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On isometry groups of a manifold without focal points

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Introduction.

This note treats a complete simply connected Riemannian manifold without focal points. We know that the class of all such manifolds is wider than that of all complete simply connected Riemannian manifolds with nonpositive sectional curvature (cf. [**10**], [**6**] and [**7**]). The former is defined by the property: No maximal geodesic has focal points along any perpendicular geodesic, or equivalently, there is a unique perpendicular geodesic from a point to a geodesic, see Proposition 2 in [**3**]. It has been observed that several results on manifolds with nonpositive sectional curvature hold for manifolds without focal points. In this note we shall, furthermore, investigate the problems of this nature.

In §1 we summarize notion and known facts which would be used later. The main object of §2 is to study about the behavior of a hyperbolic isometry and its fixed points. In manifolds with nonpositive curvature the law of cosine is powerful especially for the limit process of divergent sequences. We shall give here a crucial lemma which plays the same role as the special use of the law of cosine. The main result in §2 is

THEOREM 1. *Let M be a complete simply connected Riemannian manifold without focal points and Γ a freely acting, properly discontinuous group of isometries of M. Let D be a Dirichlet region for Γ. If $z \in M(\infty)$ is a point containing the axis of a hyperbolic element in Γ, then $\Gamma(z) \cap \bar{D} = \emptyset$.*

§3 studies the limit sets of freely acting, properly discontinuous groups Γ of isometries of M, as above, with volume $(M/\Gamma) < \infty$. §4 is devoted to extend E. Cartan's fixed point theorem proved in the case of nonpositive curvature, which leads to the conjugacy of maximal compact subgroups of a (semisimple) Lie group, to the case without focal points.

It is not yet known if the displacement function of an isometry of M, as above, is convex. If we assume the property, together with our results one could see actions of isometries of M more clearly.

1. Preliminaries.

Here we recall basic notion and known results for manifolds without focal points.

Let M be a complete simply connected Riemannian manifold without focal points. Two geodesic rays α and β are said to be *asymptotic* if $d(\alpha(t), \beta(t)) \leqq$ some constant for all $t \geqq 0$. For each geodesic ray α we denote by $\alpha(\infty)$ the class of geodesic rays asymptotic to α. $M(\infty)$ denotes the totality of all asymptote classes of geodesic rays (cf. [**2**]). Then the union $M \cup M(\infty)$, which is denoted by $\bar{M}$, has the natural topology of the closed unit disc of dimension $\dim M$ (see [**6**] and [**7**]). We call the topology the cone topology. If φ is an isometry of M and z a point in $M(\infty)$ we set $\varphi(z)=(\varphi\circ\alpha)(\infty)$, where α is any geodesic ray representing z. Since asymptotes are preserved under isometries, we obtain a well-defined mapping $\varphi: \bar{M} \to \bar{M}$ which is bijective and carries $M(\infty)$ into itself.

If Γ is a freely acting, properly discontinuous group of isometries of M, $\overline{\Gamma(p)} \cap M(\infty)$ is called the *limit set* of Γ and denoted by $L(\Gamma)$, where $\overline{\Gamma(p)}$ is the closure of the orbit of any point $p \in M$. It is to be noted that the limit set does not depend on the choice of the base point p.

All geodesics are supposed to have unit speed unless stated otherwise. For two points p and q in M, $d(p, q)$ denotes the distance between p and q.

The following facts about manifolds without focal points are useful.

FACT 1 (Proposition 4 in [**14**]). A complete Riemannian manifold has no focal points if and only if $d/dt\langle Y, Y\rangle > 0$ for $t>0$, where Y is any non-trivial Jacobi field along a geodesic with $Y(0)=0$.

FACT 2. Let M be a complete simply connected Riemannian manifold without focal points. Then any geodesic ball in M is strongly convex.

This fact follows from Fact 1 and Lemma in p. 160 [**9**].

FACT 3 (Theorem 2 and 2′ in [**6**]). Let M be as in Fact 2. If α and β are distinct geodesic rays starting at a point of M, then the length of the perpendicular from $\alpha(t)$ to β is strictly monotone increasing in t and approaches ∞ as $t \to \infty$, and so is $d(\alpha(t), \beta(t))$.

