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On construction of Siegel modular forms of degree two

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Introduction. Let κ be an odd positive integer, N a positive integer divisible by 4, and χ a character modulo N. We denote by $\mathfrak{S}_\kappa(N, \chi)$ the space of modular cusp forms of Neben-type χ and of weight $\kappa/2$ with respect to $\Gamma_0(N)$ and denote by $T_{\kappa,\chi}^N(p^2)$ the Hecke operator defined on $\mathfrak{S}_\kappa(N, \chi)$. We denote by $S_k^{(2)}(L, \psi)$ the space of Siegel modular cusp forms of Neben-type ψ and of weight k with respect to $\Gamma_0^{(2)}(L)=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \boldsymbol{Z}) \;\middle|\; C\equiv 0 \pmod{L}\right\}$. Let $T_{k,\psi}^{(2),L}(n)$ denote the Hecke operator on $S_k^{(2)}(L, \psi)$.

In this paper we discuss two problems. The first problem is a construction of Siegel modular forms of degree two from modular cusp forms of half integral weight. The second one is a construction of modular cusp forms of half integral weight from Siegel modular forms of degree two.

In §1 we show the existence of a linear mapping $\Psi_k^{M,\chi}: \mathfrak{S}_{2k-1}(\tilde{M}, \chi)\rightarrow S_k^{(2)}(M, \bar{\chi})$ where M and k are even positive integers, $\tilde{M}=\text{l.c.m.}(4, M)$ and χ is a character modulo M. In §2, using the same method as in [3], we determine Fourier coefficients of $\Psi_k^{M,\chi}(f)$ at infinity. In §3 we study a relation between Andrianov's zeta function associated with $\Psi_k^{M,\chi}(f)$ and Shimura's one associated with f, where $f\in\mathfrak{S}_{2k-1}(\tilde{M}, \chi)$. In [3], we have treated the case $M=2$.

In §4 we give a linear mapping $I_k(L, \psi): \mathcal{M}_k^{(2)}(L, \psi)\rightarrow\mathfrak{S}_{2k-1}(\tilde{L}, \psi)$ which is a generalization of the mapping given in [4] and [5], where $4\nmid L$, $\tilde{L}=\text{l.c.m.}(4, L)$ and $\mathcal{M}_k^{(2)}(L, \psi)$ denotes the Maaß's space of $S_k^{(2)}(L, \psi)$.

In the last section we present an application of the results in §1, §2, §3 and §4. *With some assumption on M we show the existence of an isomorphic mapping $\tilde{\Psi}_k^{M,\chi}$ of $\tilde{\mathfrak{S}}_{2k-1}(\tilde{M}, \chi)$ onto $\tilde{\mathcal{M}}_k^{(2)}(M, \chi)$ with the following properties: if $f\in\tilde{\mathfrak{S}}_{2k-1}(\tilde{M}, \chi)$ satisfies $T_{2k-1,\chi}^{\tilde{M}}(p^2)f=\omega_p f$ for every prime p, then $\tilde{\Psi}_k^{M,\chi}(f)$ satisfies $T_{k,\chi}^{(2),M}(n)(\tilde{\Psi}_k^{M,\chi}(f))=\tilde{\lambda}(n)(\tilde{\Psi}_k^{M,\chi}(f))$ for every positive integer n and moreover,*

$$L(2s-2k+4, \chi^2)\sum_{n=1}^{\infty}\tilde{\lambda}(n)n^{-s}$$

$$=L(s-k+1, \chi)L(s-k+2, \chi)\prod_p(1-\omega_p p^{-s}+\chi(p)^2p^{2k-3-2s})^{-1},$$

where $L(s, \chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$ *and* $\tilde{\mathfrak{S}}_{2k-1}(\tilde{M}, \chi)$ *(resp.* $\tilde{\mathcal{M}}_k^{(2)}(M, \chi)$*) is the subspace of* $\mathfrak{S}_{2k-1}(\tilde{M}, \chi)$ *(resp.* $\mathcal{M}_k^{(2)}(M, \chi)$*) spanned by the eigenfunctions of the Hecke operators* $T_{2k-1,\chi}^{\tilde{M}}(p^2)$ *(resp.* $T_{k,\chi}^{(2),M}(n)$*) for all primes* p *(resp. positive integers* n*).*

§1. Notations and preliminaries.

We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For a commutative ring A with the unity 1, we denote by A_m^n the set of all $n\times m$ matrices with entries in A. Furthermore we denote by $SL_n(A)$ (resp. $GL_n(A)$) the group of all matrices M satisfying $\det(M)=1$ (resp. $\det(M)\in A^{\times}$), where $A^{\times}$ denotes the group of all invertible elements in A, and put $A^n=A_1^n$ and $M_n(A)=A_n^n$ for the simplicity. For every $z\in\boldsymbol{C}$, we set $e[z]=\exp(2\pi iz)$ with $i=\sqrt{-1}$ and define $\sqrt{z}=z^{1/2}$ so that $-\pi/2<\arg(z^{1/2})\leqq\pi/2$. Further, we set $z^{k/2}=(\sqrt{z})^k$ for every $k\in\boldsymbol{Z}$. For each positive integer N, set

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\boldsymbol{Z})\ \middle|\ c\equiv 0 \pmod N\right\}.$$

We consider an automorphic factor $j(\gamma, z)$ of $\Gamma_0(4)$ defined by $j(\gamma, z)=\theta(\gamma(z))/\theta(z)$ for every $\gamma\in\Gamma_0(4)$, $\theta(z)=\sum_{n=-\infty}^{\infty}e[n^2z]$, where z is the variable on the complex upper half-plane $\mathfrak{H}$.

We recall the definition of modular forms of half integral weight (cf. [**10**]). Let κ be an odd positive integer, N a positive integer divisible by 4, and ω a character modulo N. A function f on $\mathfrak{H}$ is called a modular form of Neben-type ω and of weight $\kappa/2$ with respect to $\Gamma_0(N)$ satisfying the following conditions (i) and (ii);

(i) $f(\gamma(z))=\omega(d)j(\gamma, z)^{\kappa}f(z)$ for every $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_0(N)$,

(ii) f is holomorphic on $\mathfrak{H}$ and at all cusps of $\Gamma_0(N)$.

The space of such functions is denoted by $\mathfrak{G}_{\kappa}(N, \omega)$. A modular form which vanishes at all cusps is called a cusp form. We denote by $\mathfrak{S}_{\kappa}(N, \omega)$ the space of cusp forms in $\mathfrak{G}_{\kappa}(N, \omega)$.

Next we give the definition of Siegel modular forms of degree n, i.e.,

$$Sp(n, \boldsymbol{R})=\{M\in M_{2n}(\boldsymbol{R})\,|\,{}^tMJ_nM=J_n\},\ \text{where } J_n=\begin{pmatrix} 0 & E_n \\ -E_n & 0\end{pmatrix},$$

E_n means the unity of $GL_n(\boldsymbol{R})$ and ${}^tM$ denotes the transpose of M. Let $\mathfrak{H}_n$ be the complex Siegel upper half-plane of degree n, i.e., $\mathfrak{H}_n=\{Z=X+iY\,|\,X, Y\in M_n(\boldsymbol{R}), {}^tZ=Z$ and $Y>0\}$. We set $\mathfrak{H}=\mathfrak{H}_1$. We define an action of $Sp(n, \boldsymbol{R})$ on $\mathfrak{H}_n$ by

$$Z \longmapsto M\langle Z\rangle=(AZ+B)(CZ+D)^{-1}\ \left(Z\in\mathfrak{H}_n \text{ and } M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp(n, \boldsymbol{R})\right),$$

where A, B, C and D belong to $M_n(\boldsymbol{R})$. We set $Sp(n, \boldsymbol{Z})=Sp(n, \boldsymbol{R})\cap M_{2n}(\boldsymbol{Z})$. For each positive integer L, set

$$\Gamma_0^{(n)}(L)=\left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp(n, \boldsymbol{Z})\ \middle|\ A, B, C \text{ and } D\in M_n(\boldsymbol{Z}) \text{ and } C\equiv 0 \pmod{L}\right\}$$
$$\text{and } \Gamma_0^{(1)}(L)=\Gamma_0(L).$$

Let ψ be a character modulo L and let k be a positive integer. We call a holomorphic function F on $\mathfrak{H}_n$ a Siegel modular cusp form of Neben-type ψ and of weight k with respect to $\Gamma_0^{(n)}(L)$, if the following conditions are satisfied:

(i) For every $\gamma=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_0^{(n)}(L)$ and for every $Z\in\mathfrak{H}_n$,

$F(\gamma\langle Z\rangle)=\bar{\psi}(\det(A))\det(CZ+D)^k F(Z)$,

(ii) $|F(Z)|(\det(\mathrm{Im}(Z)))^{k/2}$ is bounded on $\mathfrak{H}_n$.

We denote by $S_k^{(n)}(L, \psi)$ the space of such modular forms.

Let Q be a non-degenerate symmetric $n\times n$ matrix. We denote by $O(Q)$ (resp. $O(Q)_0$) the real orthogonal group (resp. the connected component of the unity of $O(Q)$) for Q, i.e., $O(Q)=\{g\in GL_n(\boldsymbol{R})\,|\,{}^t gQg=Q\}$.

