J. Math. Soc. Japan
Vol. 35, No. 1, 1983

Equivariant function spaces and bifurcation points

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(Received May 7, 1979)
(Revised July 31, 1981)

1. Introduction.

Let E be a real Banach space, n a positive integer, $\lambda_0 \in \boldsymbol{R}^n$, and U an open subset of $E \times \boldsymbol{R}^n$ containing $(0, \lambda_0)$. We denote points of U by (x, λ). Let $T: U \to E$ be a compact linear map and $G: U \to E$ a compact map with $G = o(\|x\|)$ uniformly on bounded domain of λ. In [1], Alexander studied bifurcation of nonlinear equations of the form:

$$x = T(\lambda)x + G(x, \lambda)$$

for λ near λ_0. He constructed a topological invariant, which is an element of the stable homotopy group of spheres, and proved that the non-vanishing of this invariant implies global bifurcation from $(0, \lambda_0)$.

In this paper, we assume further that n is even and T, G are equivariant for a linear S^1-action on E which is free on $E-0$. We construct a topological invariant ω_T^s which lies in the homotopy group of the infinite general linear group over $\boldsymbol{C}$. Then Alexander's condition for bifurcation is expressed in terms of ω_T^s as follows:

$$\omega_T^s \text{ is not divisible by } \begin{array}{ll} 2 & n=8k+2, \\ b_k & n=4k,\ k \text{ is odd}, \\ b_k/2 & n=4k,\ k \text{ is even}, \end{array}$$

where b_k denotes the denominator of $B_k/4k$ for the k-th Bernoulli number B_k. Our main result is

THEOREM 1. *If ω_T^s is not zero, then $(0, \lambda_0)$ is a bifurcation point.*

The proof of this theorem is based on results in Becker and Schultz [3], concerning equivariant homotopy theory.

We give two applications in § 4. First, complex nonlinear eigenvalue problems are considered under the condition that the nonlinear term commutes with scalar multiplication by any complex number with norm one. It is shown that ω_T^s is nonzero for each characteristic value. Therefore, by our main theorem, we can say that bifurcation occurs from all characteristic values. It is known that the same conclusion holds under the quite different condition that the nonlinear term is complex analytic. See Dancer [**5**] and Schwartz [**11**]. The general complex nonlinear eigenvalue problem was studied by Ize [**9**]. He proved that global bifurcation occurs from characteristic values of odd multiplicity.

The second application treats parametrized autonomous differential systems

$$\frac{dx}{dt}=f(x, \lambda)$$

defined on an open set of $\boldsymbol{R}^n$, where n is a positive even integer and λ is a real parameter. We assume that f is equivariant for a linear S^1-action on $\boldsymbol{R}^n$ which is free on $\boldsymbol{R}^n-0$. Then, it is proved that bifurcation of periodic orbits from a stationary point occurs under a different condition from those of Alexander and Yorke [**2**] and Chow, Mallet-Paret and Yorke [**4**].

2. The main theorem.

Let S^1 be the group of complex numbers with norm one. Let X be a topological space with an S^1-action. A subset A of X is said to be *S^1-invariant* if $\alpha(A)=A$ for any $\alpha\in S^1$. For another Y with an S^1-action, a continuous map $f: X\to Y$ is *S^1-equivariant* if it commutes with any $\alpha\in S^1$. Let X and Y be metric spaces. A continuous map $f: X\to Y$ is *compact* if it maps every bounded subset of X into a compact subset of Y.

