that the estimate*

$$\|R^{ij}[g](t)\|_s \leqq C\mathcal{H}^{p_{ij}}[\|\!|g|\!\|_s](t) \tag{2.6}$$

holds for any t $(0 \leqq t \leqq \varepsilon)$, s $(0<s\leqq s_0)$ and any scalar function $g(t) \in C^0([0, \varepsilon], A_s(\Omega))$, where $p_{ij}=\max(n_j-n_i+1, 1)$, $\|\cdot\|_s$ is the supremum norm on Ω_s and $\|\!|g|\!\|_s(t)=$ $\sup\{\|g(\tau)\|_s;\ 0\leqq \tau \leqq t\}$.

COROLLARY. *If $g(t)\in C^0([0, \varepsilon], A_s(\Omega))$ satisfies $\|g(t)\|_s \leqq t^L$ for some $L> p_{ij}(\sigma-1)$, we have*

$$\|R^{ij}[g](t)\|_s \leqq C\Big(\frac{2}{c}\Big)^{p_{ij}-l}\frac{t^{L-l(\sigma-1)}}{(L-\sigma+1)\cdots(L-l\sigma+l)} \tag{2.7}$$

for any integer l such that $0\leqq l \leqq p_{ij}$.

PROOF OF LEMMA 2. For simplicity, we denote by $K=(K^{ij})_{1\leqq i,j\leqq m}$ the operator defined by

$$K[f](t, z)=\int_0^\infty e^{-A(0,z,0)s}f(\phi_\sigma(t, s), z)ds$$

and by $\tilde{A}=(\tilde{A}_{ij})_{1\leqq i,j\leqq m}$ the operator defined by $\tilde{A}[f](t, z)=(A(0, z, 0)-A(t, z, 0))f(t, z)$. Then, by the construction of approximate solutions in (2.3) we can express the resolvent matrix R by

$$R=\sum_{l=0}^\infty R_l\,,$$

$$R_0=K \quad\text{and}\quad R_l=K\tilde{A}R_{l-1} \qquad \text{for } l\geqq 1\,.$$

Therefore, to obtain Lemma 2 it is sufficient to show that the estimates

$$\|R_l^{ij}[g](t)\|_s \leqq CM^l\delta(t)^l\mathcal{H}^{p_{ij}}[\|\!|g|\!\|_s](t) \tag{2.8}$$

($l\geqq 0$) hold for some $C>0$ and $M>0$, where R_l^{ij} is the (i, j) component of R_l and $\delta(t)$ is the same as in (2.4). First, we will show the case $l=0$. Note that $A(0, z, 0)$ satisfies the following conditions: (i) $\sup\{|e^{-A(0,z,0)t}|; z\in\Omega_{s_0}\}\leqq M_1e^{-(3c/4)t}$ for some $M_1>0$, and (ii) if $p_{ij}\geqq 2$, the (i, j) component of $A(0, z, 0)^k$ vanishes identically for any k such that $0\leqq k\leqq p_{ij}-2$. (i) is clear by (3) in §1 and (ii) is verified as follows. Suppose that $p_{ij}\geqq 2$ and that the (i, j) component of $A(0, z, 0)^k$ does not vanish identically for some k such that $0\leqq k\leqq p_{ij}-2$. If $k=0$, we have $i=j$ and hence $p_{ij}=1$. But this contradicts $p_{ij}\geqq 2$. If $k=1$, we have $n_i-n_j+1\geqq 0$. Hence we obtain $p_{ij}\leqq 2$, because $n_j-n_i+1=2-(n_i-n_j+1)\leqq 2$. But this contradicts $k(=1)\leqq p_{ij}-2$. If $k\geqq 2$, there exist $j_1, \cdots, j_{k-1}\in\{1, 2, \cdots, m\}$ such that the $(j_{\nu-1}, j_\nu)$ component of $A(0, z, 0)$ does not vanish identically for $1\leqq\nu\leqq k$, where $j_0=i$ and $j_k=j$. This implies $n_{j_{\nu-1}}-n_{j_\nu}+1\geqq 0$ for $1\leqq\nu\leqq k$. Hence we obtain $p_{ij}\leqq k+1$, because $n_j-n_i+1=(k+1)-(n_{j_0}-n_{j_1}+1)-\cdots-(n_{j_{k-1}}-n_{j_k}+1)\leqq k+1$. But this contradicts $k\leqq p_{ij}-2$. In any case, denying (ii) leads us to a contradiction. Thus, (ii) is also verified. From (i) and (ii), it follows that the (i, j) component $e_{ij}(z, t)$ of $e^{-A(0,z,0)t}$ satisfies the following conditions: (iii) $\sup\{|e_{ij}(z, t)|; z\in\Omega_{s_0}\}=O(e^{-(3c/4)t})$ as $t\to+\infty$, and (iv) $\sup\{|e_{ij}(z, t)|; z\in\Omega_{s_0}\}=O(t^{p_{ij}-1})$ as $t\to+0$. Consequently, we obtain $\sup\{|e_{ij}(z, t)|; z\in\Omega_{s_0}\}\leqq M_2t^{p_{ij}-1}e^{-(c/2)t}$ for some $M_2>0$. Therefore, we have