Let $\mathcal{S}(\boldsymbol{R}^n)$ denote the space of all rapidly decreasing functions on $\boldsymbol{R}^n$. For each $f\in\mathcal{S}(\boldsymbol{R}^n)$, we define a function $\gamma(\sigma, Q)f$ on $\boldsymbol{R}^n$ by

$$(\gamma(\sigma, Q)f)(x)=\begin{cases} |c|^{-n/2}|\det(Q)|^{1/2}\int_{\boldsymbol{R}^n} e[(a\langle x, x\rangle-2\langle x, y\rangle+d\langle y, y\rangle)/2c]f(y)dy, \\ \hfill \text{if } c\neq 0, \\ |a|^{n/2}e[ab\langle x, x\rangle/2]f(ax) \qquad \text{if } c=0 \end{cases}$$

for every

$$\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\boldsymbol{R}),$$

where $\langle x, y\rangle={}^t xQy$. We can easily check $\gamma(\sigma, Q)f\in\mathcal{S}(\boldsymbol{R}^n)$ and

$$\int_{\boldsymbol{R}^n}|\gamma(\sigma, Q)f(x)|^2dx=\int_{\boldsymbol{R}^n}|f(x)|^2dx.$$

So the operator $\gamma(\sigma, Q)$ can be extended to a unitary operator on $L^2(\boldsymbol{R}^n)$, which is denoted by $\gamma(\sigma, Q)$ again. We call $\gamma(\sigma, Q)$ the Weil representation associated with Q.

Take a lattice L in $\boldsymbol{R}^n$ and set $L^*=\{x\in\boldsymbol{R}^n\,|\,\langle x, y\rangle\in\boldsymbol{Z} \text{ for every } y\in L\}$. We assume that $L^*\supset L$.

For every $f\in\mathcal{S}(\boldsymbol{R}^n)$, define a series $\theta(f: h)$ $(h\in L^*/L)$ by $\theta(f: h)=\sum_{x\in L} f(h+x)$.

The following theorem was proved by Shintani [11].

THEOREM A. *Suppose that* $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z})$. *Then*

$$\theta(\gamma(\sigma, Q)f: h)=\sum_k c(h, k)_\sigma \theta(f: k),$$

where k *runs over* L^*/L *and*

$$c(h, k)_\sigma=\begin{cases} \delta_{h,ak}e[ab\langle h, h\rangle/2] & \text{if } c=0, \\ \sqrt{|\det(Q)|}^{-1}\left(\int_{\boldsymbol{R}^n/L}dx\right)^{-1}|c|^{-n/2} & \\ \quad\cdot\sum_{\gamma\in L/cL} e[(a\langle h+\gamma, h+\gamma\rangle-2\langle k, h+\gamma\rangle+d\langle k, k\rangle)/2] & \text{otherwise.}\end{cases}$$

Moreover assume that c *is even,* $cL^*\subset L$, $cd\neq 0$, $d>0$ *and* $c\langle x, x\rangle\equiv 0 \pmod 2$ *for every* $x\in L^*$. *Then*

$$\sqrt{i}^{-(p-q)\mathrm{sgn}(cd)}c(h, k)_\sigma=\delta_{h,dk}e[ab\langle h, h\rangle/2]\left(\frac{-1}{d}\right)^{n/2}\left(\frac{-2c}{d}\right)^n\left(\frac{D}{d}\right),$$

where p *(resp.* q*) is the number of positive (resp. negative) eigenvalues of* Q, $D=\det(\langle\lambda_i, \lambda_j\rangle_{1\leq i,j\leq n})$ *and* $\{\lambda_1, \cdots, \lambda_n\}$ *is a* $\boldsymbol{Z}$*-base of* L.

The group $GL_n(\boldsymbol{R})$ operates on $L^2(\boldsymbol{R}^n)$ in the following manner: $Tf(x)=|\det(T)|^{-1/2}f(T^{-1}x)$ for every $T\in GL_n(\boldsymbol{R})$ and for every $f\in L^2(\boldsymbol{R}^n)$.

Now we consider

$$Q_0=\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad\text{and}\quad Q_1=\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For a positive integer K, set $L(K)=\{{}^t(x_1, x_2, Kx_3, x_4, \sqrt{2}x_5)\,|\,x_i\in\boldsymbol{Z}\}$ and $L_1=\{{}^t(x_1, x_2, \sqrt{2}x_3)\,|\,x_i\in\boldsymbol{Z}\}$. Let M be an even positive integer and let χ be a character modulo M with $\chi(-1)=1$. We consider an isomorphism $\tau: L(1)/L(M)\to\boldsymbol{Z}/M\boldsymbol{Z}$ given by $\tau(x+L(M))=x_3+M\boldsymbol{Z}$, where $x={}^t(\cdots, x_3, \cdots)\in L(1)$. By virtue of Lemma 1.1 in [3], we have

$$\tau(\rho(g)^{-1}x)=\det(A)\tau(x) \tag{1.1}$$

for every $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_0^{(2)}(M)$ and $x\in L(1)/L(M)$, where ρ is the isomorphism of $Sp(2, \boldsymbol{R})/\{\pm E_4\}$ onto $O(Q_0)_0$ given in [3, §1]. Set

$$f_k(x)=\langle x, {}^t(-i, i, 1, -1, 0)\rangle^k\exp\left(-\pi\sum_{i=1}^5 x_i^2\right)$$

for all $x={}^t(x_1, \cdots, x_5)\in\boldsymbol{R}^5$. Then we have

$$\rho(\kappa)f_k=(\det(A-Bi))^k f_k \tag{1.2}$$

for all $\kappa=\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\in K_2$, where $K_2=\{M\in Sp(2, \boldsymbol{R})\,|\,M\langle iE_2\rangle=iE_2\}$.

We consider a theta series $\theta_k^{M,\chi}(z, g)$ on $\mathfrak{H}\times Sp(2, \boldsymbol{R})$ given by $\theta_k^{M,\chi}(z, g)=v^{-(2k-1)/4}\sum_l \chi(\tau(l))\sum_h \rho(g)\gamma(\sigma_z, Q_0)f_k(l+h)$ for every $z=u+iv\in\mathfrak{H}$ and for every $g\in Sp(2, \boldsymbol{R})$, where l (resp. h) runs over $L(1)/L(M)$ (resp. $L(M)$) and

$$\sigma_z=\begin{pmatrix} \sqrt{v} & u\sqrt{v}^{-1} \\ 0 & \sqrt{v}^{-1} \end{pmatrix}.$$

Then, by (1.1), (1.2) and Theorem A, we get the following lemma.

LEMMA 1.

(i) $\theta_k^{M,\chi}(\sigma(z), g)=\chi(d)j(\sigma, z)^{2k-1}\theta_k^{M,\chi}(z, g)$ *for every* $\sigma=\begin{pmatrix} * & * \\ * & d \end{pmatrix}\in\Gamma_0(\tilde{M})$.

(ii) $\theta_k^{M,\chi}(z, \gamma g)=\bar{\chi}(\det(A))\theta_k^{M,\chi}(z, g)$ *for every* $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_0^{(2)}(M)$.

(iii) $\theta_k^{M,\chi}(z, g\kappa)=(\det(A-Bi))^k\theta_k^{M,\chi}(z, g)$ *for every* $\kappa=\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\in K_2$, *where* $K_2=\{g\in Sp(2, \boldsymbol{R})\,|\,g(\langle iE_2\rangle)=iE_2\}$.

For a function $f\in\mathfrak{S}_{2k-1}(\tilde{M}, \chi)$, we define a function $\Psi_k^{M,\chi}(f)$ on $\mathfrak{H}_2$ by

$$\Psi_k^{M,\chi}(f)(Z)=J(g, iE_2)^k\int_{D_0(\tilde{M})} v^{(2k-1)/2}f(z)\bar{\theta}_k^{M,\chi}(z, g)v^{-2}du\,dv$$

with $Z=g\langle iE_2\rangle$, where $D_0(\tilde{M})$ is a fundamental domain for $\Gamma_0(\tilde{M})$ and $J(g, iE_2)=\det(Ci+D)$ with $g=\begin{pmatrix} * & * \\ C & D \end{pmatrix}\in Sp(2, \boldsymbol{R})$. By virtue of (i) and (iii) in Lemma 1, the above function is well-defined. The property (ii) of Lemma 1 shows $\Psi_k^{M,\chi}(f)(\gamma\langle Z\rangle)=\chi(\det(A))\det(CZ+D)^k\Psi_k^{M,\chi}(f)(Z)$ for every $\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_0^{(2)}(M)$. We can observe that $\Psi_k^{M,\chi}(f)$ is holomorphic on $\mathfrak{H}_2$ (see [8] and [**9**]). Therefore we see that $\Psi_k^{M,\chi}(f)$ is a Siegel modular form of Neben-type $\bar{\chi}$ and of weight k with respect to $\Gamma_0^{(2)}(M)$.

§ 2. Explicit calculation of the Fourier coefficients of $\Psi_k^{M,\chi}(f)$.