Now let E be a real Banach space with an S^1-action $x\to\alpha\cdot x$, $\alpha\in S^1$, which is linear and *free*, that is, $\alpha\cdot x$ is linear with respect to x and $\alpha\cdot x\neq x$ for any $\alpha\neq 1$, $x\neq 0$. Let S^1 act on $E\times\boldsymbol{R}^n$ by $(x, \lambda)\to(\alpha\cdot x, \lambda)$, where n is a positive even integer. Let U be an S^1-invariant open neighborhood of $(0, \lambda_0)$ in $E\times\boldsymbol{R}^n$. We consider a nonlinear equation:

$$x=T(\lambda)x+G(x, \lambda) \tag{2.1}$$

defined in U, where

(2.2)
i) $T(\lambda)x$ and $G(x, \lambda)$ are compact maps from U to E,
ii) $T(\lambda)x$ is linear with respect to x, S^1-equivariant and the map $\lambda\to T(\lambda)$ is continuous with respect to the operator norm,
iii) $G(x, \lambda)=o(\|x\|)$ uniformly for λ near λ_0,
iv) G is S^1-equivariant.

The point $(0, \lambda_0)$ is called a *bifurcation point* if it is an accumulation point of the set of solutions (x, λ) of (2.1) with $x\neq 0$.

If we define a bounded linear operator $J: E\to E$ by $Jx=\sqrt{-1}\cdot x$ and an S^1-invariant norm $|||\ |||$ by $|||x|||=\int_{S^1}\|\alpha\cdot x\|$, then we obtain a complex Banach space $(E, J, |||\ |||)$ denoted also by E. Furthermore, from Mostow [**10**, p. 31], the action

$\alpha\cdot x$ coincides with scalar multiplication. Note that assumption (2.2) ii) implies that $T(\lambda)$ is a complex linear operator of E.

Now suppose that $I-T(\lambda)$ is invertible for λ near λ_0, except possibly $\lambda=\lambda_0$. Then the restriction of the map $I-T(\lambda)$ to a small sphere around λ_0 determines an element ω_T of the homotopy group $\pi_{n-1}(GL_c(E))$, where $GL_c(E)$ denotes the topological group of invertible complex linear operators of the form "I+compact". We define an element ω_T^s of the homotopy group $\pi_{n-1}(GL_C)$, where $GL_C=\lim_m GL(\boldsymbol{C}^m)$: the infinite general linear group, as follows. If E is infinite dimensional, ω_T^s is the image of ω_T under the isomorphism induced from the homotopy equivalence $GL_c(E)\to GL_C$ given by Elworthy and Tromba [**6**, Theorem 1.3]. If E is finite dimensional ω_T^s is the stabilization of the image of ω_T under the natural isomorphism $\pi_{n-1}(GL_c(E))\to\pi_{n-1}(GL(\boldsymbol{C}^d))$, where d is the dimension of E. It is well known that $\pi_{n-1}(GL_C)$ is isomorphic to the group of integers. We now state the main theorem.

THEOREM 1. *If ω_T^s is not zero, then $(0, \lambda_0)$ is a bifurcation point.*

REMARK. Let $i: GL_C\to GL_R$ be the natural inclusion. Then ω_T^s is mapped to Alexander's invariant γ_T^s by the homomorphism $i_*: \pi_{n-1}(GL_C)\to\pi_{n-1}(GL_R)$. Alexander's condition for bifurcation is that $J(\gamma_T^s)\neq 0$, where J is the J-homomorphism from $\pi_*(GL_R)$ to π_*^s: the stable homotopy group of spheres. This condition is equivalent to the one that

$$\omega_T^s \text{ is not divisible by } \begin{cases} 2 & n=8k+2, \\ b_k & n=4k,\ k \text{ is odd}, \\ b_k/2 & n=4k,\ k \text{ is even}. \end{cases}$$

Here the number b_k is the denominator of $B_k/4k$ for the k-th Bernoulli number. For the definition of Bernoulli numbers, see [**1**, p. 38].

3. Proof of Theorem 1.

For the proof, it is enough to consider the case where E is infinite dimensional. The reason is as follows. Suppose that E is finite dimensional. Choose an infinite dimensional Banach space E'. Denote by $P: E\oplus E'\to E$ the projection, $J: E\to E\oplus E'$ the inclusion and by I' the identity on E'. Then the solution set $(I-T-G)^{-1}(0)$ may be identified with the set $(I\oplus I'-J\circ T\circ P-J\circ G\circ P)^{-1}(0, 0)$ by $x\to(x, 0)$ and it is easily shown that $\omega_T^s=\omega_{J\circ T\circ P}^s$. Therefore we need consider only the infinite dimensional case.