$$\begin{aligned}\|K^{ij}[g](t)\| &= \left\|\int_0^\infty e_{ij}(z, s)g(\phi_\sigma(t, s), z)ds\right\| \\ &\leqq M_2\int_0^\infty s^{p_{ij}-1}e^{-(c/2)s}|||g|||(\phi_\sigma(t, s))ds \\ &\leqq M_2(p_{ij}-1)!\,\mathcal{H}^{p_{ij}}[|||g|||](t), \end{aligned}\tag{2.9}$$

where $\|\cdot\|=\|\cdot\|_s$ and $|||\cdot|||=|||\cdot|||_s$. Here we used (2.5). If we choose $C=\max\{M_2(p_{ij}-1)!; 1\leqq i, j\leqq m\}$, (2.9) immediately leads us to (2.8). Thus, (2.8) is proved for $l=0$. Now, we will show (2.8) for the general case by induction on l. Suppose that (2.8) is valid for $l=k$. Then, for any i, $j_1(=j(1))$, $j_2(=j(2))$ and j we have

$$\begin{aligned}\|K^{ij_1}\tilde{A}_{j_1j_2}R_k^{j_2j}[g](t)\|_s &\leqq C\mathcal{H}^{p_{ij(1)}}[|||\tilde{A}_{j_1j_2}R_k^{j_2j}[g]|||_s](t) \\ &\leqq C\delta(t)\mathcal{H}^{p_{ij(1)}}[|||R_k^{j_2j}[g]|||_s](t) \\ &\leqq C\delta(t)CM^k\delta(t)^k\mathcal{H}^{p_{ij(1)}+p_{j(2)j}}[|||g|||_s](t). \end{aligned}\tag{2.10}$$

Here we remark the following facts: (v) if $n_{j_1}-n_{j_2}+1<0$, we have $\tilde{A}_{j_1j_2}=0$, and (vi) if $n_{j_1}-n_{j_2}+1\geqq 0$, we have $p_{ij_1}+p_{j_2j}\geqq p_{ij}$ because $(n_{j_1}-n_i+1)+(n_j-n_{j_2}+1)=(n_j-n_i+1)+(n_{j_1}-n_{j_2}+1)\geqq(n_j-n_i+1)$. Therefore, applying (2.10), (v) and (vi) to the summation $R_{k+1}^{ij}=\sum_{j_1,j_2=1}^{m}K^{ij_1}\tilde{A}_{j_1j_2}R_k^{j_2j}$ we have

$$\|R^{ij}_{k+1}[g](t)\|_s \leqq CM_{ij}M^k\delta(t)^{k+1}\mathcal{H}^{p_{ij}}[\|\|g\|\|_s](t), \tag{2.11}$$

$$M_{ij}=\sum_{\substack{1\leqq j_1, j_2\leqq m\\ n_{j_1}-n_{j_2}+1\geqq 0}} C\left(\frac{2}{c}\right)^{p_{ij(1)}+p_{j(2)j}-p_{ij}}.$$

If we choose $M=\max\{M_{ij};\ 1\leqq i,\ j\leqq m\}$, it follows from (2.11) that (2.8) is also valid for $l=k+1$. Thus, (2.8) is obtained for any $l\geqq 0$. Q. E. D.