FACT 4 (Corollary 1 to Theorem 4 in [**6**]). Let M be as in Fact 2. If α and β are geodesic rays asymptotic to each other. Then $d(\alpha(t), \beta(t))$ is non-increasing in t.

FACT 5 (§3 in [**6**]). Let M be as in Fact 2. For every point $p \in M$ we can arrange the Euclidean metric of T_pM so that the exponential mapping is distance increasing.

2. Hyperbolic isometries.

Let M be a complete simply connected Riemannian manifold without focal points. For an isometry φ of M we denote by d_{φ} the corresponding displacement function defined by $d_{\varphi}(p)=d(p, \varphi(p))$. φ is said to be *elliptic, hyperbolic* or *parabolic* if d_{φ} attains zero minimum, a positive minimum or no minimum, respectively. Brouwer's fixed point theorem tells us that hyperbolic or parabolic isometries have fixed points in $M(\infty)$. It is well-known that a non-elliptic isometry φ is hyperbolic if and only if φ translates a geodesic.

The following lemma would be one of the key tools in manifolds without focal points. The result corresponds to the special use of the law of cosine in manifolds with nonpositive curvature.

LEMMA 2.1. *Let M be a complete simply connected Riemannian manifold without focal points. Let $p\in M$ and τ a geodesic ray starting at p. Suppose that $\{q_n\}$ is a sequence of points of M with properties:*

(1) $\lim_{n\to\infty} d(p, q_n)=\infty$, *and*

(2) *For a constant $c>0$, $d(q_n, \tau)\leqq c$ for all n.*

Then

$$\lim q_n=\tau(\infty)\in M(\infty).$$

PROOF. Let σ_n denotes the geodesic segment connecting p to q_n with $\sigma_n(0)=p$. Let t_n be the positive number determined by $\sigma_n(t_n)=q_n$. By passing through a subsequence if necessary, we may assume that $t_1<t_2<\cdots$ and $\lim_{n\to\infty} t_n=\infty$ by the property (1). It follows from the Fact 3 that

$$d(\sigma_n(t), \tau)\leqq d(\sigma_n(t_n), \tau)=d(q_n, \tau) \qquad \text{for } t,\ 0\leqq t\leqq t_n .$$

Therefore we can replace the property (2) as follows:

(2′) For a constant $c>0$,

$$d(\sigma_n(t), \tau)\leqq c \qquad \text{for all } n \text{ and } t,\ 0\leqq t\leqq t_n .$$

We know that any subsequence $\{\sigma'_m(0)\}$ of $\{\sigma'_n(0)\}$ has a subsequence $\{\sigma'_k(0)\}$ which converges, say, to v. Assuming $v\neq\tau'(0)$, we shall induce a contradiction.

First of all we extend each geodesic segment σ_n to the geodesic ray beyond q_n. Let σ be the geodesic ray such that $\sigma(0)=p$ and $\sigma'(0)=v$. By our assumption, σ and τ are distinct, so using Fact 3 we can choose a positive number T such that

$$d(\sigma(t), \tau)\geqq 3c \qquad \text{for } t\geqq T .$$

We fix T chosen above. Let B be the geodesic ball centered at $\sigma(T)$ with radius c ($c>0$ as in the property (2)). For $a\in B$ we denote by a' the foot of the perpendicular from a to τ. Then

$$
\begin{aligned}
d(a, a') &\geqq d(\sigma(T), a') - d(a, \sigma(T)) \\
&\geqq d(\sigma(T), \tau) - c \\
&\geqq 3c - c \\
&= 2c .
\end{aligned}
$$

Since $\sigma_k \to \sigma$ (parameterwise) as $n \to \infty$, there is an integer N_0 such that $\sigma_k(T) \in B$ for $k \geqq N_0$. Choose an integer N_1 so that $t_k > T$ if $k \geqq N_1$. Let $N = \max\{N_0, N_1\}$. Then for $k \geqq N$, $\sigma_k(T) \in B$ and the length of the perpendicular from $\sigma_k(T)$ to τ is $\geqq 2c$. This fact contradicts the property (2′) since $t_k > T$.