We prepare several lemmas. The following one generalizes Lemma 3.2 in Geba [**7**]. Since the proof is quite analogous, we omit it.

LEMMA 1. *Let X be a closed bounded subset of E and Λ a compact metric space. Let $f: X\times\Lambda\to E$ be a continuous map such that the map $I-f: X\times\Lambda\to E$*

defined by $(I-f)(x, \lambda)=x-f(x, \lambda)$ *is compact. Then the set* $f(X\times\Lambda)$ *is closed in* E.

Let a and b be positive numbers. Denote by S the sphere of radius a in E and by S^{n-1} the sphere of radius b in $\boldsymbol{R}^n$. Then $S\times S^{n-1}$ is an S^1-invariant subset of $E\times\boldsymbol{R}^n$. If X is a topological space with an S^1-action and Y is another one, then we denote by $F(X, Y)_{S^1}$ the set of S^1-equivariant maps from X to Y. Put

$$F_{E,n}=\{f\in F(S\times S^{n-1}, E-0)_{S^1} \mid I-f\colon S\times S^{n-1}\to E \text{ is compact}\},$$

where the map $I-f$ is defined by $(I-f)(x, \lambda)=x-f(x, \lambda)$. We say two elements f_0, f_1 of $F_{E,n}$ are homotopic if there exists an $F\in F(S\times S^{n-1}\times[0, 1], E-0)_{S^1}$ such that $F(x, \lambda, 0)=f_0(x, \lambda)$, $F(x, \lambda, 1)=f_1(x, \lambda)$ and the map $I-F\colon S\times S^{n-1}\times[0, 1]\to E$ defined by $(I-F)(x, \lambda, t)=x-F(x, \lambda, t)$ is compact. We denote by $[S\times S^{n-1}, E-0]^c_{S^1}$ the set of all homotopy classes of elements of $F_{E,n}$.

Let Γ denote the set of all finite dimensional subspaces of the complex Banach space E. For $L\in\Gamma$, denote by $S(L)$ the sphere of radius a of L. For $L, M\in\Gamma$ with $L\subset M$, we define a map

$$\sigma_{L,M}\colon [S(L)\times S^{n-1}, L-0]_{S^1} \longrightarrow [S(M)\times S^{n-1}, M-0]_{S^1},$$

where $[\ ,\ ]_{S^1}$ denotes S^1-homotopy classes of S^1-equivariant maps, as follows. Choose a subspace N of M such that $M=L\oplus N$. For $f\in F(S(L)\times S^{n-1}, L-0)_{S^1}$, define a map $\tilde{f}\in F(L\times S^{n-1}, L)_{S^1}$ by

$$\begin{aligned}\tilde{f}(x, \lambda)&=|||x|||f(x/|||x|||, \lambda) && x\neq 0,\\ &=0 && x=0.\end{aligned}$$

Furthermore, define a map $\sigma_{L,M,N}(f)\in F(S(M)\times S^{n-1}, M-0)_{S^1}$ as to be the restriction of the map $\tilde{f}\oplus I_N$ to $S(M)\times S^{n-1}$, where $\oplus$ denotes direct product of maps and I_N is the identity of N. Then we obtain a map $\sigma_{L,M,N}\colon F(S(L)\times S^{n-1}, L-0)_{S^1}\to F(S(M)\times S^{n-1}, M-0)_{S^1}$. Clearly $\sigma_{L,M,N}$ induces a map

$$\sigma'_{L,M,N}\colon [S(L)\times S^{n-1}, L-0]_{S^1}\to[S(M)\times S^{n-1}, M-0]_{S^1}.$$