PROOF OF COROLLARY. Put $h_L(t)=t^L$. Then we have $\|\|g\|\|_s(t)\leqq h_L(t)$. Therefore, by Lemma 2 we obtain

$$\|R^{ij}[g](t)\|_s\leqq C\mathcal{H}^{p_{ij}}[h_L](t)$$
$$\leqq C\left(\frac{2}{c}\right)^{p_{ij}-l}\mathcal{H}^l[h_L](t). \tag{2.12}$$

On the other hand, we have the following estimate

$$\mathcal{H}[h_L](t)\leqq\int_0^\infty(\phi_\sigma(t,\ s))^L ds=\frac{t^{L-\sigma+1}}{(L-\sigma+1)}. \tag{2.13}$$

Hence, applying (2.13) to (2.12) l-times we can easily obtain the estimate (2.7). Q. E. D.

§ 3. Proof of Theorem 1.

We shall solve (S) by the method of successive approximations. Assume that ε (>0) is sufficiently small. Then, for given $f(t)\in C^0([0,\ T],\ A_{s_0}(\Omega))$ we can define $u^{(p)}\in C^0([0,\ \varepsilon],\ A_s(\Omega))\cap C^1((0,\ \varepsilon],\ A_s(\Omega))$ $(0<s<s_0)$ by the solution of

$$t^\sigma D_t u^{(p)}+A(t,\ z,\ 0)u^{(p)}=(A(t,\ z,\ 0)-A(t,\ z,\ t^\rho D_z))u^{(p-1)}+f$$

inductively on $p\geqq 0$, where $u^{(-1)}=0$. In other words, $u^{(p)}$ $(p\geqq 0)$ are defined by

$$u^{(0)}=R[f] \quad\text{and}\quad u^{(p)}=R[Bu^{(p-1)}+f] \qquad \text{for } p\geqq 1, \tag{3.1}$$

where R is the resolvent matrix in § 2 and B is the differential operator defined by $Bu(t,\ z)=(A(t,\ z,\ 0)-A(t,\ z,\ t^\rho D_z))u(t,\ z)$. Put $v^{(p)}=u^{(p)}-u^{(p-1)}$ for $p\geqq 0$. Then, to obtain the existence of solutions of (S) we have only to show that the series $\sum_p v^{(p)}$ is convergent in $C^0([0,\ \varepsilon],\ A_s(\Omega))$. Further, it will suffice to show that the series $\sum_p v^{(dp+q)}$ is convergent in $C^0([0,\ \varepsilon],\ A_s(\Omega))$ for $d=\max\{n_i-n_j+1;\ 1\leqq i,\ j\leqq m\}$ and $q=d^2+\max\{n_i;\ 1\leqq i\leqq m\}$, because $\sum_{p\geqq q} v^{(p)}$ is expressed formally as follows:

$$\sum_{p\geqq q} v^{(p)}=(1+(RB)+\cdots+(RB)^{d-1})[\sum_p v^{(dp+q)}].$$

Therefore, from now on we will discuss only the convergence of $\sum_p v^{(dp+q)}$.

