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Quasi-continuous functions and Hunt processes

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Quasi-continuous functions have been studied in various frameworks of potential theory, especially Dirichlet spaces (cf. [3] and its references). In that case, it has been shown that these functions are continuous on the paths of the associated symmetric Hunt process (cf. [3] § 4-3). We show that this property is in fact a characterization that extends to arbitrary stochastic processes having right continuous paths with left limits. We apply these results to characterize the semigroups associated with Hunt processes. The characterization of quasi-continuous functions was given in [5], for Hunt processes only, with a different proof. The characterization of the semigroups associated with Hunt processes is given here for the first time. (These results were announced in a note at the Comptes rendus [4]).

1. Quasi-continuous functions associated with a stochastic process.

Let $\{X_t(\omega), t\in R^+\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{A}, P)$, taking its values in a metrizable locally compact space E. We assume that the trajectories of X_t are right continuous with left limits on $(0, \infty)$.

1-1. *Capacity.*

Define, for any A Borel subset of E, the contact time $\tau_A(\omega)=\inf(t\geqq 0, X_t(\omega)$ or $X_{t-}(\omega)\in A)$. By convention, we set $\tau_A=\infty$ if neither X_t nor X_{t-} hit A.

PROPOSITION 1. *τ_A is a. s. measurable and for any $\alpha>0$, $\mathrm{Cap}_\alpha(A)=E(e^{-\alpha\tau_A})$ is a Choquet capacity on K.*

As in the case of Hunt processes, this is an easy consequence of the capacitability theorem. First, remark that if G is open, τ_G is $\mathcal{A}$ measurable (since $\tau_G=\inf(t\in Q^+, X_t\in G)$). If K is compact and intersection of a decreasing sequence of open set G_n, $\tau_K=\lim\uparrow\tau_{G_n}$. Conversely, if an open set G is the union of an increasing sequence of compact sets K_n, $\tau_G=\lim\downarrow\tau_{K_n}$. Finally, if K and K' are two compact sets, $\tau_{K\cap K'}\geqq\tau_K\vee\tau_{K'}$, and $\tau_{K\cup K'}=\tau_K\wedge\tau_{K'}$. Therefore, for any $\alpha>0$, $e^{-\alpha\tau_{K\cup K'}}+e^{-\alpha\tau_{K\cap K'}}\leqq e^{-\alpha\tau_K}+e^{-\alpha\tau_{K'}}$. We can now apply the capacitability theorem

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(cf. [2] § III-39) to extend the Choquet capacity $\mathrm{Cap}_\alpha(K)=E(e^{-\alpha\tau_K})$ on the whole Borel σ-algebra. For any open set G, $\mathrm{Cap}_\alpha(G)=E(e^{-\alpha\tau_G})$ and since the capacity is right continuous, for any Borel set A, there is a decreasing sequence G_n of open sets containing A and an increasing sequence K_n of compact sets contained in A such that $\mathrm{Cap}_\alpha(A)=\lim\uparrow\mathrm{Cap}_\alpha(K_n)=\lim\downarrow\mathrm{Cap}_\alpha(G_n)$. But this implies that τ_{G_n} and τ_{K_n} have a.s. the same limit as $n\uparrow+\infty$, and since $\tau_{G_n}\leqq\tau_A\leqq\tau_{K_n}$, this limit is a.s. equal to τ_A. Moreover, $\mathrm{Cap}_\alpha(A)=E(e^{-\alpha\tau_A})$.

The capacities Cap_α are equivalent in the sense that for any decreasing sequence of Borel set, $\mathrm{Cap}_\alpha(A_n)\downarrow 0$ if and only if $\mathrm{Cap}_\beta(A_n)\downarrow 0$. Indeed, both are equivalent to $\tau_{A_n}\uparrow\infty$ a.s. We shall denote one of these capacities by Cap. "Quasi-everywhere" means except on a set of zero capacity.