This map does not depend on the choice of N. This is verified as follows. Let N' be another one. Then an element z of $S(M)$ is expressed as $z=x_1+y_1=x_2+y_2$, where $x_1, x_2\in L$, $y_1\in N$ and $y_2\in N'$. For t with $0\leqq t\leqq 1$, put $x_t=tx_1+(1-t)x_2$, $y_t=ty_1+(1-t)y_2$ and define a map $\sigma_t(f)\in F(S(M)\times S^{n-1}, M-0)_{S^1}$ by

$$\sigma_t(f)(z, \lambda)=\tilde{f}(x_t, \lambda)+y_t.$$

Then $\sigma_t(f)$ gives a homotopy between $\tilde{f}\oplus I_N|_{S(M)\times S^{n-1}}$ and $\tilde{f}\oplus I_{N'}|_{S(M)\times S^{n-1}}$. Hence $\sigma'_{L,M,N}=\sigma'_{L,M,N'}$. Therefore if we put $\sigma_{L,M}=\sigma'_{L,M,N}$, it is well defined. We have also a map

$$\sigma_L: [S(L)\times S^{n-1}, L-0]_{S^1} \longrightarrow [S\times S^{n-1}, E-0]^c_{S^1}$$

induced from $f\to\tilde{f}\oplus I_N|_{S\times S^{n-1}}$, where $E=L\oplus N$. Put $\sigma=\lim_{L\in\Gamma} \sigma_L$.

LEMMA 2. *$\sigma: \lim_{L\in\Gamma} [S(L)\times S^{n-1}, L-0]_{S^1}\to[S\times S^{n-1}, E-0]^c_{S^1}$ is injective.*

PROOF. Let f and g be two elements of $F(S(L)\times S^{n-1}, L-0)_{S^1}$. Suppose that $\sigma_L([f])=\sigma_L([g])$, where $[\ ]$ denotes the homotopy class. Then if we choose an N with $E=L\oplus N$, there exists a homotopy $h\in F(S\times S^{n-1}\times[0, 1], E-0)_{S^1}$ between $\tilde{f}\oplus I_N|_{S\times S^{n-1}}$ and $\tilde{g}\oplus I_N|_{S\times S^{n-1}}$ with $I-h: S\times S^{n-1}\times[0, 1]\to E$ compact. Let ε be the distance between the set $h(S\times S^{n-1}\times[0, 1])$ and the origin of E. ε is positive, since by Lemma 1 $h(S\times S^{n-1}\times[0, 1])$ is closed in E. Using the well known approximation lemma for compact maps ([7, Lemma 3.1]), there exist an $M\in\Gamma$ with $L\subset M$ and a compact map $K: S\times S^{n-1}\times[0, 1]\to M$ which is an ε-approximation of $I-h$. Then the map $I-K$ is an ε-approximation of h. Put

$$h'(x, \lambda, t)=\int_{\alpha\in S^1}\bar{\alpha}(I-K)(\alpha x, \lambda, t)\,.$$

Then h' is an S^1-equivariant ε-approximation of h. This implies that $[h_i'|_{S(M)\times S^{n-1}}]=[h_i|_{S(M)\times S^{n-1}}]$ in $[S(M)\times S^{n-1}, M-0]_{S^1}$, where $i=0, 1$ and $h_i=h(\ ,\ , i)$, $h_i'=h'(\ ,\ , i)$. Since

$$[h_0|_{S(M)\times S^{n-1}}]=[\tilde{f}\oplus I_{M\cap N}|_{S(M)\times S^{n-1}}]=\sigma_{L,M}([f])\,,$$
$$[h_1|_{S(M)\times S^{n-1}}]=[\tilde{g}\oplus I_{M\cap N}|_{S(M)\times S^{n-1}}]=\sigma_{L,M}([g])\,,$$
$$[h_0'|_{S(M)\times S^{n-1}}]=[h_1'|_{S(M)\times S^{n-1}}]\,,$$

this implies that $\sigma_{L,M}([f])=\sigma_{L,M}([g])$. This completes the proof. q.e.d.