Recall that $(A(t, z, 0)-A(t, z, t^{\rho}D_z))$ has the form

$$A(t, z, 0)-A(t, z, t^{\rho}D_z)=\sum_{1\leq l\leq d}\sum_{|\alpha|=l}A_{\alpha}(t, z)(t^{\rho}D_z)^{\alpha} \tag{3.2}$$

for some $A_{\alpha}(t, z)$. For simplicity, we denote by $B^{(l)}=(B_{ij}^{(l)})_{1\leq i,j\leq m}$ the matrix of differential operators of order l defined by $B^{(l)}u(t, z)=\sum_{|\alpha|=l}A_{\alpha}(t, z)(t^{\rho}D_z)^{\alpha}u(t, z)$. Since $B=\sum_{l=1}^{d}B^{(l)}$, by (3.1) we can express $v^{(dp+q)}$ by

$$v^{(dp+q)}=\sum_{1\leq l_1,\cdots,l_d\leq d}RB^{(l_1)}\cdots RB^{(l_d)}[v^{(d(p-1)+q)}]. \tag{3.3}$$

Now, we will estimate $\|v^{(dp+q)}(t)\|_s$ by induction on p. Let $v_i^{(dp+q)}$ be the i-th component of $v^{(dp+q)}$ and introduce the notation $\mu(l, i)=d^2\rho+l(\rho-\sigma+1)+n_i(\sigma-1)$. Then we have

LEMMA 3. *Let s_1 be a positive number $(0<s_1<s_0)$. Then, there is a positive constant C such that the estimate*

$$\|v_i^{(q)}(t)\|_s\leqq Ct^{\mu(0,i)} \tag{3.4}$$

holds for any i, t $(0\leqq t\leqq\varepsilon)$ and s $(0<s<s_1)$.

PROOF. For any $i, j, j_1, \cdots, j_q, k_1, \cdots, k_q$ and $l_1, \cdots, l_q$,

$$\|R^{ij_1}B_{j_1k_1}^{(l_1)}\cdots R^{k_{q-1}j_q}B_{j_qk_q}^{(l_q)}R^{k_qj}[f_j](t)\|_s\leqq C_1t^{(l_1+\cdots+l_q)\rho} \tag{3.5}$$

holds for some $C_1>0$, where f_j is the j-th component of f. Since $(l_1+\cdots+l_q)\rho\geqq q\rho\geqq d^2\rho+n_i\rho>d^2\rho+n_i(\sigma-1)=\mu(0, i)$, (3.5) immediately leads us to (3.4).

Q. E. D.

In general, we have the following lemma.

LEMMA 4. *Let $L\geqq 0$ and assume that $w(t)={}^t(w_1(t), \cdots, w_m(t))\in C^0([0, \varepsilon], A_s(\Omega))$ satisfies the estimate*

$$\|w_i(t)\|_s\leqq\frac{t^{\mu(L,i)}}{(s_1-s)^L}(L+1)^{n_i} \tag{3.6}$$

for any i, t $(0\leqq t\leqq\varepsilon)$ and s $(0<s<s_1)$. Then, there is a positive constant M independent of L and $w(t)$ such that the estimate

$$\begin{aligned}&\|R^{ij_1}B_{j_1k_1}^{(l_1)}\cdots R^{k_{d-1}j_d}B_{j_dk}^{(l_d)}[w_k](t)\|_s\\&\leqq M^{l_1+\cdots+l_d}\frac{t^{\mu(L+l_1+\cdots+l_d,i)}}{(s_1-s)^{L+l_1+\cdots+l_d}}(L+l_1+\cdots+l_d+1)^{n_i}\end{aligned} \tag{3.7}$$

holds for any $i, j_1, \cdots, j_d, k_1, \cdots, k_d(=k), l_1, \cdots, l_d, t$ $(0\leqq t\leqq\varepsilon)$ and s $(0<s<s_1)$.