We can now conclude that $\lim_{n\to\infty} \sigma'_n(0) = \tau'(0)$. Therefore $\lim_{n\to\infty} q_n = \tau(\infty)$. Q.E.D.

LEMMA 2.2. *Let M be a complete simply connected Riemannian manifold without focal points. If φ is a hyperbolic isometry of M which translates a geodesic α, then for a suitable orientation of α, $\varphi^n(p) \to \alpha(\infty)$ and $\varphi^{-n}(p) \to \alpha(-\infty)$ for any point $p \in M$ as $n \to \infty$.*

PROOF. Let φ translate α by $a \neq 0$. Reversing the orientation of α if necessary, we may assume that $a > 0$. For every integer n we have $d(\varphi^n(p), \alpha(na)) = d(p, \alpha(0))$. The sequence $\{\varphi^n(p)\}$ satisfies the properties in Lemma 2.1, since

$$
\begin{aligned}
d(\alpha(0), \varphi^n(p)) &\geqq d(\alpha(0), \alpha(na)) - d(\varphi^n(p), \alpha(na)) \\
&= d(\alpha(0), \alpha(na)) - d(p, \alpha(0)) .
\end{aligned}
$$

The result follows. Q. E. D.

As an immediate application of Lemma 2.2 we have the following.

PROPOSITION 2.3. *Let M be a complete simply connected Riemannian manifold without focal points. Suppose φ is a hyperbolic isometry of M. Then*

(1) *All axes of φ have the same end points.*

(2) *If an isometry ψ commutes with a positive power of φ, then ψ fixes the two end points of the axes of φ.*

Again let M be a complete simply connected Riemannian manifold without focal points. If γ is a geodesic ray in M we have a function F_t defined by $F_t(p) = d(p, \gamma(t)) - t$. The limit $F_\gamma = \lim_{t\to\infty} F_t$ is called a *Busemann function* at $\gamma(\infty) \in M(\infty)$. A Busemann function at $z \in M(\infty)$ is determined uniquely up to a constant and can be viewed as a distance function with respect to z. Busemann functions are at least of class C^2 (Theorem 2 in [5]). *Horospheres* with center z are defined to be the level sets of a Busemann function at z. It should be noted that if γ is a geodesic ray, then $F_\gamma(\gamma(0)) = 0$ and $F_\gamma(\gamma(t)) < 0$ for $t > 0$. If Σ is a horosphere with center $z \in M(\infty)$ which passes through $p \in M$ and γ a geodesic ray representing z, then the set $\{q \in M;\ F_\gamma(q) < F_\gamma(p)\}$, which is called the *horodisc* bounded by Σ, is convex (Theorem 2 in [5]).

For any two distinct points p and q of M the *bisector* $M(p, q)=\{m\in M; d(p, m)=d(m, q)\}$ is the locus of the centers of all spheres through p and q. A similar statement holds for horospheres. Precisely, denote $\overline{M(p, q)}$ the closure of $M(p, q)$ in the cone topology. Then

LEMMA 2.4. *Let M be a complete simply connected Riemannian manifold without focal points. Let p and q be distinct points in M. Then a point m of $\overline{M}$ is the center of a sphere or horosphere through p and q if and only if $m\in \overline{M(p, q)}$.*

PROOF. Define a function $f: M\to \boldsymbol{R}$ by $f(m)=d(p, m)-d(m, q)$. Then for $m_1, m_2\in M$ we have

$$|f(m_1)-f(m_2)|\leqq 2d(m_1, m_2)\,.$$

Therefore f is uniformly continuous, and extends to $M(\infty)$ continuously. It is clear that $f^{-1}(0)\cap M=M(p, q)$. The continuity of f to $M(\infty)$ implies $\overline{M(p, q)}\subset f^{-1}(0)$. Note that if F is a Busemann function at $z\in M(\infty)$, then $f(z)=F(p)-F(q)$.