For a continuous map $f: S^{n-1}\to GL_c(E)$, define a map $\phi(f): S\times S^{n-1}\to E-0$ by $\phi(f)(x, \lambda)=f(\lambda)x$. Then it is not difficult to verify that $I-\phi(f)$ is compact. Hence $\phi(f)$ is an element of $F(S\times S^{n-1}, E-0)^c_{S^1}$. ϕ induces a map from the homotopy set $[S^{n-1}, GL_c(E)]$ to $[S\times S^{n-1}, E-0]^c_{S^1}$. We denote it by the same letter ϕ.

LEMMA 3. *The map $\phi: [S^{n-1}, GL_c(E)]\to[S\times S^{n-1}, E-0]^c_{S^1}$ is injective.*

PROOF. For $L\in\Gamma$, denote by $GL(L)$ the general linear group of L and by $F_{S^1}(L)$ the topological space of S^1-equivariant maps from $S(L)$ to $L-0$ with the uniform topology. For another $M\in\Gamma$ with $L\subset M$, define a map

$$\sigma^1_{L,M}: [S^{n-1}, F_{S^1}(L)] \longrightarrow [S^{n-1}, F_{S^1}(M)]$$

by $\sigma^1_{L,M}([f])=[(\tilde{f}\oplus I_N|_{S(M)})]$, where $M=L\oplus N$ and the map $\tilde{f}: S^{n-1}\to F(L, L)_{S^1}$ is defined by $\tilde{f}(\lambda)(x)=|||x|||f(\lambda)\,(x/|||x|||)$ for $x\neq 0$, $f(\lambda)(0)=0$. We also define maps

$$\sigma^2_{L,M}: [S^{n-1}, GL(L)] \longrightarrow [S^{n-1}, GL(M)]$$

and

$$\sigma^2_L: [S^{n-1}, GL(L)] \longrightarrow [S^{n-1}, GL_c(E)]$$

by $\sigma_{L,M}^2([T])=[T\oplus I_N]$ and $\sigma_L^2([T])=[T\oplus I_{N'}]$ where $M=L\oplus N$, $E=L\oplus N'$. These maps are well defined by the same reason as that for $\sigma_{L,M}$. It is easy to see that $\{[S^{n-1}, F_{S^1}(L)], \sigma_{L,M}^1\}$ and $\{[S^{n-1}, GL(L)], \sigma_{L,M}^2\}$ form inductive systems. Now we have the following commutative diagram:

$$\begin{tikzcd} \lim\limits_{L\in\Gamma}[S^{n-1}, GL(L)] \arrow[r, "\phi^1"] \arrow[d, "\sigma^2"] & \lim\limits_{L\in\Gamma}[S^{n-1}, F_{S^1}(L)] \arrow[r, "\iota"] & \lim\limits_{L\in\Gamma}[S(L)\times S^{n-1}, L-0]_{S^1} \arrow[d, "\sigma"] \\ {[S^{n-1}, GL_c(E)]} \arrow[rr, "\phi"] & & {[S\times S^{n-1}, E-0]_{S^1}^c}, \end{tikzcd}$$