1-2. *Quasi-uniform convergence.*

a) Following [3], we call nest an increasing sequence of closed sets F_n such that $\mathrm{Cap}(E-F_n)$ decreases to 0. We say that a function f is quasi-continuous if and only if there exists a nest $\{F_n\}$ such that the restriction of f to each F_n is continuous. We say that a sequence of functions f_n converges towards f quasi-uniformly if and only if there exists a nest $\{F_n\}$ such that f_n converges towards f uniformly on each F_n. Clearly, a quasi-uniform limit of quasi-continuous functions is quasi-continuous and by Urysohn's theorem, any quasi-continuous function is a quasi-uniform limit of continuous functions. Quasi-continuous functions are nearly Borel, i.e. equal to a Borel function quasi-everywhere.

b) One can give two alternative criteria for the quasi-uniform convergence of a sequence f_n of nearly Borel functions toward a nearly Borel function f. First one can check that the convergence occurs if and only if $\lim_{N\to\infty}\downarrow\mathrm{Cap}(\bigcup_{n\geqq N}\{|f_n-f|\geqq\varepsilon\})=0$ for all $\varepsilon>0$. Second, this condition is clearly identical to $\lim_{N\to\infty}\tau_N^\varepsilon=\infty$ a.s. with $\tau_N^\varepsilon=\inf(t>0,\ |f_N-f|(X_t)\geqq\varepsilon$ or $|f_N-f|(X_{t-})\geqq\varepsilon)$ and this statement is equivalent to the a.s. uniform convergence on time segments of the processes $f_n(X_t)$ and $f_n(X_{t-})$ towards $f(X_t)$ and $f(X_{t-})$. This stochastic characterisation will be fundamental in the following.

c) It follows from the first criterion of quasi uniform convergence that it will occur if for some convergent series with strictly positive terms ε_n, $\sum_n\mathrm{Cap}(\{|f-f_n|>\varepsilon_n\})$ is finite. We say that f_n converges towards f in capacity if and only if, for any $\varepsilon>0$, $\lim_{n\to\infty}\mathrm{Cap}(\{|f_n-f|>\varepsilon\})$ is zero. By the preceding remark, the convergence in capacity implies the quasi-uniform convergence for a subsequence.

1-3. *Stochastic characterization of quasi-continuous functions.*

THEOREM 1. *A function f is quasi-continuous if and only if f is nearly Borel and a.s. continuous with X, i.e. $f(X_t)$ is a.s. right continuous and $f(X_{t-})$ a.s. left continuous.*

If f is quasi-continuous on a nest $\{F_n\}$, f is continuous with X on each time segment $[0, \tau_{F_n^c})$ and $\tau_{F_n^c}$ increases towards $+\infty$ a. s. as $n \uparrow +\infty$. The converse is less obvious. First remark that if f is continuous with X, $\tau_{\{|f|>n\}} \uparrow +\infty$ as $n \uparrow +\infty$, and therefore f is the quasi-uniform limit of the truncated $f_n=(f \wedge n) \vee (-n)$. So it is enough to prove that bounded nearly Borel functions continuous with X are quasi-continuous. We shall denote the algebra of these functions by B. Define on bounded Borel functions the semi-norm: $\|f\|=E(\sup_{t \geq 0} e^{-t}|f(X_t)| \vee e^{-t}|f(X_{t-})|)$. We have obviously $\|1_A\|=\mathrm{Cap}_1(A)$ for any Borel set and therefore, $\mathrm{Cap}_1(|f| \geqq \varepsilon) \leqq \|f\|/\varepsilon$. It follows that the functions of norm zero are the functions null quasi-everywhere and that convergence in norm implies convergence in capacity.

LEMMA 1. *If f_n is a sequence of functions of B decreasing to 0 quasi-everywhere, f_n decreases to 0 in norm and quasi-uniformly.*

The q. e. convergence implies a. s. the convergence of $f_n(X_t)$ and $f_n(X_{t-})$ to 0. A slight generalization of Dini's theorem shows that the convergence is uniform on every time segment.

This lemma and Daniell's theorem (cf. [2] § III-35) shows that any positive continuous linear form on $(B, \| \ \|)$ extends into a positive bounded Borel measure on E, charging no set of zero capacity. But B is a lattice and $|f| \leqq |g|$ implies $\|f\| \leqq \|g\|$. Therefore any continuous linear form on B is the difference of two positive continuous linear forms (cf. [1] p. 28). It is therefore induced by a bounded Radon measure charging no set of zero capacity. Then, it is null on B as soon as it is null on the space C_K of continuous functions with compact support. By Hahn Banach theorem, C_K is therefore dense in B. Since the convergence in norm implies the convergence in capacity, and therefore the quasi-uniform convergence for a subsequence, we can conclude the proof.