PROOF. Note that $B_{jk}^{(l)}=0$, if $n_j-n_k+1<l$. Therefore, in the proof given below we may assume that $n_{j_\nu}-n_{k_\nu}+1\geqq l_\nu$ for $1\leqq\nu\leqq d$. By Cauchy's inequality and (3.6), we have

$$\|B_{j_d k}^{(l_d)}[w_k](t)\|_s \leqq M_1 \frac{t^{l_d\rho}}{\eta^{l_d}} \|w_k(t)\|_{s+\eta}$$

$$\leqq M_1 \frac{t^{\mu(L,k)+l_d\rho}}{\eta^{l_d}(s_1-s-\eta)^L} (L+1)^{n_k}$$

for any η $(0<\eta<s_1-s)$, where M_1 is a positive constant which depends only on the coefficients of $B_{j_d k}^{(l_d)}$. If we choose $\eta=(s_1-s)/(L+1)$, it follows that the estimate

$$\|B_{j_d k}^{(l_d)}[w_k](t)\|_s \leqq M_1 e \frac{t^{\mu(L,k)+l_d\rho}}{(s_1-s)^{L+l_d}} (L+1)^{l_d}(L+1)^{n_k} \tag{3.8}$$

holds. Therefore, applying (2.7) to (3.8) we obtain

$$\|R^{k_{d-1}j_d} B_{j_d k}^{(l_d)}[w_k](t)\|_s$$

$$\leqq C_1 M_1 e \frac{t^{\mu(L,k)+l_d\rho-a_d(\sigma-1)}}{(s_1-s)^{L+l_d}} \frac{(L+1)^{l_d}(L+1)^{n_k}}{(L(\rho-\sigma+1)+l_d\rho)^{a_d}}$$

for any a_d such that $0\leqq a_d \leqq p_{k_{d-1}j_d}$, where C_1 is the constant in (2.7). Hence, applying the same argument d-times we can obtain the estimate

$$\|R^{ij_1} B_{j_1 k_1}^{(l_1)} \cdots R^{k_{d-1}j_d} B_{j_d k}^{(l_d)}[w_k](t)\|_s$$

$$\leqq (C_1 M_1 e)^d \frac{t^{\mu(L,k)+(l_1+\cdots+l_d)\rho-(a_1+\cdots+a_d)(\sigma-1)}}{(s_1-s)^{L+l_1+\cdots+l_d}}$$

$$\times \frac{(L+l_2+\cdots+l_d+1)^{l_1}\cdots(L+1)^{l_d}(L+1)^{n_k}}{(L(\rho-\sigma+1)+(l_1+\cdots+l_d)\rho)^{a_1}\cdots(L(\rho-\sigma+1)+l_d\rho)^{a_d}} \tag{3.9}$$

for any $a_1, \cdots, a_d$ such that $0\leqq a_\nu \leqq p_{k_{\nu-1}j_\nu}$ for $1\leqq\nu\leqq d$, where $k_0=i$. If we choose $a_1, \cdots, a_d$ such that $a_1+\cdots+a_d=l_1+\cdots+l_d+n_k-n_i$, then (3.9) immediately leads us to (3.7) because $\mu(L,k)+(l_1+\cdots+l_d)\rho-(a_1+\cdots+a_d)(\sigma-1) =\mu(L+l_1+\cdots+l_d, i)$. Therefore, to complete the proof it is sufficient to show that such a choice is really possible. Put $\varphi(x_1, \cdots, x_d)=x_1+\cdots+x_d$ and $b_\nu=\max(l_\nu+n_{k_\nu}-n_{k_{\nu-1}}, 0)$ for $1\leqq\nu\leqq d$, where $k_0=i$ and $k_d=k$. Then, we have

$$\varphi(0, \cdots, 0)<l_1+\cdots+l_d+n_k-n_i \leqq \varphi(b_1, \cdots, b_d). \tag{3.10}$$

Since we have assumed that $n_{j_\nu}-n_{k_\nu}+1\geqq l_\nu$ for $1\leqq\nu\leqq d$, we also have $0\leqq b_\nu\leqq p_{k_{\nu-1}j_\nu}$ for $1\leqq\nu\leqq d$. Therefore, by (3.10) we can choose $a_1, \cdots, a_d$ such that $\varphi(a_1, \cdots, a_d)=l_1+\cdots+l_d+n_k-n_i$ and $0\leqq a_\nu\leqq b_\nu$ for $1\leqq\nu\leqq d$. Thus, the possibility of the choice of $a_1, \cdots, a_d$ is guaranteed. Q. E. D.