where ϕ^1 is the map induced from the natural inclusion $GL(L)\to F_{S^1}(L)$, $\sigma^2=\lim\limits_{L\in\Gamma}\sigma_L^2$ and ι is the limit of the bijective maps $[S^{n-1}, F_{S^1}(L)]\to[S(L)\times S^{n-1}, L-0]_{S^1}$ which are defined in the same way as ϕ. It is easy to see that ι is bijective. According to Elworthy and Tromba [**6**, Theorem 1.3], σ^2 is a bijection. σ is injective by Lemma 2. Therefore the proof of the lemma is completed if we show that ϕ^1 is injective. This is done by using results in Becker and Schultz [**3**]. For a positive integer m, let $F_{S^1}(\boldsymbol{C}^m)$ denote the topological space of S^1-equivariant maps from the unit sphere of $\boldsymbol{C}^m$ to $\boldsymbol{C}^m-0$, where the S^1-action is given by scalar multiplication. In [**3**], it is shown that the map $\pi_*(GL_C)\to\pi_*(\lim\limits_m F_{S^1}(\boldsymbol{C}^m))$ induced by the inclusion is injective (Theorem 11.1) and that the pairs $(GL_C, GL(\boldsymbol{C}^m))$ and $(\lim\limits_m F_{S^1}(\boldsymbol{C}^m), F_{S^1}(\boldsymbol{C}^m))$ are $2m$ and $(2m-2)$-connected respectively (p. 25). These implies that the spaces $GL(\boldsymbol{C}^m)$ and $F_{S^1}(\boldsymbol{C}^m)$ are arcwise connected and that the map $\pi_{n-1}(GL(\boldsymbol{C}^m))\to\pi_{n-1}(F_{S^1}(\boldsymbol{C}^m))$ is injective for m sufficiently large. The spaces $GL(\boldsymbol{C}^m)$ and $F_{S^1}(\boldsymbol{C}^m)$ are simple, since they have the homotopy type of arcwise connected topological semi-groups. Therefore the homotopy sets $[S^{n-1}, GL(\boldsymbol{C}^m)]$ and $[S^{n-1}, F_{S^1}(\boldsymbol{C}^m)]$ may be identified with the homotopy groups $\pi_{n-1}(GL(\boldsymbol{C}^m))$ and $\pi_{n-1}(F_{S^1}(\boldsymbol{C}^m))$ respectively ([**8**, Lemma 16.11]). Hence the map $[S^{n-1}, GL(\boldsymbol{C}^m)]\to[S^{n-1}, F_{S^1}(\boldsymbol{C}^m)]$ is injective if m is sufficiently large. This implies that the map $[S^{n-1}, GL(L)]\to[S^{n-1}, F_{S^1}(L)]$ is injective if the dimension of L is sufficiently large. This shows that ϕ^1 is injective. Thus the proof is completed. q. e. d.

Now we prove Theorem 1. We assume that $\lambda_0=0$ without loss of generality. Suppose that (0, 0) is not a bifurcation point. Then there exists a small positive number b such that

$$x-T(\lambda)x-G(x, \lambda)\neq 0 \qquad \text{for} \quad 0<|||x|||\leqq b,\ |\lambda|\leqq b.$$

Furthermore by assumption (2.2) iii), we see that there exists a number a with $0<a\leqq b$ such that

$$x-T(\lambda)x-tG(x, \lambda)\neq 0 \qquad \text{for} \quad |||x|||=a, |\lambda|=b, 0\leqq t\leqq 1.$$

These imply that

$$[I-T(\lambda)]=[I-T(\lambda)-G(\ ,\lambda)]=[I-T(0)-G(\ ,0)] \quad \text{in} \quad [S\times S^{n-1}, E-0]^{c}_{S^1}.$$

We prove that

$$[I-T(0)-G(\ ,0)]=[I]$$

as follows. By Lemma 1 and the approximation lemma, there exist an $L\in\Gamma$ and an S^1-equivariant compact map $K: S\to L$ such that $[I-K\circ p]=[I-T(0)-G(\ ,0)]$, where $p: S\times S^{n-1}\to S$ is the projection. Both the homotopy classes $[I|_{S(L)\times S^{n-1}}]$ and $[I-K\circ p|_{S(L)\times S^{n-1}}]$ in $[S(L)\times S^{n-1}, L-0]_{S^1}$ are mapped to the unit element of the homotopy group $\pi_{n-1}(F_{S^1}(L))$ under the bijection $[S(L)\times S^{n-1}, L-0]_{S^1}\to \pi_{n-1}(F_{S^1}(L))$ given in the proof of Lemma 3. Hence $[I|_{S(L)\times S^{n-1}}]=[I-K\circ p|_{S(L)\times S^{n-1}}]$. Since

$$\sigma_L([I|_{S(L)\times S^{n-1}}])=[I] \quad \text{and} \quad \sigma_L([I-K\circ p|_{S(L)\times S^{n-1}}])=[I-K\circ p],$$

this shows that $[I]=[I-K\circ p]$. Thus the desired result is proved.