2. Application to the theory of Hunt processes.

2-1. *Quasi-continuous functions for Hunt processes.*

In this section and in the following one, we require in addition that X_t is a Hunt process with fixed initial distribution m. Denoting by Q_t its transition kernels, we prove the following:

THEOREM 2. *If f is bounded and quasi-continuous, $Q_t f$ is quasi-continuous and $\lim_{t \downarrow 0} Q_t f=f$ quasi-uniformly.*

Let $V_\alpha=\int_0^\infty e^{-\alpha t} Q_t \, dt$ be the resolvent associated with Q_t.

Let $\mathcal{V}$ be the space of functions of the form $V_\alpha f$, with f nearly Borel and bounded. $\mathcal{V}$ is independent of α by the resolvent equation.

LEMMA 2. *$\mathcal{V}$ is a dense subspace of B.*

Let us check first that each element of $\mathcal{V}$ is continuous with X. It is well known that $V_\alpha f(X_t)$ is right continuous and quasi-left continuous. By the section theorem (cf. [2] § IV-85) $V_\alpha f(X_{t-})$ is the left continuous version of $V_\alpha f(X_t)$ (since it is enough to check it at predictable times). Therefore, $\mathcal{V}$ is a subspace of B. To prove that $\mathcal{V}$ is dense in B, it is enough to prove that a Radon measure is null as soon as it is null on $\mathcal{V}$ (cf. proof of Theorem 1). But this follows from the fact that any f in C_K is the pointwise limit of $\alpha V_\alpha f$ as $\alpha \to \infty$.

LEMMA 3. *If a sequence of nearly Borel functions uniformly bounded by some constant M converges quasi-uniformly towards f, $|f-f_n|$ is dominated by a sequence g_n of 1 excessive functions decreasing to 0 quasi-uniformly.*

Remark first that g_n decreases to 0 quasi-uniformly as soon as $m(g_n) \downarrow 0$, by Doob's inequality applied to the supermartingale $e^{-t} g_n(X_t)$. To construct g_n, define $N_p = \inf(n, \mathrm{Cap}_1(\{\sup_{m \geq n} |f_m - f| \geqq 2^{-p}\}) < 2^{-p})$ and $\phi_p = 2^{-p} + M\psi_p$, where ψ_p is the 1-potential $E_x(e^{-\tau_{G_p}})$ of an open set G_p such that $\{\sup_{m \geq N_p} |f_m - f| \geqq 2^{-p}\}$ is included in G_p and $\mathrm{Cap}_1(G_p) < 2^{-p}$. We have $m(\phi_p) < (M+1)2^{-p}$ and $|f_n - f| < \phi_p$ for all $n \geqq N_p$. Define $g_n = (\sum_{k \geq p} \phi_k)\, M$ for $N_p \leqq n < N_{p+1}$.

PROOF OF THE THEOREM. First remark that if g is a quasi-continuous (i. e. regular) α-excessive function, $e^{-\alpha t} Q_t g \uparrow g$ quasi-uniformly by Lemma 1. The bounded quasi-continuous function f is by Lemma 2 the quasi-uniform limit of a sequence f_n in $\mathcal{V}$. By truncation, we can make $|f_n|$ bounded by $M = \|f\|_\infty$. Still f_n is the difference of two regular potentials and therefore $Q_u f_n$ converges towards f_n quasi-uniformly as $u \downarrow 0$. We can associate to $|f - f_n|$ the dominating sequence g_n of Lemma 3. First, $|Q_u f_n - Q_u f| \leqq Q_u(|f - f_n|) \leqq e^u g_n$ and therefore, $Q_u f$ is quasi-continuous. Set $\|h\|_{t,\omega} = \sup_{0 \leq s \leq t} (|h(X_s(\omega))| + |h(X_{s-}(\omega))|)$. For any t and $\varepsilon > 0$ and almost all ω, there is an integer $N(t, \omega, \varepsilon)$ such that $\|g_N\|_{t,\omega} < \varepsilon$ and also $y(N, \omega, t, \varepsilon)$ such that $\|Q_u f_N - f_N\|_{t,\omega} < \varepsilon$ for all $u < y$. Since $|f - Q_u f| \leqq (1+e^u) g_N + |Q_u f_N - f_N|$, we get $\|f - Q_u f\|_{t,\omega} \leqq \varepsilon(2+e)$ for $u < y(N) \wedge 1$.