On the other hand, let Σ be the horosphere with center $z\in M(\infty)$ which passes through p and q. We have to prove that $z\in\overline{M(p, q)}$. Let γ_{pz} and γ_{qz} be the geodesic rays from p to z and q to z, respectively. Set $p_t=\gamma_{pz}(t)$ and $q_t=\gamma_{qz}(t)$ for $t>0$. The function $t\mapsto d(q, p_t)-d(p, p_t)$ $(=-f(p_t))$ decreases strictly to zero. Hence $f(p_t)<0$ for all $t>0$. Similarly $f(q_t)>0$ for all $t>0$. Therefore the geodesic segments σ_t from p_t to q_t intersect $M(p, q)$. Choose points of intersection $r_t=\sigma_t\cap M(p, q)$. Then $d(r_t, p_t)<d(p_t, q_t)\leqq d(p, q)$. Here the last inequality follows from Fact 4. Thus the sequence $\{r_n\}$, $n\in \boldsymbol{N}$, converges to z by Lemma 2.1. Q. E. D.

$M(p, q)$ is a C^∞ submanifold of M of codimension one since it is the zero level surface of the function $m\mapsto d(p, m)-d(q, m)$, which is C^∞ on $M-\{p, q\}$. Note that the gradient of the function $m\mapsto d(p, m)-d(q, m)$ is nonzero at any point m in $M(p, q)$ since the gradients of $m\mapsto d(p, m)$ and $m\mapsto d(q, m)$ point radially outward from p and q respectively if $m\neq p$ and $m\neq q$. It is known that $M(p, q)$ is diffeomorphic to $\boldsymbol{R}^{n-1}$, where $n=\dim M$ (see [**15**] Proposition 2.6); the set $\{m\in M; d(p, m)\leqq d(q, m)\}$ is starshaped relative to p since the function $m\mapsto d(p, m)-d(q, m)$ is nondecreasing on geodesics starting at p.

Let Γ be a freely acting, properly discontinuous group of isometries of M. For any point $p\in M$ the canonical fundamental region, called a *Dirichlet region*, for Γ with center p is the set

$$D_p=\{m\in M;\ d(p, m)\leqq d(\varphi(p), m)\quad \text{for all } \varphi\in\Gamma\}.$$

D_p has properties: (1) $\Gamma D_p=M$, (2) The interior of D_p does not intersect any of its Γ-images, (3) For each side s of D_p there is another side $\bar{s}$ and an element $\varphi\in\Gamma$ such that $\varphi(\bar{s})=s$, and φD_p is a Dirichlet region for Γ adjacent to

D_p along $\bar{s}$, and (4) $\varphi D_p = D_{\varphi(p)}$ for $\varphi \in \Gamma$.

$\bar{D}_p$ denotes the closure of D_p in the cone topology.

LEMMA 2.5. *Let M be a complete simply connected Riemannian manifold without focal points and Γ a freely acting, properly discontinuous group of isometries of M. Let D be a Dirichlet region for Γ with center p. Let $z \in M(\infty)$. Let Σ be the horosphere with center z which passes through p, and B the horodisc bounded by Σ. Then $z \in \bar{D}$ if and only if $\Gamma(p) \subset M - B$.*

PROOF. Suppose that $\varphi(p) \in B$ for some $\varphi \in \Gamma$. Then $\overline{M(p, \varphi(p))} \not\ni z$ by Lemma 2.4. The extended geodesic segment α from p to $\varphi(p)$ beyond $\varphi(p)$ strikes Σ at a point, say q (possibly z). If $q = z$, it is clear that $\overline{M(p, \varphi(p))}$ separates z from $\bar{D}$. Hence we assume that $q \neq z$. We recall that $\overline{M(p, \varphi(p))}$ divides $\bar{M}$ into two components. We have that $M(p, q) \cap M(p, \varphi(p)) = \emptyset$ since for each $m \in M$ the function $\alpha \ni r \mapsto d^2(r, m)$ is strictly convex (cf. [3] Lemma 1). Moreover $z \in \overline{M(p, q)}$ by Lemma 2.4. From these informations it follows that $\overline{M(p, \varphi(p))}$ separates z from $\bar{D}$. Thus $z \notin \bar{D}$.