After preparing some theta series, we shall show that for an element

$$g=\begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix}\in Sp(2, \boldsymbol{R}),$$

$\theta_k^{M,\chi}(z, g)$ can be split into a convenient form. Set $Y=yY_1$, where

$$Y_1=\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix},$$

$\det(Y_1)=1$, $y>0$ and $y_1>0$. For a non-negative integer ε, we consider three theta series defined by

$$\theta_{1,\varepsilon}(z: Y_1)=v^{(-\varepsilon+2)/2}\sum_l H_\varepsilon(\sqrt{2\pi v}(y_1, -y_3, -\sqrt{2}\,y_2)l)e[(u^tlQ_1l+iv^tlR(Y_1)l)/2],$$

$$\theta^*_{1,\varepsilon}(z: Y_1)=v^{(-\varepsilon+2)/2}\sum_{l'} H_\varepsilon(\sqrt{2\pi v}(y_1, -y_3, -\sqrt{2}\,y_2)l')e[(u^tl'Q_1l'+iv^tl'R(Y_1)l')/2]$$

and

$$\theta_{2,\varepsilon}^{M,\chi}(z: y)=v^{(1-\varepsilon)/2}\sum_{m,n=-\infty}^{\infty}\chi(m)\exp(-2\pi imnu-\pi v(y^2m^2+y^{-2}n^2))H_\varepsilon(\sqrt{2\pi v}(my-ny^{-1}))$$

for each $z=u+iv\in\mathfrak{H}$, where $H_\varepsilon(x)=(-1)^\varepsilon\exp(x^2/2)\dfrac{d^\varepsilon}{dx^\varepsilon}(\exp(-x^2/2))$,

$$L_1^*=\{y\in\boldsymbol{R}^3\,|\,{}^tyQ_1x\in\boldsymbol{Z} \text{ for all } x\in L_1\},$$

$$R(Y_1)=\begin{pmatrix} y_1^2 & -y_2^2 & -\sqrt{2}\,y_1y_2 \\ -y_2^2 & y_3^2 & \sqrt{2}\,y_2y_3 \\ -\sqrt{2}\,y_1y_2 & \sqrt{2}\,y_2y_3 & 1+2y_2^2 \end{pmatrix}$$

and the summation $\sum_l$ (resp. $\sum_{l'}$) is taken over all the elements in L_1 (resp. $2L_1^*$). Set $\Gamma_\infty=\left\{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\middle|\, n\in\boldsymbol{Z}\right\}$. Using the Poisson summation formula, we get

$$\text{(2.1)}\qquad \theta_{2,\varepsilon}^{M,\chi}(-1/\tilde{M}z: y)$$
$$=2(\sqrt{2\pi}\,iyv^{-1}z)^\varepsilon y\sum_{n=1}^{\infty}\chi(n)n^\varepsilon\sum_\gamma\chi(d)\overline{J(\gamma, z)}^\varepsilon k^{\tilde{M}}(\gamma(z); n, y),$$

where $\gamma=\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ runs over $\Gamma_\infty\backslash\Gamma_0(\tilde{M})$, $J(\gamma, z)=(cz+d)$ and $k^{\tilde{M}}(z; n, y)=\exp(-\pi n^2y^2/\tilde{M}v)$ (see [7, p. 152]). The following lemma was proved in [3, Lemma 3.1].

LEMMA 2.1. *Suppose that* σ *belongs to* $\Gamma_0(4)$. *Then*

$$\theta_{1,\varepsilon}(\sigma(z): Y_1)=j(\sigma, z)^{2\varepsilon-1}\theta_{1,\varepsilon}(z: Y_1)$$

and

$$(4z)^{-(2\varepsilon-1)/2}\theta_{1,\varepsilon}(-1/4z: Y_1)=\sqrt{i/2}\,2^{2-\varepsilon}\theta^*_{1,\varepsilon}(z: Y_1).$$

For every $f\in\mathfrak{S}_{2k-1}(\tilde{M}, \chi)$, we define a function $f|[W_{\tilde{M}}]_{2k-1}$ of $\mathfrak{S}_{2k-1}\left(\tilde{M}, \bar{\chi}\left(\dfrac{\tilde{M}}{*}\right)\right)$ by $f|[W_{\tilde{M}}]_{2k-1}(z)=f(-1/\tilde{M}z)(\tilde{M}^{1/4}(-iz)^{1/2})^{-(2k-1)}$. Then $f|[W_{\tilde{M}}]_{2k-1}$ has the

Fourier expansion $f|[W_{\tilde{M}}]_{2k-1}(z)=\sum_{n=1}^{\infty} a^{(0)}(n)e[nz]$ at ∞. Set

$$P_2'=\left\{T=\begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix}\middle| T\geqq 0 \text{ and } (n_1, n_2, n_3)\in \boldsymbol{Z}_3^1\right\}$$

and $P_2=\{T\in P_2' \mid T>0\}$. For every $T=\begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix}\in P_2$, set $e(T)=\text{l. c. m.}\,(n_1, n_2, n_3)$ and $N(T)=4\det(T)$. Now we define $c_f(T)$ by $c_f(T)=\sum_m \bar{\chi}(m)m^{k-1}a^{(0)}(\tilde{M}N(T)/4m^2)$, where the summation $\sum_m$ is taken over all positive integers m with $m|e(T)$. We prove the following theorem.

THEOREM 1. *Under the above notations, let f be an element of $\mathfrak{S}_{2k-1}(\tilde{M}, \chi)$. Suppose k (>5) is even. Then the Fourier expansion of $\Psi_k^{M,\chi}(f)$ at infinity has the form $\Psi_k^{M,\chi}(f)(Z)=c\sum_T c_f(T)e[\mathrm{tr}(TZ)]$, where c $(\neq 0)$ is a constant not depending upon f and T runs over all $T\in P_2$.*

PROOF. Since $\Psi_k^{M,\chi}(f)$ is a Siegel modular form with respect to $\Gamma_0^{(2)}(M)$, we have the following Fourier expansion $\Psi_k^{M,\chi}(f)(Z)=\sum_T c(T)e[\mathrm{tr}(TZ)]$ at infinity, where the sum $\sum_T$ is taken over all $T\in P_2'$. Set $Z=iY$ with $Y=yY_1$, $y>0$, $\det(Y_1)=1$ and $Y_1>0$. Then

$$(*)\qquad \Psi_k^{M,\chi}(f)(iY)=\sum_{T>0} c(T)\exp(-2\pi\,\mathrm{tr}(TY))\,.$$

Set

$$g=\begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix}.$$

Then, by the same method as in [3, § 3], we have

$$\theta_k^{M,\chi}(z, g)=\sqrt{2\pi}^{-k}\sum_{\varepsilon=0}^{k} {}_kC_\varepsilon(-i)^\varepsilon\theta_{1,\varepsilon}(z: Y_1)\theta_{2,k-\varepsilon}^{M,\chi}(z: y)\,.$$

This shows that

$$\begin{aligned}
&\Psi_k^{M,\chi}(f)(iY)\\
&=(\sqrt{2\pi}y)^{-k}\sum_{\varepsilon=0}^{k} {}_kC_\varepsilon i^\varepsilon\int_{D_0(\tilde{M})} v^{(2k-1)/2}f(z)\bar{\theta}_{1,\varepsilon}(z: Y_1)\bar{\theta}_{2,k-\varepsilon}^{M,\chi}(z: y)v^{-2}dudv\\
&=(\sqrt{2\pi}y)^{-k}\sum_{\varepsilon=0}^{k} {}_kC_\varepsilon i^\varepsilon\int_{D_0(\tilde{M})} (v/\tilde{M}|z|^2)^{(2k-1)/2}f(-1/\tilde{M}z)\bar{\theta}_{1,\varepsilon}(-1/\tilde{M}z: Y_1)\\
&\qquad\cdot\bar{\theta}_{2,k-\varepsilon}^{M,\chi}(-1/\tilde{M}z: y)v^{-2}dudv\,.
\end{aligned}$$

By Lemma 2.1 and (2.1), we have

$$\begin{aligned}
&\Psi_k^{M,\chi}(f)(iY)\\
&=c'\sum_{\varepsilon=0}^{k} {}_kC_\varepsilon(\tilde{M}/2\sqrt{2\pi})^\varepsilon y^{1-\varepsilon}\int_{D_0(\tilde{M})} v^{\varepsilon-1/2}\sum_{m=1}^{\infty}\bar{\chi}(m)m^{k-\varepsilon}\sum_\gamma \bar{\chi}(d)J(\gamma, z)^{k-\varepsilon}k^{\tilde{M}}(\gamma(z), m, y)\\
&\qquad\cdot\bar{\theta}_{1,\varepsilon}^{*}((\tilde{M}/4)z: Y_1)f|[W_{\tilde{M}}]_{2k-1}(z)v^{-2}dudv\,,
\end{aligned}$$

where $\sum_{\gamma}$ is the sum taken over all $\gamma\in\Gamma_{\infty}\backslash\Gamma_0(\tilde{M})$. By virtue of Lemma 2.1 and [10, Proposition 1.3 and 1.4], we see that $f|[W_{\tilde{M}}]_{2k-1}$ belongs to $\mathfrak{S}_{2k-1}\left(\tilde{M}, \bar{\chi}\left(\frac{\tilde{M}}{*}\right)\right)$ and $\theta^*_{1,\varepsilon}((\tilde{M}/4)\gamma(z):\ Y_1)=\left(\frac{\tilde{M}}{d}\right)j(\gamma, z)^{2\varepsilon-1}\theta^*_{1,\varepsilon}((\tilde{M}/4)z:\ Y_1)$ for every $\gamma=\begin{pmatrix} * & * \\ * & d\end{pmatrix}$ $\in\Gamma_0(\tilde{M})$. It follows from those that