Let $i: \pi_{n-1}(GL_c(E))\to[S^{n-1}, GL_c(E)]$ be the natural identification. Then, since

$$(\phi\circ i)(\omega_T)=[I-T(\lambda)] \quad \text{and} \quad (\phi\circ i)(0)=[I],$$

the above arguments show that $(\phi\circ i)(\omega_T)=(\phi\circ i)(0)$. Therefore, by Lemma 3, $\omega_T=0$. Hence $\omega_T^{\sharp}=0$. This contradicts the assumption and we completes the proof of Theorem 1.

4. Applications.

We give two applications of Theorem 1 in this section. Let E be a complex Banach space and U an open set of $E\times \boldsymbol{C}$. We consider a nonlinear equation

$$x=T(\lambda)x+G(x, \lambda) \tag{4.1}$$

defined in U, where T and G satisfy the following:

(4.2)
i) $T(\lambda)x$ is a compact operator from $E\times\boldsymbol{C}$ to E. $T(\lambda)$ depends analytically on λ. There exists $\lambda_1\in\boldsymbol{C}$ such that $I-T(\lambda_1)$ is invertible.
ii) $G: U\to E$ is a compact map with $G(x, \lambda)=o(\|x\|)$ uniformly on bounded λ.
iii) $\alpha(U)=U$ and $G(\alpha x, \lambda)=\alpha G(x, \lambda)$ for any $\alpha\in S^1$.

It is well known that every bifurcation point is a characteristic value, that is, $I-T(\lambda_0)$ is not invertible. Suppose that λ_0 is a characteristic value. Then, by the argument similar to Ize [**9**, p. 89-91], $\omega_T^{\sharp}$ is equal to a positive integer called the multiplicity of λ_0. Therefore we obtain the following:

THEOREM 2. *A parameter value is a bifurcation point if and only if it is a characteristic value.*

Next, we apply our main theorem to Hopf bifurcation problem. We fix a free linear S^1-action on $\boldsymbol{R}^n$, where n is a positive even integer. Let U be an S^1-invariant open neighborhood of 0 in $\boldsymbol{R}^n$. Consider a parametrized autonomous differential system:

$$\frac{dx}{dt}=f(x, \lambda) \tag{4.3}$$

for $x\in U$ and λ in some real interval Λ. Here $f: U\times \Lambda\to \boldsymbol{R}^n$ is a continuous map such that

(4.4)
i) $f(0, \lambda)=0$,
ii) f is differentiable in x at $(0, \lambda)$ and $L(\lambda)=f_x(0, \lambda)$ is continuous in λ,
iii) f is S^1-equivariant in x.

Let $\overline{\boldsymbol{R}}^n$ denote the $n/2$-dimensional complex vector space defined in § 2. Then the assumption (4.4) iii) implies that $L(\lambda)$ is complex linear.

Now we assume that for any $\lambda\in\Lambda$ and any initial value $x\in U$, the system (4.3) has a unique solution $H(x, \lambda, t)$ for t in some real interval. Suppose that for $\lambda_0\in\Lambda$, $L(\lambda_0)$ is invertible and that, as a complex linear operator on $\overline{\boldsymbol{R}}^n$, $L(\lambda_0)$ has an eigenvalue $\sqrt{-1}\beta$ with $\beta\neq 0$, $\beta\in\boldsymbol{R}$. Put

$$M(\beta)=\{\sqrt{-1}m\beta \mid m \text{ is an integer, } \sqrt{-1}m\beta \text{ is an eigenvalue of } L(\lambda_0)\}.$$