Applying Lemma 4 to (3.3) p-times, we can obtain

$$\|v_i^{(dp+q)}(t)\|_s \leqq \sum_{\substack{l=l_1+\cdots+l_{dp} \\ 1\leqq l_1,\cdots,l_{dp}\leqq d}} CM^l \frac{t^{\mu(l,i)}}{(s_1-s)^l}(l+1)^{n_i}$$

$$\leqq \sum_{dp\leqq l\leqq d^2p} Ct^{d^2\rho+n_i(\sigma-1)}\left(\frac{2Mt^{\rho-\sigma+1}}{(s_1-s)}\right)^l (l+1)^{n_i} \tag{3.11}$$

for any i and p. If we choose ε (>0) such that

$$\varepsilon<\left(\frac{s_1-s}{2M}\right)^{1/(\rho-\sigma+1)},$$

it follows from (3.11) that the series $\sum_p v^{(dp+q)}$ is convergent in $C^0([0, \varepsilon], A_s(\Omega))$. Hence, we may conclude that the series $\sum_p v^{(p)}$ is also convergent in $C^0([0, \varepsilon], A_s(\Omega))$. This implies that the sequence $\{u^{(p)}\}$ converges to a function u in $C^0([0, \varepsilon], A_s(\Omega))$. Since u satisfies $u=R[Bu+f]$, it follows that u becomes a genuine solution of (S) in $C^0([0, \varepsilon], A_s(\Omega))\cap C^1((0, \varepsilon], A_s(\Omega))$. Thus, the existence part of Theorem 1 is obtained. The uniqueness can be proved in the same way. Therefore, we may omit the details.

§4. Proof of Theorem 2.

Recall that any distribution $u\in\mathcal{D}'(\boldsymbol{R}^n)$ with compact support can be regarded as an analytic functional $\tilde{u}$ on $\boldsymbol{C}^n$ by the following definition

$$\langle\tilde{u}, \theta\rangle=\langle u, \theta|_{\boldsymbol{R}^n}\rangle_{\mathcal{E}'\times C^\infty}, \quad \theta\in\mathcal{H}(\boldsymbol{C}^n),$$

where $\mathcal{H}(\boldsymbol{C}^n)$ is the space of all entire functions on $\boldsymbol{C}^n$. Therefore, Theorem 2 is obtained from the following proposition which is a Cauchy-Kowalewski type theorem for analytic functionals. Let Ω be a bounded open set in $\boldsymbol{R}^n$. We define $F_s(\Omega)$ by the closure of $\mathcal{H}(\boldsymbol{C}^n)$ in $A_s(\Omega)$ and $F_s'(\Omega)$ by the dual space of $F_s(\Omega)$ as a Banach space. Note that the system $\{F_s'(\Omega)\}$ becomes an increasing scale of Banach spaces. Therefore, by the same argument as in §§2 and 3 we can obtain

PROPOSITION. *For any s_1, s $(0<s_1<s<s_0)$, there is a positive number ε $(0<\varepsilon<T)$ which satisfies the following: for any $f(t)\in C^0([0, T], F_{s_1}'(\Omega))$ there exists a unique solution $u(t)\in C^0([0, \varepsilon], F_s'(\Omega))\cap C^1((0, \varepsilon], F_s'(\Omega))$ of* (S) *satisfying $t^\sigma u_t'(t)\in C^0([0, \varepsilon], F_s'(\Omega))$.*

Here we used the same notations as in Baouendi-Goulaouic [**1**], so that we can follow their argument directly. Hence, using the above proposition instead of Proposition 2 of [**1**], we can easily obtain Theorem 2 by the same argument as in the proof of Theorem 4 of [**1**]. Therefore, we may omit the details.

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