Conversely, we suppose that $\Gamma(p) \subset M - B$. Let β be the geodesic ray with unit speed such that $\beta(0) = p$ and $\beta(\infty) = z$. For any point $q \in M - B$, the function $t \mapsto d(q, \beta(t)) - d(p, \beta(t))$ is decreasing strictly to zero. Therefore β does not intersect $M(p, \varphi(p))$ for any $\varphi \in \Gamma$, $\varphi \neq 1$. Also by Lemma 2.4 $\overline{M(p, \varphi(p))} \not\ni z$ for any $\varphi \in \Gamma$. Therefore, for every $\varphi \in \Gamma$ the closure of the set $\{m \in M;\ d(p, m) \leqq d(m, \varphi(p))\}$ contains $\beta \cup \{z\}$ since the set is starshaped. Thus $z \in \bar{D}$. Q.E.D.

We are ready to prove the following theorem.

THEOREM 1. *Let M be a complete simply connected Riemannian manifold without focal points and Γ a freely acting, properly discontinuous group of isometries of M. Let D be a Dirichlet region for Γ with center p. If $z \in M(\infty)$ is a point containing the axis of a hyperbolic element φ in Γ, then $\Gamma(z) \cap \bar{D} = \emptyset$.*

PROOF. Let α be the axis of φ such that $\alpha(\infty) = z$. Let φ translates α by $a \neq 0$. Suppose $a > 0$. By Lemma 2.2 $\varphi^n(p) \to z$ as $n \to \infty$. For simplicity, we parametrize α as follows: Let Σ denote the horosphere through p with center z. Define the intersection of α with Σ to be the point $\alpha(0)$. Let B_t denote the geodesic ball centered at $\alpha(t)$ with radius t. If B denotes the horodisc bounded by Σ, then $B_t \subset B$ for all $t > 0$. Choose a positive integer N so that $na > d(p, \alpha(0))$ for $n \geqq N$. Then $\varphi^n(p) \in B_{na}$ for $n \geqq N$. If $a < 0$, $\varphi^{-n}(p) \in B_{-na}$ for $n \geqq N$. Therefore it follows from Lemma 2.5 that $z \notin \bar{D}$. For $\psi \in \Gamma$, $\psi(z)$ is a fixed point of a hyperbolic isometry $\psi\varphi\psi^{-1}$, and hence $\psi(z) \notin \bar{D}$ as before.

Q. E. D.

REMARK. Contrary to Theorem 1, for a fixed point $z \in M(\infty)$ of a parabolic isometry φ of Γ we can prove that $z \in \bar{D}$ under a certain condition. Seemingly the condition itself requires to restrict curvatures. So I consider this problem would belong to a different field. The result would be published elsewhere.

3. Limit set of a discrete group Γ with volume (M/Γ) finite.

In this section we shall prove the following theorem:

THEOREM 2. *Let M be a complete simply connected Riemannian manifold without focal points and Γ a freely acting properly discontinuous group of isometries of M. If* volume (M/Γ) *is finite, the limit set $L(\Gamma)$ is the whole boundary $M(\infty)$.*

The proof is essentially the same as that of Satz in §4 in [12], where he proved the statement for a manifold with nonpositive curvature. We shall give the complete proof.

Let M and Γ be as in the theorem. D denotes a Dirichlet region for Γ in M. Consider the unit sphere bundle SM of M and let $\pi: SM \to M$ be the natural projection. $\tilde{D}=\pi^{-1}D$ is a Dirichlet region for Γ in SM and has finite measure. For a Borel set A of M we denote by $\mu(A)$ its measure. If G_t is the geodesic flow of SM, then the volume element of SM is invariant under G_t (Liouville) and G_t commutes with isometries. Therefore $G_t\tilde{D}$ is a Dirichlet region for Γ in SM and $\mu(G_t\tilde{D})=\mu(\tilde{D})$ for all t.

LEMMA 3.1. *For all open sets $U\neq\emptyset$ of SM there is an infinite sequence $\{t_i\}$ with $t_i\to\infty$ such that $G_{t_i}U\cap\Gamma U\neq\emptyset$ for all t_i.*

In fact, $\mu(G_t\Gamma U\cap\tilde{D})=\mu(\Gamma U\cap G_t^{-1}\tilde{D})=\mu(\Gamma U\cap\tilde{D})$. Therefore $\mu(G_t\Gamma U\cap\tilde{D})\geqq 0$ is independent of t. Since $\tilde{D}$ has finite measure, for $t_0>0$ all $G_{2nt_0}(\Gamma U)$, $n\in\boldsymbol{N}$, are not disjoint. Hence $G_t(\Gamma U)\cap\Gamma U\neq\emptyset$ for $t=2(m-n)t_0>t_0$, and $G_{t_1}U\cap\Gamma U\neq\emptyset$ for some $t_1\geqq 2t_0$. Repeating the argument we have the assertion. Q. E. D.