Let p denote the number of elements of $M(\beta)$ counted with multiplicity. Cover each element z of $M(\beta)$ by a disc D_z which contains no eigenvalue of $L(\lambda_0)$ except z. We denote by $M_\lambda(\beta)$ the set of eigenvalues of $L(\lambda)$ which lie in the union of the disc D_z, $z\in M(\beta)$. If λ is sufficiently close to λ_0, then

$$M_\lambda(\beta)=\{\alpha_i(\lambda)+\sqrt{-1}\,\beta_i(\lambda) \mid i=1, 2, \cdots, p\}, \tag{4.5}$$

where $\alpha_i(\lambda)$, $\beta_i(\lambda)$ are real valued continuous functions. We assume that $\alpha_i(\lambda)\neq 0$ for any $\lambda\neq\lambda_0$. Note that $\alpha_i(\lambda_0)=0$ and that $\beta_i(\lambda_0)/\beta$ is an integer. Define $u_\pm$ to be the number of integers i such that

$$\alpha_i(\lambda_0\pm\varepsilon)\cdot\beta_i(\lambda_0)>0,$$

where ε is an arbitrary sufficiently small positive number.

THEOREM 3. *Under the above assumptions, if $u_+-u_-\neq 0$, then $(\lambda_0, 2\pi/|\beta|)$ is a bifurcation point, i.e., for every $\varepsilon>0$ there exist $(x_\varepsilon, \lambda_\varepsilon)\in U\times\Lambda$ and $t_\varepsilon>0$ such that $\|x_\varepsilon\|<\varepsilon$, $x_\varepsilon\neq 0$, $|(\lambda_\varepsilon, t_\varepsilon)-(\lambda_0, 2\pi/|\beta|)|<\varepsilon$ and x_ε lies on a closed orbit of the equation $dx/dt=f(x, \lambda_\varepsilon)$ with period t_ε.*

PROOF. By Alexander and Yorke [2, Lemma 3.1], H is written as $H(x, \lambda, t)=\exp\{tL(\lambda)\}x+G(x, \lambda, t)$, where G is S^1-equivariant in x, $G(x, \lambda, t)=o(\|x\|)$. Put $T(\lambda, t)=\exp\{tL(\lambda)\}$. From the assumption, $I-T(\lambda, t)$ is invertible for (λ, t) near $(\lambda_0, 2\pi/|\beta|)$. Therefore ω_T^s is defined and it is easy to see that ω_T^s is mapped to the integer u_+-u_- under the isomorphism $\pi_{n-1}(GL_C)\to \boldsymbol{Z}$. Thus by Theorem 1, the conclusion of this theorem holds. q. e. d.

In case where f is not necessarily S^1-equivariant, a sufficient condition for global Hopf bifurcation was obtained by Alexander and Yorke [2]. Let $r_+(r_-)$ be the number of conjugate pairs of eigenvalues of $L(\lambda)$ crossing the imaginary axis from left to right (from right to left) as λ increases past λ_0. They proved that global bifurcation of periodic orbits occurs if r_+-r_- is odd. Later Chow, Mallet-Paret and Yorke [4] obtained an improvement of this result which states that global bifurcation of periodic orbits occurs if simply $r_+-r_-\neq 0$. The following example shows that our theorem gives new information about occurrence of Hopf bifurcation, in case where $r_+-r_-=0$.

EXAMPLE.

$$\frac{dz}{dt}=(\lambda+\sqrt{-1})z+\sum a_{ijkl}z^i w^j \bar{z}^k \bar{w}^l$$

$$\frac{dw}{dt}=(-\lambda-\sqrt{-1})w+\sum b_{ijkl}z^i w^j \bar{z}^k \bar{w}^l,$$

where z, w, a_{ijkl} and b_{ijkl} are complex numbers and only finite members of a_{ijkl} and b_{ijkl} are nonzero. The sum is taken over non-negative integers i, j, k and l such that $i+j-k-l=1$, $i+j+k+l>1$. In this case, the map given in the right-hand side of the equation is S^1-equivariant for scalar multiplication. We put $\lambda_0=0$, $\beta=1$. Then, since $r_+=r_-=1$, this example does not satisfy the sufficient conditions for bifurcation of [2] and [4]. But $u_+=2$ and $u_-=0$. Thus $u_+-u_-\neq 0$. Therefore it follows from our theorem that local bifurcation of periodic orbits from the stationary point $(z, w)=(0, 0)$ occurs.

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