$$\begin{aligned}
&\Psi_k^{M,\chi}(f)(iY)\\
&=c'\sum_{\varepsilon=0}^{k}{}_kC_\varepsilon(\tilde{M}/2\sqrt{2\pi})^\varepsilon y^{1-\varepsilon}\int_0^\infty\int_0^1\sum_{m=1}^{\infty}\bar{\chi}(m)m^{k-\varepsilon}v^{\varepsilon-1/2}k^{\tilde{M}}(z, m, y)\\
&\qquad\cdot\bar{\theta}^*_{1,\varepsilon}((\tilde{M}/4)z:Y_1)f|[W_{\tilde{M}}]_{2k-1}(z)v^{-2}dudv\\
&=c'\sum_{\varepsilon=0}^{k}{}_kC_\varepsilon(\tilde{M}/2\sqrt{2\pi})^\varepsilon y^{1-\varepsilon}(\tilde{M}/4)^{(2-\varepsilon)/2}\int_0^\infty v^{(\varepsilon-3)/2}\sum_{m=1}^{\infty}\bar{\chi}(m)m^{k-\varepsilon}\\
&\qquad\cdot\sum_{l'}a^{(0)}(\tilde{M}\langle l', l'\rangle_1/8)H_\varepsilon(\sqrt{2\pi(\tilde{M}/4)v}(y_1, -y_3, -\sqrt{2}y_2)l')\\
&\qquad\cdot\exp\{(-\pi m^2y^2/\tilde{M}v)-(\tilde{M}\pi v/4)({}^tl'R(Y_1)l'+\langle l', l'\rangle_1)\}\,dv\,,
\end{aligned}$$

where $\langle l', l'\rangle_1={}^tl'Q_1l'$. We have ${}^tl'R(Y_1)l'+\langle l', l'\rangle_1=\{(y_1, -y_3, -\sqrt{2}y_2)l'\}^2$. Now we can check the following formula:

$$\int_0^\infty v^{(\varepsilon-3)/2}\exp(-\alpha v-\beta v^{-1})H_\varepsilon(\sqrt{2\alpha v})dv=\beta^{(\varepsilon-1)/2}\sqrt{\pi}\,\sqrt{2^\varepsilon}\exp(-2\sqrt{\alpha\beta})$$

for each $\alpha, \beta>0$. Consequently we have

$$(**)\qquad \begin{aligned}\Psi_k^{M,\chi}(f)(iY)&=c''\sum_{m=1}^{\infty}\bar{\chi}(m)m^{k-1}\sum a^{(0)}(\tilde{M}N(T)/4m^2)\exp(-2\pi m|\mathrm{tr}(TY)|)\\
&=c''\sum_{T}c_f(T)\exp(-2\pi\ \mathrm{tr}(TY))\,,\end{aligned}$$

where T runs over all $T\in P_2$. Put $t_i=\exp(-2\pi y_i)$ for $Y=\begin{pmatrix} y_1 & y_2\\ y_2 & y_3\end{pmatrix}$. By the equalities (*) and (**), we have

$$(***)\qquad \sum c\left(\begin{pmatrix} n_1 & n_2/2\\ n_2/2 & n_3\end{pmatrix}\right)t_1^{n_1}t_2^{n_2}t_3^{n_3}=c''\sum c_f\left(\begin{pmatrix} n_1' & n_2'/2\\ n_2'/2 & n_3'\end{pmatrix}\right)t_1^{n_1'}t_2^{n_2'}t_3^{n_3'}$$

for $t_1, t_3\in(0, \exp(-\pi))$ and $t_2\in(\exp(-\pi), 1)$, where $\sum$ (resp. $\sum'$) is the sum over all (n_1, n_2, n_3) (resp. (n_1', n_2', n_3')) $\in \boldsymbol{Z}^3_3$ under the condition $\begin{pmatrix} n_1 & n_2/2\\ n_2/2 & n_3\end{pmatrix}\geqq 0$ $\left(\text{resp. } \begin{pmatrix} n_1' & n_2'/2\\ n_2'/2 & n_3'\end{pmatrix}>0\right)$. By the equality (***), we have $\Psi_k^{M,\chi}(f)(Z)=\tilde{c}\sum c_f(T)e[\mathrm{tr}(TZ)]$, where $\tilde{c}$ $(\neq 0)$ is a constant not depending upon the choice of f (cf. [3, § 3]). By the same ideas as those in [8, § 6], we can verify that $\Psi_k^{M,\chi}(f)$ is a cusp form. We may omit the details.

§ 3. Hecke operators and Euler products.

Let $T^{N}_{\kappa,\omega}(p^2)$ (resp. $T^{(2),L}_{k,\psi}(n)$) be the Hecke operator on $\mathfrak{S}_\kappa(N, \omega)$ (resp. on $S_k^{(2)}(L, \psi)$) for all primes p (resp. for all positive integers n) (cf. [1], [6] and [10]). In the following we shall investigate the two cases: (1) $\tilde{M}\neq M$ and M is divided by the conductor of $\boldsymbol{Q}(\sqrt{2M})$ and (2) $\tilde{M}=M$. In the case (1) (resp. case (2)), we consider a linear mapping $\Phi_k^{M,\chi}: \mathfrak{S}_{2k-1}(\tilde{M}, \chi)\to S_k^{(2)}(M, \chi_{2M})$ (resp. $\Phi_k'^{M,\chi}: \mathfrak{S}_{2k-1}(\tilde{M}, \chi)\to S_k^{(2)}(M, \chi_M)$) defined by $\Phi_k^{M,\chi}(f)=\Psi_k^{M,\bar{\chi}_{2M}}(f|[W_{\tilde{M}}]_{2k-1})$ (resp. $\Phi_k'^{M,\chi}(f)=\Psi_k^{M,\bar{\chi}_M}(f|[W_{\tilde{M}}]_{2k-1})$), where $\chi_t=\chi\left(\frac{t}{*}\right)$. Now we shall prove the following theorem.

THEOREM 2. *In the above case* (1) (*resp. case* (2)), *if* $f\in\mathfrak{S}_{2k-1}(\tilde{M}, \chi)$ *satisfies* $T^{\tilde{M}}_{2k-1,\chi}(p^2)f=\omega_p f$ *for all primes* p, *then* $\Phi_k^{M,\chi}(f)$ (*resp.* $\Phi_k'^{M,\chi}(f)$) *is a common eigenfunction of* $T^{(2),M}_{k,\chi_{2M}}(n)$ (*resp.* $T^{(2),M}_{k,\chi_M}(n)$) *for all* n, *i.e.*, $T^{(2),M}_{k,\chi_{2M}}(n)\Phi_k^{M,\chi}(f)=\lambda(n)\Phi_k^{M,\chi}(f)$ (*resp.* $T^{(2),M}_{k,\chi_M}(n)\Phi_k'^{M,\chi}(f)=\lambda'(n)\Phi_k'^{M,\chi}(f)$). *Furthermore*

$$L(2s-2k+4, \chi_{2M}^2)\sum_{n=1}^{\infty}\lambda(n)n^{-s}$$
$$=L(s-k+1, \chi_{2M})L(s-k+2, \chi_{2M})\prod_p(1-\omega_p p^{-s}+\chi(p)^2p^{2k-3-2s})^{-1}$$

$$(\textit{resp.}\ L(2s-2k+4, \chi_M^2)\sum_{n=1}^{\infty}\lambda'(n)n^{-s}$$
$$=L(s-k+1, \chi_M)L(s-k+2, \chi_M)\prod_p(1-\omega_p p^{-s}+\chi(p)^2p^{2k-3-2s})^{-1}),$$

where $L(s, *)$ *denotes the Dirichlet* L *function and product* $\prod_p$ *is taken over all primes* p.

Before proving the above theorem, we recall some lemmas (cf. [3]). Let U be the set of all complex-valued functions ϕ defined on P_2 with the property $\phi(\gamma T{}^t\gamma)=\phi(T)$ for every $\gamma\in SL_2(\boldsymbol{Z})$. For every $\phi\in U$, we define

$$T_a\left(SL_2(\boldsymbol{Z})\begin{pmatrix}1 & 0\\ 0 & m\end{pmatrix}SL_2(\boldsymbol{Z})\right)\phi(T)=\sum_{d=1}^{l}\phi(\sigma_d T{}^t\sigma_d),$$

where $SL_2(\boldsymbol{Z})\begin{pmatrix}1 & 0\\ 0 & m\end{pmatrix}SL_2(\boldsymbol{Z})=\bigcup_{d=1}^{l}SL_2(\boldsymbol{Z})\sigma_d$ (a disjoint union). For each positive integer m, the operators $\varDelta^+(m)$, $\varDelta^-(m)$ and $\Pi(m)$ on U are defined by $\varDelta^+(m)\phi(T)=\phi(mT)$, $\varDelta^-(m)\phi(T)=\phi(m^{-1}T)$ or 0 according as $m|e(T)$ or $m\nmid e(T)$ and

$$\Pi(m)=T_a\left(SL_2(\boldsymbol{Z})\begin{pmatrix}1 & 0\\ 0 & m\end{pmatrix}SL_2(\boldsymbol{Z})\right)\varDelta^-(m).$$

The following lemma has been proved by Andrianov [1, Proposition 2.1.2] and Matsuda [6].