LEMMA 3.2. *The set*

$$L=\{v\in SM;\ \exists\{t_i\}\ \textit{with}\ t_i\to\infty,\ \exists\{\varphi_i\}\subset\Gamma\ \textit{such that}\ \lim G_{t_i}\varphi_{i*}(v)=v\}$$

is dense in SM.

PROOF. Let $v\in SM$ and U an open neighborhood of v. Let V_1 be an open subset of U with bounded diameter. By Lemma 3.1, $V_1\cap G_{t_1}{}^{-1}\varphi_1{}^{-1}V_1\neq\emptyset$ for some $t_1>1$ and for some $\varphi_1\in\Gamma$. We define sequences $\{V_k\}$ of open sets and $\{\varphi_k\}\subset\Gamma$ so that

$$\overline{V_{k+1}}\subset V_k\cap G_{t_k}{}^{-1}\varphi_k{}^{-1}(V_k)\ \text{ and }\ \text{diameter}(V_{k+1})\leqq\frac{1}{2}\text{diameter}(V_k).$$

Then diameter$(V_k)\to 0$ as $k\to\infty$. Let v_0 be the limit of $\{V_k\}$. Then $v_0\in U$. Since $v_0\in V_{k+1}$, $G_{t_k}\varphi_{k*}(v_0)\in V_k$ and the sequence $\{G_{t_k}\varphi_{k*}(v_0)\}$ converges to v_0. Therefore $v_0\in U\cap L$. Q. E. D.

PROOF OF THE THEOREM. Let $z\in M(\infty)$, $p\in M$ and τ the geodesic ray connecting p to z. Owing to Lemma 3.2 we have a sequence $\{v_n\}\subset L$ with $v_n\to\tau'(0)$ as $n\to\infty$. Then there are sequences $\{\varphi_n\}\subset\Gamma$ and $\{t_n\}$ with $t_n\to\infty$ such that $\lim G_{t_n}\varphi_{n*}(v_n)=\tau'(0)$. It follows that

$$\lim d(\pi G_{t_n}\varphi_{n*}(v_n),\ \pi v_n)=0\,.$$

Putting $p_n=\pi v_n$ and $q_n=\pi G_{t_n}(v_n)$, we have that

$$\lim d(q_n,\ \varphi_n^{-1}p_n)=0\,.$$

Thus, to prove that $\lim \varphi_n^{-1}(p)=z$ (and hence $z\in L(\Gamma)$), it suffices to show that $\lim q_n=z$.

Since $\lim d(p,\ p_n)=0$, $\lim d(p_n,\ q_n)=\lim t_n=\infty$ and

$$d(p,\ q_n)\geqq d(p_n,\ q_n)-d(p,\ p_n)\,,$$

it follows that $\lim d(p,\ q_n)=\infty$.

If τ_n denotes the geodesic ray with the initial condition v_n, and σ_n the geodesic segment connecting p to q_n with $\sigma_n(0)=p$, then

$$d(\tau(1),\ \sigma_n(1))\leqq d(\tau(1),\ \tau_n(1))+d(\tau_n(1),\ \sigma_n(1))\,.$$

If $d(p,\ q_n)-d(p_n,\ q_n)=l\geqq 0$, then

$$d(\sigma_n(1),\ \tau_n(1))\leqq d(\sigma_n(1),\ \sigma_n(1+l))+d(\sigma_n(1+l),\ \tau_n(1))$$
$$\leqq l+d(p_n,\ \sigma_n(l))\,.$$

Since $l\leqq d(p,\ p_n)$, it follows that $\sigma_n(1)\to\tau(1)$ as $n\to\infty$. For the case $l<0$, take $\tau_n(1-l)$ and $\tau_n(-l)$ instead of $\sigma_n(1+l)$ and $\sigma_n(l)$ respectively. The similar argument yields the same conclusion. Fact 5 gives rise to the inequality

$$\tilde{d}(\tau'(0),\ \sigma_n'(0))\leqq d(\tau(1),\ \sigma_n(1))\,,$$

where $\tilde{d}$ denotes the distance function with respect to the new metric of T_pM. Therefore $\sigma_n'(0)$ converges to $\tau'(0)$ as $n\to\infty$. Thus $\lim q_n=z$. Q. E. D.