LEMMA A. *Let* $F(Z)=\sum_{T} c(T)e[\mathrm{tr}(TZ)]\in S_{k}^{(2)}(L, \psi)$ *and let* p *be a prime. Then* $T_{k,\psi}^{(2);L}(p^{n})F(Z)=\sum_{T} c(p^{n}: T)e[\mathrm{tr}(TZ)]$ *and*

$$c(p^{n}: T)=\begin{cases} c(p^{n}T) & \text{if } p\mid L, \\ \sum p^{(k-2)\beta+(2k-3)\gamma}\psi(p)^{\beta+2\gamma}\Delta^{-}(p^{\gamma})\Pi(p^{\beta})\Delta^{+}(p^{\alpha})c(T) & \text{if } p\nmid L, \end{cases}$$

where the summation $\sum$ *is taken over all* $(\alpha, \beta, \gamma)\in \boldsymbol{Z}_{3}^{1}$ *with* $\alpha, \beta, \gamma\geqq 0$ *and* $\alpha+\beta+\gamma=n$.

For each $T\in P_{2}$, we denote by $d(T)$ the discriminant of the imaginary quadratic field $\boldsymbol{Q}(\sqrt{-N(T)})$. It is well-known that $-N(T)=d(T)f^{2}$ with a positive integer f. For a prime p, we have the following equality:

$$SL_{2}(\boldsymbol{Z})\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}SL_{2}(\boldsymbol{Z})=\bigcup_{i=1}^{p+1} SL_{2}(\boldsymbol{Z})\sigma_{i} \quad \text{(a disjoint union).}$$

The following lemma was shown in [**3**, Lemma 4.1].

LEMMA B. *Suppose that* $T\in P_{2}$ *satisfies* $e(T)=1$. *Then the following assertions hold:*

(1) *Among* $(p+1)$ *matrices* $\{\sigma_{i}T^{t}\sigma_{i}\}_{i=1}^{p+1}$, *there are* $p-\left(\frac{d(T)}{p}\right)$ *matrices with* $e(\sigma_{i}T^{t}\sigma_{i})=1$ *and* $1+\left(\frac{d(T)}{p}\right)$ *matrices with* $e(\sigma_{i}T^{t}\sigma_{i})=p$, *if* f *is prime to* p.

(2) *Among* $(p+1)$ *matrices* $\{\sigma_{i}T^{t}\sigma_{i}\}_{i=1}^{p+1}$, *there are* p *matrices with* $e(\sigma_{i}T^{t}\sigma_{i})=1$ *and one matrix with* $e(\sigma_{i}T^{t}\sigma_{i})=p^{2}$, *if* f *is divisible by* p.

PROOF OF THEOREM 2. Since our proofs of the both cases are same, we prove the assertions only in the case (1). It is sufficient to show the following equalities (3.1), (3.2) and (3.3):

$$T_{k,\chi_{2M}}^{(2);M}(p)\Phi_{k}^{\prime M,\chi}(f)=\omega_{p}\Phi_{k}^{\prime M,\chi}(f) \quad \text{if } p\mid M, \tag{3.1}$$

$$T_{k,\chi_{2M}}^{(2);M}(p)\Phi_{k}^{\prime M,\chi}(f)=(\omega_{p}+\chi_{2M}(p)(p^{k-1}+p^{k-2}))\Phi_{k}^{\prime M,\chi}(f) \tag{3.2}$$

and

$$T_{k,\chi_{2M}}^{(2);M}(p^{2})\Phi_{k}^{\prime M,\chi}(f)=(\omega_{p}^{2}+\chi_{2M}(p)(p^{k-1}+p^{k-2})\omega_{p}+\chi(p)^{2}p^{2k-2})\Phi_{k}^{\prime M,\chi}(f) \quad \text{if } p\nmid M. \tag{3.3}$$

First of all, we verify the equality (3.2). We see that (3.2) is equivalent to the following equality (3.2)′:

$$c(p: T)=(\omega_{p}+\chi_{2M}(p)(p^{k-1}+p^{k-2}))c(T) \quad \text{for all } T\in P_{2}, \tag{3.2$'$}$$

where $T_{k,\chi_{2M}}^{(2);M}(p)\Phi_{k}^{\prime M,\chi}(f)=\sum_{T} c(p: T)e[\mathrm{tr}(TZ)]$ and $\Phi_{k}^{\prime M,\chi}(f)=\sum_{T} c(T)e[\mathrm{tr}(TZ)]$. Therefore we prove (3.2)′. By Lemma A, we have $c(p: T)=p^{k-2}\chi_{2M}(p)(\Pi(p)c)(T)+p^{2k-3}(\chi_{2M}(p))^{2}(\Delta^{-}(p)c)(T)+(\Delta^{+}(p)c)(T)$. We calculate $c(p: T)$ step by step. Let α be the greatest number satisfying $p^{\alpha}\mid e(T)$. Set $e=e(T)$, $N=N(T)$ and $T=eT_{0}$.

Let D be the discriminant of the imaginary quadratic field $\boldsymbol{Q}(\sqrt{-N(T_0)})$. Clearly $-N(T_0)=Df^2$, where f is a certain positive integer. Since the Fourier coefficient $c(T)$ of $\Phi_k'^{M,\chi}(f)$ is determined by only e and N, we can write $c(T)=c(e, N)$.

First, we prove (3.2)$'$ under the conditions $\alpha\geqq 1$ and $p\nmid f$. By virtue of Lemma B, we see that $p^{k-2}\chi_{2M}(p)(\Pi(p)c)(T)=p^{k-2}\chi_{2M}(p)\sum_{i=1}^{p+1}c((e/p)\sigma_i T_0{}^t\sigma_i)=p^{k-2}\chi_{2M}(p)\left\{\left(p-\left(\frac{D}{p}\right)\right)c(e/p, N)+\left(1+\left(\frac{D}{p}\right)\right)c(e, N)\right\}$, where

$$SL_2(\boldsymbol{Z})\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}SL_2(\boldsymbol{Z})=\bigcup_{i=1}^{p+1}SL_2(\boldsymbol{Z})\sigma_i \quad \text{(a disjoint union).}$$

By virtue of Theorem 1, we have

$$\begin{aligned}
&p^{k-2}\chi_{2M}(p)(\Pi(p)c)(T)\\
&=p^{k-2}\chi_{2M}(p)\left\{\left(p-\left(\frac{D}{p}\right)\right)\sum_{d\mid e/p}d^{k-1}\chi_{2M}(d)a(MN/2d^2)\right.\\
&\qquad\left.+\left(1+\left(\frac{D}{p}\right)\right)\sum_{d\mid e}d^{k-1}\chi_{2M}(d)a(MN/2d^2)\right\}\\
&=p^{k-1}\chi_{2M}(p)\sum_{d\mid e/p}d^{k-1}\chi_{2M}(d)a(MN/2d^2)+p^{k-2}\chi_{2M}(p)c(T)\\
&\quad+p^{k-2}\chi_{2M}(p)\left(\frac{D}{p}\right)p^{\alpha(k-1)}\chi_{2M}(p^\alpha)\sum_{d'\mid e_0}(d')^{k-1}\chi_{2M}(d')a(-(M/2)D(fe_0/d')^2),
\end{aligned}$$

where $f(z)=\sum_{n=1}^{\infty}a(n)e[nz]$ and $e=p^\alpha e_0$ $((p, e_0)=1)$. On the other hand, by Lemma B, we have $(\Delta^+(p)c)(T)=c(pe, p^2N)$. By virtue of Theorem 1, we see that

$$\begin{aligned}
(\Delta^+(p)c)(T)=&\sum_{d\mid e}d^{k-1}\chi_{2M}(d)a(p^2MN/2d^2)+p^{k-1}\chi_{2M}(p)\sum_{d\mid e}d^{k-1}\chi_{2M}(d)a(MN/2d^2)\\
&-p^{k-1}\chi_{2M}(p)\sum_{d\mid e/p}d^{k-1}\chi_{2M}(d)a(MN/2d^2).
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_{d\mid e}d^{k-1}\chi_{2M}(d)a(p^2MN/2d^2)\\
&=\sum_{i=0}^{\alpha-1}\sum_{d\mid e_0}(p^id')^{k-1}\chi_{2M}(p^id')a(p^{2\alpha+2}e_0^2(-D)(M/2)f^2/(p^id')^2)\\
&\quad+\chi_{2M}(p^\alpha)p^{\alpha(k-1)}\sum_{d'\mid e_0}(d')^{k-1}\chi_{2M}(d')a(-(M/2)Dp^2(fe_0/d')^2).
\end{aligned}$$

By [**10**, Corollary 1.8], we have

$$\sum_{d|e} d^{k-1}\chi_{2M}(d)a(p^2MN/2d^2)$$

$$=\sum_{i=0}^{\alpha-1}\sum_{d'|e_0}(p^id')^{k-1}\chi_{2M}(p^id')\Big\{\omega_p a(-(M/2)Dp^{2(\alpha-i)}(e_0f/d')^2)$$

$$-\Big(\chi(p)\Big(\frac{-1}{p}\Big)\Big)^2 p^{2k-3}a(-(M/2)Dp^{2(\alpha-i)-2}(e_0f/d')^2)\Big\}$$

$$+\chi_{2M}(p^\alpha)p^{\alpha(k-1)}\sum_{d|e_0}(d')^{k-1}\chi_{2M}(d')\Big\{\omega_p-\chi_{2M}(p)\Big(\frac{D}{p}\Big)p^{k-2}\Big\}a(-(M/2)D(fe_0/d')^2)$$

$$=\omega_p c(T)-\chi_{2M}(p)^2p^{2k-3}(\Delta^-(p)c)(T)$$

$$-p^{k-2}\chi_{2M}(p)\Big(\frac{D}{p}\Big)\chi_{2M}(p^\alpha)p^{\alpha(k-1)}\sum_{d'|e_0}(d')^{k-1}\chi_{2M}(d')a(-(M/2)D(fe_0/d')^2)\,.$$

Thus we obtain (3.2)′ in the case $\alpha\geqq 1$ and $(p, f)=1$.