Let M and Γ be as in the theorem. For $p\in M$ and $\varphi\in\Gamma$, let $\gamma:[0,1]\to M$ be the unique geodesic with $\gamma(0)=p$ and $\gamma(1)=\varphi(p)$, parametrized proportionally to arc-length. Then we have a fundamental vector field v defined by $v_\varphi(p)=\gamma'(0)$. v_φ is nowhere zero if $\varphi\neq 1$.

COROLLARY 3.3. *Let M be a complete simply connected Riemannian manifold without focal points and Γ a freely acting, properly discontinuous group of isometries of M with* volume$(M/\Gamma)<\infty$. *Then at any point of M the set*

$$\{v_\varphi(p)/\|v_\varphi(p)\|\in T_pM;\ 1\neq\varphi\in\Gamma\}$$

is dense in $\{v\in T_pM;\ \|v\|=1\}\subset SM$.

This is an immediate consequence of Theorem 2.

REMARK. As an application of Lemma 2.1 we can prove the following statement which is the special case of the above corollary and answers a question raised by K. Uesu: Let N be a compact Riemannian manifold without focal points and let p be a point of N. Then in $S_pN=\{v\in T_pN;\ \|v\|=1\}$ we can find a dense subset W so that every geodesic issuing from p in the direction $w\in W$ is closed. See also Lemma 1 in § 1 [**13**].

4. Fixed point theorem.

Let N be a compact topological space and $C(N)$ the algebra of $\boldsymbol{R}$-valued continuous functions on N. With supremum norm it is a Banach space. An element of the dual space $C^*(N)$ is called a *Radon measure* or simply *measure*. For $f\in C(N)$ and $\mu\in C^*(N)$, $\mu(f)\in\boldsymbol{R}$ is called the *integral* of f with respect to μ and is denoted by $\int_N f(x)d\mu(x)$. A nonzero measure μ is said to be *positive* if $\mu(f)\geqq 0$ for any nonnegative $f\in C(N)$. These notion extend to vector valued maps from N as well. That is: Let $\boldsymbol{R}^n$ be an n-dimensional Euclidean space. Consider a continuous map $X\colon N\to\boldsymbol{R}^n$. If $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $\boldsymbol{R}^n$, then $X(x)=\sum_{i=1}^n \alpha^i(x)e_i$, where the α^i are continuous functions from N to $\boldsymbol{R}$. If μ is a measure on N, we define the integral of X with respect to μ by

$$\int_N X(x)d\mu(x)=\sum_{i=1}^n\left(\int_N \alpha^i(x)d\mu(x)\right)e_i\,.$$

Now, we assume that M is a complete simply connected Riemannian manifold without focal points and f is a continuous map from N to M. We can choose a strongly convex geodesic ball B in M so that $f(N)\subseteqq B$ by Fact 2. Let us consider a continuous vector field $x\mapsto\exp_p^{-1}f(x)$ on N for each $p\in B$.

We are required to choose a positive measure μ on N. Our current purpose is to show that the tangent vector field V defined by $V_p=\int_N \exp_p^{-1}f(x)d\mu(x)$ has only one zero in B. V is differentiable.