Secondly, we consider the case $\alpha\geqq 1$ and $p|f$. Let β be the greatest number satisfying $p^\beta|f$. By Lemma B, we can check

$$p^{k-2}\chi_{2M}(p)(\Pi(p)c)(T)$$

$$=p^{k-2}\chi_{2M}(p)\{pc(e/p,\ N)+c(pe,\ N)\}$$

$$=p^{k-1}\chi_{2M}(p)\sum_{d|e/p}d^{k-1}\chi_{2M}(d)a(MN/2d^2)+p^{k-2}\chi_{2M}(p)c(T)$$

$$+p^{2k-3}\chi_{2M}(p)^2\sum_{d|e}d^{k-1}\chi_{2M}(d)a((M/2)N/p^2d^2)-p^{2k-3}\chi_{2M}(p)^2(\Delta^-(p)c)(T)\,.$$

On the other hand, we see that

$$(\Delta^+(p)c)(T)$$

$$=\sum_{d|e}d^{k-1}\chi_{2M}(d)a(p^2MN/2d^2)+p^{k-1}\chi_{2M}(p)c(T)$$

$$-p^{k-1}\chi_{2M}(p)\sum_{d|e/p}d^{k-1}\chi_{2M}(d)a(MN/2d^2)$$

$$=\sum_{i=0}^{\alpha}\sum_{d'|e_0}(p^id')^{k-1}\chi_{2M}(d')\chi_{2M}(p^i)a(p^{2(\alpha+\beta-i)+2}(-MD/2)(e_0f_1/d')^2)$$

$$+p^{k-1}\chi_{2M}(p)c(T)-p^{k-1}\chi_{2M}(p)\sum_{d|e/p}d^{k-1}\chi_{2M}(d)a(MN/2d^2)\quad (f=p^\beta f_1).$$

Using [**10**, Corollary 1.8], we can check

$$(\Delta^+(p)c)(T)$$

$$=\sum_{i=0}^{\alpha}\sum_{d'|e_0}(p^id')^{k-1}\chi_{2M}(d')\chi_{2M}(p^i)\Big\{\omega_p a(p^{2(\alpha+\beta-i)}(-MD/2)(e_0f_1/d')^2)$$

$$-\chi(p)^2\Big(\frac{-1}{p}\Big)^2p^{2k-3}a(p^{2(\alpha+\beta-i)-2}(-MD/2)(e_0f_1/d')^2)\Big\}$$

$$+p^{k-1}\chi_{2M}(p)c(T)-p^{k-1}\chi_{2M}(p)\sum_{d|e/p}d^{k-1}\chi_{2M}(d)a(MN/2d^2)$$

$$=\omega_p c(T)-\chi(p)^2 p^{2k-3}\sum_{d|e} d^{k-1}\chi_{2M}(d)a(MN/2p^2d^2)+p^{k-1}\chi_{2M}(p)c(T)$$
$$-p^{k-1}\chi_{2M}(p)\sum_{d|e/p} d^{k-1}\chi_{2M}(d)a(MN/2d^2)\,.$$

Hence we obtain (3.2)′ in the case $\alpha\geqq 1$ and $p|f$.

Thirdly, we consider the case $\alpha=0$ and $(p, f)=1$. By Lemma B, we see that $p^{k-2}\chi_{2M}(p)(\Pi(p)c)(T)=\chi_{2M}(p)p^{k-2}\left(1+\left(\frac{D}{p}\right)\right)c(T)$. An immediate computation shows that

$$(\Delta^+(p)c)(T)=\sum_{d|e} d^{k-1}\chi_{2M}(d)a(p^2(-MD/2)(ef/d^2))+p^{k-1}\chi_{2M}(p)c(T)$$
$$=\omega_p c(T)-p^{k-2}\chi(p)\left(\frac{2M}{p}\right)\left(\frac{D}{p}\right)c(T)+p^{k-1}\chi_{2M}(p)c(T)\,.$$

Therefore we obtain (3.2)′ in the case $\alpha=0$ and $(p, f)=1$.

Fourthly, we consider the remaining case. Set $f=p^{\beta}f_1$, where $(p, f_1)=1$. Now we see that

$$p^{k-2}\chi_{2M}(p)(\Pi(p)c)(T)=p^{k-1}\chi_{2M}(p)c(pe,\, N)$$
$$=p^{k-2}\chi_{2M}(p)c(T)+p^{2k-3}\chi_{2M}(p)^2\sum_{d|e} d^{k-1}\chi_{2M}(d)a(MN/2p^2d^2)$$

and

$$(\Delta^+(p)c)(T)$$
$$=\sum_{d|e} d^{k-1}\chi_{2M}(d)\Big\{\omega_p a(p^{2\beta}(-MD/2)(ef_1/d)^2)$$
$$-\left(\chi(p)\left(\frac{-1}{p}\right)\right)^2 p^{2k-3}a(p^{2\beta-2}(-MD/2)(ef_1/d)^2)\Big\}+p^{k-1}\chi_{2M}(p)c(T)$$
$$=\omega_p c(T)-\chi(p)^2p^{2k-3}\sum_{d|e} d^{k-1}\chi_{2M}(d)a(MN/2p^2d^2)+p^{k-1}\chi_{2M}(p)c(T)\,.$$

Therefore we have (3.2)′ in the case $(p, e)=1$ and $p|f$. Consequently, we have completed the proof of (3.2). By a similar argument, we can prove (3.1) and (3.3). Therefore we obtain the desired results.

§ 4. Maaß's space.

Consider a function $F(Z)=\sum_{T>0} c(T)e[\mathrm{tr}(TZ)]\in S_k^{(2)}(L, \psi)$ whose coefficients $c(T)$ satisfy

$$c\left(\begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix}\right)=\sum_m \psi(m)m^{k-1}c\left(\begin{pmatrix} 1 & n_2/2m \\ n_2/2m & n_1n_3/m^2 \end{pmatrix}\right),$$

where m runs over all positive integers with $m|n_1$, $m|n_2$ and $m|n_3$. Denote by

$\mathcal{M}_k^{(2)}(L, \psi)$ the subspace of $S_k^{(2)}(L, \psi)$ of all such F. We call $\mathcal{M}_k^{(2)}(L, \psi)$ Maaß's space. It should be noted that $\mathcal{M}_k^{(2)}(L, \psi)$ is an invariant space of all Hecke operators $T_{k,\psi}^{(2),L}(n)$.

In this section we discuss a generalization of Maaß's theorem (cf. [4] and [5]). For this purpose we introduce two theta functions $\theta_1(z_1, z_2)$ and $\theta_2(z_1, z_2)$ which are defined by

$$\theta_h(z_1, z_2)=\sum_{n=-\infty}^{\infty}\exp(\pi i(z_1(\sqrt{2}n+\sqrt{2}h/2)^2+2\sqrt{2}(\sqrt{2}n+\sqrt{2}h/2)z_2)) \qquad (z_1\in\mathfrak{H} \text{ and } z_2\in\boldsymbol{C}).$$

For each $x\in\boldsymbol{R}$, $z_1\in\mathfrak{H}$ and $z_2\in\boldsymbol{C}$, set $g(x:z_1, z_2)=\exp(\pi i(z_1x^2+2\sqrt{2}xz_2))$. By our definition of the Weil representation $\gamma(\sigma, 1)$,

$$\begin{aligned}&\gamma(\sigma, 1)g(x;z_1, z_2)\\ &=|c|^{-1/2}\int_{-\infty}^{\infty}e[(ax^2-2xy+dy^2)/2c]g(y;z_1, z_2)dy\\ &=|c|^{-1/2}\exp(\pi iax^2/c)\int_{-\infty}^{\infty}\exp\{-(-\pi i((cz_1+d)/c))y^2+2(\pi i(\sqrt{2}z_2-x/c))y\}dy\\ &=\varepsilon(\sigma)(cz_1+d)^{-1/2}g(x;\sigma(z_1), z_2/(cz_1+d))e[-z_2^2c/(cz_1+d)]\end{aligned}$$

for all $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\boldsymbol{R})$ $(c\neq 0)$, where $\varepsilon(\sigma)=\sqrt{i}^{\,\mathrm{sgn}\,c}$.

Take two lattices $\sqrt{2}\boldsymbol{Z}$ and $(\sqrt{2}/2)\boldsymbol{Z}$ in $\boldsymbol{R}$. Now we see easily that $\theta_h(z_1, z_2)=\sum_{l\in\sqrt{2}\boldsymbol{Z}}g(l+\sqrt{2}h/2;z_1, z_2)$. By Theorem A, we can verify the following lemma.

LEMMA 4.1. *Let* $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ *be an element of* $\Gamma_0(4)$ *and let* γ *be a positive integer satisfying* $(4, \gamma)\neq 4$. *Then*

(i) $\theta_h(\sigma(z_1), z_2/(cz_1+d))=e[abh/4]j(\sigma, z_1)e[cz_2^2/(cz_1+d)]\theta_h(z_1, z_2)$

and

(ii) $\theta_h(z_1/(\gamma z_1+1), z_2/(\gamma z_1+1))=\sqrt{i}^{-1}\sqrt{\gamma z_1+1}\,e[\gamma z_2^2/(\gamma z_1+1)]\sum_{k=0}^{1}c(h, k)_\gamma\theta_k(z_1, z_2)$,

where $c(0, 1)_\gamma$ $(=c(1, 0)_\gamma)$ *equals* $(\sqrt{i}/2)\left(1+\left(\frac{-1}{\gamma}\right)i\right)$ *or* $\sqrt{i}$ *according as* $(4, \gamma)=1$ *or* $(4, \gamma)=2$ *and* $c(0, 0)_\gamma$ $(=c(1, 1)_\gamma)$ *equals* 0 *or* $\sqrt{i}\left(1-\left(\frac{-1}{\gamma}\right)i\right)\Big/2$ *according as* $(4, \gamma)=2$ *or* $(4, \gamma)=1$.

Let F be an element of $\mathcal{M}_k^{(2)}(L, \psi)$. Then we have the expansion of the form $F\left(\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}\right)=\sum_{m=1}^{\infty} F_m(z_1, z_2)e[mz_3]$. For a matrix $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the positive determinant D, set

$$(F_1|_k\sigma)(z_1, z_2)=F_1(\sigma(z_1), \sqrt{D}\, z_2/(cz_1+d))(cz_1+d)^{-k}e[-cz_2^2/(cz_1+d)]\,.$$

For each positive integer m, we define a function

$$(F_1|_kH(m))(z_1, z_2)=m^{k-1}\sum_{\nu=1}^{n} \psi(S_\nu)(F_1|_kS_\nu)(z_1, z_2)\,,$$

where $\{S_\nu\}_{\nu=1}^{n}=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\,\middle|\, ad=m,\ d>0 \text{ and } b\in \boldsymbol{Z}/d\boldsymbol{Z}\right\}$ and $\psi\left(\begin{pmatrix} a_\nu & * \\ * & * \end{pmatrix}\right)=\psi(a_\nu)$.

An obvious modification of Maaß's arguments in [4] shows the following relations

(4.1) $$(F_1|_kH(m))(z_1, z_2)=F_m(z_1, z_2/\sqrt{m}) \quad \text{and} \quad F_1(z_1, z_2)\equiv 0$$

if and only if $F_m(z_1, z_2)\equiv 0$ for every positive m.

Let us consider an embedding of $SL_2(\boldsymbol{R})$ into $Sp(2, \boldsymbol{R})$ defined by

$$i(\sigma)=\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \left(\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

Since $i(\sigma)\left\langle\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}\right\rangle=\begin{pmatrix} \sigma(z_1) & z_2/(cz_1+d) \\ z_2/(cz_1+d) & z_3-cz_2^2/(cz_1+d) \end{pmatrix}$, we have

$$\sum_{m=1}^{\infty} F_m(\sigma(z_1), z_2/(cz_1+d))e[-mcz_2^2/(cz_1+d)]e[mz_3]$$

$$=\bar{\psi}(a)(cz_1+d)^k\sum_{m=1}^{\infty} F_m(z_1, z_2)e[mz_3]$$

for every $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(L)$. Comparing the coefficients of $e[z_3]$ in both sides of the above equality, we have

(4.2) $$F_1(\sigma(z_1), z_2/(cz_1+d))=e[cz_2^2/(cz_1+d)]\bar{\psi}(a)(cz_1+d)^kF_1(z_1, z_2)$$

for every $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(L)$. By our definition of F_1, F_1 has the expansion of the form $F_1(z_1, z_2)=\sum_{h=0}^{1} c_h(z_1)\theta_h(z_1, z_2)$. It should be noted that

(4.3) $$c_0(z_1)=\sum_{n=1}^{\infty} c\left(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\right)e[nz_1] \quad \text{and} \quad F_1(z_1, z_2)\equiv 0$$

if and only if $c_h(z_1)\equiv 0$ for $h=0$ and 1. By virtue of Lemma 4.1 and (4.2), we can verify the following relations

$$F_1(\sigma(z_1), z_2/(cz_1+d))e[-cz_2^2/(cz_1+d)]$$
$$=\sum_{h=0}^{1} c_h(\sigma(z_1))e[abh/4]j(\sigma, z_1)\theta_h(z_1, z_2)$$
$$=\bar{\psi}(a)(cz_1+d)^k \sum_{h=0}^{1} c_h(z_1)\theta_h(z_1, z_2) \quad \text{for every } \sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(\tilde{L})$$

and

$$F_1(z_1/(Lz_1+1), z_2/(Lz_1+1))e[-Lz_2^2/(Lz_1+1)]$$
$$=\sum_{h=0}^{1} c_h(z_1/(Lz_1+1))\sqrt{i}^{\,-\operatorname{sgn} L}\sqrt{Lz_1+1}\sum_{k=0}^{1} c(h, k)_L\theta_k(z_1, z_2)$$
$$=(Lz_1+1)^k \sum_{h=0}^{1} c_h(z_1)\theta_h(z_1, z_2), \quad \text{where } \tilde{L}=\text{l. c. m.}(4, L).$$

Therefore we obtain the following transformation formula

$$\text{(4.4)}\quad \begin{aligned} &c_0(\sigma(z_1))=\bar{\psi}(a)j(\sigma, z_1)^{2k-1}c_0(z_1) \quad \text{for every } \sigma=\begin{pmatrix} a & * \\ * & * \end{pmatrix}\in\Gamma_0(\tilde{L}), \\ &c_0(z_1/(Lz_1+1))c(0, 0)_L+c_1(z_1/(Lz_1+1))c(1, 0)_L=\sqrt{i}^{\,\operatorname{sgn} L}\sqrt{Lz_1+1}^{\,2k-1}c_0(z_1). \end{aligned}$$

Noting that $c(T)=O(N(T)^{k/2})$, we have $c_0(z_1)=\sum_{n=1}^{\infty} c\left(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\right)e[nz_1]\in\mathfrak{S}_{2k-1}(\tilde{L}, \psi)$. Consider a mapping $I_k(L, \psi)F=c_0$ for every $F\in\mathcal{M}_k^{(2)}(L, \psi)$. By the fact that $c(1, 0)_L=0$ and the relations (4.1), (4.3) and (4.4), the linear mapping $I_k(L, \psi)$ is injective. A relation between $I_k(L, \psi)$ and Hecke operators on $\mathcal{M}_k^{(2)}(L, \psi)$ and $\mathfrak{S}_{2k-1}(\tilde{L}, \psi)$ can be stated as follows:

THEOREM 3. *Suppose that k is even and $4\nmid L$. Then the mapping $I_k(L, \psi)$ is injective, and, if F satisfies the relations $T_k^{(2),L}(p)F=\lambda(p)F$ for every odd prime p, then $T_{2k-1,\psi}^{\tilde{L}}(p^2)(I_k(L, \psi)F)=(\lambda(p)-\psi(p)(p^{k-2}+p^{k-1}))I_k(L, \psi)F$ for every odd prime p. Moreover, when $(4, L)=2$, the above last assertion is satisfied for all primes p.*

PROOF OF THEOREM 3. We prove the latter assertion. We note that $c_0(z)=\sum_{n=1}^{\infty} c\left(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\right)e[nz]$. Set $F(z)=\sum_{T>0} c(T)e[\operatorname{tr}(TZ)]$ and $(T_{k,\psi}^{(2),L}(p)F)(Z)=\sum_{T>0} c(p:T)\cdot e[\operatorname{tr}(TZ)]$. By virtue of Lemma A,

$$c(p:T)=p^{k-2}\psi(p)(\Pi(p)c)(T)+p^{2k-3}\psi(p)^2(\Delta^-(p)c)(T)+(\Delta^+(p)c)(T).$$

This shows that

$$c\left(p:\begin{pmatrix}1&0\\0&n\end{pmatrix}\right)=c\left(\begin{pmatrix}p&0\\0&pn\end{pmatrix}\right)+p^{k-2}\psi(p)\sum_{i=1}^{p+1}c\left(p^{-1}\sigma_i\begin{pmatrix}1&0\\0&n\end{pmatrix}{}^t\sigma_i\right),$$

where $SL_2(\boldsymbol{Z})\begin{pmatrix}1&0\\0&p\end{pmatrix}SL_2(\boldsymbol{Z})=\bigcup_{i=1}^{p+1}SL_2(\boldsymbol{Z})\sigma_i$. Define $\tilde{c}(e, N)=c(T)$ with $e(T)=e$ and $N(T)=N$. Since $F\in\mathcal{M}_k^{(2)}(L, \psi)$, our definition of $\tilde{c}(e, N)$ is meaningful.