PROPOSITION 4.1. *The vector field V on B has only one zero in B, and hence in M.*

To prove the proposition we shall follow the technique used in [8]. For $A\in T_pM$ with $\|A\|=1$, let $\gamma(u)=\exp_p uA$. We consider the family of geodesic $c(t, u)=\exp_{\gamma(u)}((1-t)\exp_{\gamma(u)}{}^{-1}f(x))$, $x\in N$. Denoting by $Y_u(t)$ the Jacobi field along the geodesic $t\mapsto c(t, u)$ defined by $Y_u(t)=\dfrac{\partial}{\partial u}c(t, u)$, then $Y_u(1)=\gamma'(u)$ and $\dfrac{\nabla}{dt}Y_u\Big|_{t=1}=-\dfrac{\nabla}{du}(\exp_{\gamma(u)}{}^{-1}f(x))$. Since $\dfrac{\nabla}{du}(\exp_{\gamma(u)}{}^{-1}f(x))$ is continuous in x and $\gamma(u)$ simultaneously on $N\times B$, $\int_N \dfrac{\nabla}{du}(\exp_{\gamma(u)}{}^{-1}f(x))d\mu(x)$ exists and $\dfrac{\nabla}{du}V_{\gamma(u)}=$

$\int_N \frac{\nabla}{du}(\exp_{\gamma(u)}{}^{-1}f(x))d\mu(x)$. It follows from Fact 1 that

$$\left\langle Y_u(1), \frac{d}{dt}Y_u\Big|_{t=1}\right\rangle = \left\langle \gamma'(u), -\frac{\nabla}{du}(\exp_{\gamma(u)}{}^{-1}f(x))\right\rangle > 0 .$$

Therefore, integrating over N, we have $\left\langle \gamma'(u), -\frac{\nabla}{du}V_{\gamma(u)}\right\rangle > 0$, because μ is positive. Namely, we have the following lemma:

LEMMA 4.2. *Along any geodesic γ issuing from a zero of V the $\gamma'(u)$-component of $-V$ is strictly increasing.*

From this lemma we have some consequences. First we claim that V has at most one (isolated) zero. In fact, suppose that there are two zeros p and q in B. Join p and q by a geodesic segment γ with $\gamma(0)=p$. Then by the lemma, the γ'-component of $-V$ is strictly increasing, which implies that q cannot be a zero of V. Also from the lemma it follows that the index of $-V$ at an (isolated) zero is $+1$.

On the other hand, since B is a convex geodesic ball and the continuous vector field V is an average over vectors pointing inward on the boundary of B, the index of $-V$ must be $+1$ by Hopf's theorem, which guarantees the existence of zeros of V. Thus our proposition is proved. Q. E. D.

We call the zero of V the *center of mass* of $f(N)$ with respect to μ.

As an application of Proposition 4.1 we can prove

THEOREM 3. *Every compact group G of isometries of a complete simply connected Riemannian manifold M without focal points has a fixed point.*

PROOF. We choose any point $p_0 \in M$ and define a map $f: G \to M$ by $f(a) = a(p_0)$ for $a \in G$. Let μ be a Haar measure on G. Proposition 4.1, with $N=G$, guarantees the existence of the center of mass of $f(G)$ with respect to μ. We claim that this center of mass is a fixed point of G. We prove the invariance of the center of mass of $f(G)$ under G by using the left invariance of the Haar measure μ. Let $\varphi \in G$. Suppose that p is the center of mass of $f(G)$ with respect to μ. Recalling that $\varphi_* \cdot \exp_p{}^{-1} = \exp_{\varphi(p)}{}^{-1} \cdot \varphi$, we have

$$\begin{aligned}\int_G \exp_{\varphi(p)}{}^{-1}f(x)d\mu(x) &= \int_G \exp_{\varphi(p)}{}^{-1}x(p_0)d\mu(x)\\ &= \int_G \varphi_* \exp_p{}^{-1}(\varphi^{-1}x)(p_0)d\mu(x)\\ &= \varphi_* \int_G \exp_p{}^{-1}(\varphi^{-1}x)(p_0)d\mu(\varphi^{-1}x)\\ &= \varphi_* \int_G \exp_p{}^{-1}f(x)d\mu(x)\\ &= 0 .\end{aligned}$$

Q. E. D.

We also have the following immediate corollary of the theorem.

COROLLARY 4.3. *Let M be a connected, simply connected homogeneous Riemannian manifold without focal points. Let G be a closed subgroup of the group of isometries of M. Assume that G is transitive so that $M=G/H$, where H is the isotropy subgroup of G at a point of M. Then H is a maximal compact subgroup of G (and every maximal compact subgroup of G is conjugate to H).*

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