Applying Lemma B, we obtain

$$\sum_{i=1}^{p+1}c\left(p^{-1}\sigma_i\begin{pmatrix}1&0\\0&n\end{pmatrix}{}^t\sigma_i\right)=\begin{cases}\left(1+\left(\dfrac{d}{p}\right)\right)\tilde{c}(1, 4n) & \text{if } (p, f)=1,\\ \tilde{c}(p, 4n) & \text{otherwise,}\end{cases}$$

where d is the discriminant of the imaginary quadratic field $\boldsymbol{Q}(\sqrt{-4n})$ and f is the positive integer satisfying $-4n=df^2$. By our definition, we see that $\tilde{c}(1, 4n)=c\left(\begin{pmatrix}1&0\\0&n\end{pmatrix}\right)$ and $\tilde{c}(p, 4n)=c\left(\begin{pmatrix}1&0\\0&n\end{pmatrix}\right)+p^{k-1}\psi(p)c\left(\begin{pmatrix}1&0\\0&n/p^2\end{pmatrix}\right)$ if $p\neq 2$ and $p|f$. First we consider the case $p\neq 2$. Noting that $1+\left(\frac{-n}{p}\right)=1+\left(\frac{d}{p}\right)$ or 1 according as $(p, f)=1$ or not, we obtain $\sum_{i=1}^{p+1}c\left(p^{-1}\sigma_i\begin{pmatrix}1&0\\0&n\end{pmatrix}{}^t\sigma_i\right)=\left(1+\left(\frac{-n}{p}\right)\right)$ $\cdot c\left(\begin{pmatrix}1&0\\0&n\end{pmatrix}\right)+p^{k-2}\psi(p)c\left(\begin{pmatrix}1&0\\0&n/p^2\end{pmatrix}\right)$, where $c\left(\begin{pmatrix}1&0\\0&n/p^2\end{pmatrix}\right)=0$ if $p^2\nmid n$. Therefore, we see that $c\left(p:\begin{pmatrix}1&0\\0&n\end{pmatrix}\right)=c\left(\begin{pmatrix}1&0\\0&p^2n\end{pmatrix}\right)+\left\{p^{k-1}+p^{k-2}\left(1+\left(\frac{-n}{p}\right)\right)\right\}\psi(p)c\left(\begin{pmatrix}1&0\\0&n\end{pmatrix}\right)$ $+p^{2k-3}\psi(p)^2c\left(\begin{pmatrix}1&0\\0&n/p^2\end{pmatrix}\right)$. It follows from this that

$$(4.5)\quad I_k(L, \psi)(T_{k,\psi}^{(2)}(p)F)=T_{2k-1,\psi}^{L}(p^2)(I_k(L, \psi)F)$$
$$+(p^{k-1}+p^{k-2})\psi(p)I_k(L, \psi)F \quad \text{for every prime } p\neq 2.$$

Next we consider the case $p=2$ and $(4, L)=2$. We have

$$c\left(2:\begin{pmatrix}1&0\\0&n\end{pmatrix}\right)=c\left(\begin{pmatrix}2&0\\0&2n\end{pmatrix}\right)=c\left(\begin{pmatrix}1&0\\0&4n\end{pmatrix}\right).$$

Hence

$$(4.6)\qquad I_k(L, \psi)(T_{k,\psi}^{(2),L}(2)F)=T_{2k-1,\psi}^{L}(2^2)(I_k(L, \psi)F).$$

By (4.5) and (4.6), we obtain the desired results.

The above theorem was first proved in [5] for $\mathcal{M}_k^{(2)}(1, 1_\psi)$, where 1_ψ denotes the trivial character modulo 1.

§ 5. An application.

Let M be a positive integer satisfying $(4, M)=2$ and let χ be a character modulo M. Consider the natural isomorphism $(\boldsymbol{Z}/M\boldsymbol{Z})^{\times}\cong\prod_{p\mid M}(\boldsymbol{Z}/M_p\boldsymbol{Z})^{\times}$, where M_p denotes the p-factor of M. We denote by $(\chi^2)_p$ the induced character modulo M_p of χ^2.

In the following we assume that

(5.1) $\tilde{M}$ is divided by the conductor of $\boldsymbol{Q}(\sqrt{2M})$ and $M_p=m_p$ or $M_p=p$ for all primes p $(p\mid M)$, where m_p means the conductor of $(\chi^2)_p$. Let χ be a character modulo M. Let $f=\sum_{n=1}^{\infty}a(n)e[nz]$

$$(a(1)\neq 0)\qquad \left(\text{resp. } F(Z)=\sum_{T>0}c(T)e[\mathrm{tr}(TZ)]\left(c\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\neq 0\right)\right)$$

be an element of $\mathfrak{S}_{2k-1}(\tilde{M},\chi)$ (resp. $\mathcal{M}_k^{(2)}(M,\chi)$) such that $T_{2k-1,\chi}^{\tilde{M}}(p^2)f=\omega_p f$ (resp. $T_{k,\chi}^{(2),M}(n)F=\lambda(n)F$) for all primes p (resp. positive integers n). We denote by $\tilde{\mathfrak{S}}_{2k-1}(\tilde{M},\chi)$ (resp. $\tilde{\mathcal{M}}_k^{(2)}(M,\chi)$) the vector space spanned by all such $f_1, f_2, \cdots, f_n$ (resp. $F_1, F_2, \cdots, F_m$), where $f_i(z)=\sum_{n=1}^{\infty}a_i(n)e[nz]$ and $F_j(Z)=\sum_{T>0}c_j(T)e[\mathrm{tr}(TZ)]$ with $a_i(1)=1$ and $c_j\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=1$.

Now we show that $\tilde{I}_k(M,\chi)$ is an isomorphic mapping between $\tilde{\mathcal{M}}_k^{(2)}(M,\chi)$ and $\tilde{\mathfrak{S}}_{2k-1}(\tilde{M},\chi)$, where $\tilde{I}_k(M,\chi)$ denotes the restriction of $I_k(M,\chi)$ to $\tilde{\mathcal{M}}_k^{(2)}(M,\chi)$. By Theorem 3, we see easily that $\tilde{I}_k(M,\chi)$ is injective. So we prove $\tilde{I}_k(M,\chi)$ is surjective.

For every f_i, set $\tilde{F}_i=\tilde{\Psi}_k^{M,\chi}(f_i)=\Psi_k^{M,\chi}\left(\left(f_i\left|\left[\Delta_1\begin{pmatrix}1&0\\0&M/2\end{pmatrix}\Delta_1\right]_{2k-1}\right)\right|[W_{2M}]_{2k-1}\right)$, where $f_i\left|\left[\Delta_1\begin{pmatrix}1&0\\0&M/2\end{pmatrix}\Delta_1\right]_{2k-1}\right.(z)=\sum_{n=1}^{\infty}a_i((M/2)n)e[nz]\in\mathfrak{S}_{2k-1}(\tilde{M},\chi_{2M})$ (cf. [**10**, Proposition 1.5]).

Using Theorem 1 and [**10**, (i) of Corollary 1.8], we have $\tilde{F}_i(Z)=\sum_{T>0}c_i(T)\cdot e[\mathrm{tr}(TZ)]$, where $c_i(T)=a_i((M/2)^2)\sum_m \chi(m)m^{k-1}a_i(N(T)/m^2)$ with m running over all positive integers under the condition $m\mid e(T)$. This shows that $I_k(M,\chi)\tilde{F}_i=a_i((M/2)^2)a_i(4)f_i$. Set $T_{2k-1,\chi}^{\tilde{M}}(p^2)f_i=(\omega_p)_i f_i$. Note that $a_i(1)=1$. By [**7**, Theorem] and [**10**, Theorem 1.9], we see that $\prod_p(1-(\omega_p)_i p^{-s}+\chi(p)^2p^{2k-3-2s})^{-1}$ is the zeta function associated with a cusp form of $S_{2k-2}^{(1)}(M,\chi^2)$. Since f_i is an eigenfunction of Hecke operators $T_{2k-1,\chi}^{\tilde{M}}(p^2)$ and $a_i(1)=1$, we have $a_i((M/2)^2)a_i(4)=\prod_{p\mid M}(\omega_p)_i$ (cf. [**10**, (i) of Corollary 1.8]). The assumption about M shows $(\omega_p)_i\neq 0$ for all p with $p\mid M$ (cf. [**2**, Theorem 4.6.17]). So we see that $a_i((M/2)^2)a_i(4)\neq 0$.

By virtue of [**10**, Proposition 1.5 and Theorem 1.7], $f_i \left| \left[\Delta_1 \begin{pmatrix} 1 & 0 \\ 0 & M/2 \end{pmatrix} \Delta_1 \right]_{2k-1} \right.$ is an eigenfunction of Hecke operators $T^{\tilde{M}}_{2k-1, \chi_{2M}}(p^2)$ for every prime p. Therefore, by Theorem 2, we can verify that $\tilde{F}_i$ is an eigenfunction of all Hecke operators $T^{(2), M}_{k, \chi}(n)$. Consequently we obtain $\tilde{F}_i = cF_j$ for some j and c $(\neq 0)$. So we have $\tilde{I}_k(M, \chi)F_j = c' f_i$ $(c' \neq 0)$, which yields our assertion.

Thus, if we define a mapping $\tilde{\Psi}^{M, \chi}_k$ of $\tilde{\mathfrak{S}}_{2k-1}(M, \chi)$ onto $\tilde{\mathcal{M}}^{(2)}_k(M, \chi)$ by $\tilde{\Psi}^{M, \chi}_k(f_i)$ for each f_i, we have the following theorem.

THEOREM 4. *Suppose that M satisfies the above assumption* (5.1). *Then the mapping $\tilde{\Psi}^{M, \chi}_k$ is an isomorphic mapping between $\tilde{\mathfrak{S}}_{2k-1}(M, \chi)$ and $\tilde{\mathcal{M}}^{(2)}_k(M, \chi)$, and if f satisfies $T^{(2), M}_{k, \chi}(n)(\tilde{\Psi}^{M, \chi}_k(f)) = \tilde{\lambda}(n)(\tilde{\Psi}^{M, \chi}_k(f))$ for every positive integer n. Furthermore*

$$L(2s-2k+4, \chi^2) \sum_{n=1}^{\infty} \tilde{\lambda}(n) n^{-s}$$

$$= L(s-k+1, \chi) L(s-k+2, \chi) \prod_p (1 - \omega_p p^{-s} + \chi(p)^2 p^{2k-3-2s})^{-1}.$$

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