2-2. *Semi-groups associated with Hunt processes.*

Let $(E, \mathcal{B})$ be a metrizable locally compact space and its Borel σ-algebra. Let m be a positive measure on $\mathcal{B}$. We shall say that a Hunt process on E is based on m if and only if its resolvent kernels V_α are such that mV_α is equivalent to m. For a given Hunt process, this property is verified by measures of the form μV_α. V_α induces then a resolvent of operators U_α on $L^\infty(m)$ and the transition semigroup Q_t induces a semigroup of operators P_t. Both are submarkovian and continuous for bounded a. e. convergence.

The purpose of this section is to give a necessary and sufficient condition for such a semi-group to be associated with a Hunt process based on m. Since only the equivalence class of m is involved, we may assume $m(1) = 1$.

Let P_t be a semi-group of positive submarkovian operators on $L^\infty(m)$, continuous for bounded a. e. convergence, and U_α the associated resolvent. We assume that mU_α is equivalent to m, i. e. that $U_\alpha|f|=0$ m a. e. implies $|f|=0$ a. e. This implies that for any α-supermedian function g, $\lim_{\beta\to\infty}\uparrow\beta U_{\alpha+\beta}g=g$ a. e. (since $m(U_\alpha\beta U_{\alpha+\beta}f)=m(U_\alpha f-U_{\alpha+\beta}f)$ increases toward $m(U_\alpha f)$).

We shall say that a sequence f_n in $L^\infty(m)$ is converging almost uniformly towards f if and only if $|f-f_n|$ is dominated by a sequence of bounded α-supermedian functions g_n decreasing towards 0 almost surely. Let $\mathcal{U}$ be the space of all $U_\alpha f$, with f in $L^\infty(m)$. Let $\mathcal{C}$ be the adherence of $\mathcal{U}$ for almost uniform convergence. One checks easily that $P_tf\to f$ almost uniformly as $t\downarrow 0$ if f belongs to $\mathcal{C}$. We say that P_t is regular if and only if the following condition holds.

(R) $\mathcal{C}_K$ is contained and dense in $\mathcal{C}$.

Define for any open set G and $\alpha>0$ $R_\alpha^G=\text{ess inf}(g,\ g\ \alpha\text{-supermedian},\ g\geqq 1$ a. e. on G). Let S_α be the cone of bounded α-supermedian functions and S_α^n be $\{h\in L^\infty(m),\ h\geqq 0$ and $e^{-\alpha 2^{-n}}P_{2^{-n}}h\leqq h\}$. Clearly S_α^n is decreasing with n and $\bigcap S_\alpha^n=S_\alpha$. Set $R_{\alpha,n}^G=\text{ess inf}(h,\ h\in S_\alpha^n,\ h\geqq 1$ a. e. on G). We have $R_\alpha^G=\lim\uparrow R_{\alpha,n}^G$. Since $R_{\alpha,n}^G$ is the capacitary potential associated with the transition function $e^{-\alpha 2^{-n}}P_{2^{-n}}$, we have the strong subadditivity relation: $R_{\alpha,n}^{G\cup G'}+R_{\alpha,n}^{G\cap G'}\leqq R_{\alpha,n}^G+R_{\alpha,n}^{G'}$. (Consider the hitting times for the associated Markov chains), and this relation extends to R_α^G. Set $\text{Cap}_\alpha(G)=m(R_\alpha^G)$. One checks as in Chapter 3 of [3] that Cap_α extends into a Choquet capacity. Capacities associated with different measures m or different α's are equivalent.

We now come to our characterization theorem.

THEOREM 3. *P_t is associated with a Hunt process if and only if it is regular.*

Necessity. Suppose X_t is associated with a Hunt process X_t. We can use the results of Section 2-1 for X_t with initial distribution m. The class modulo m of each α-excessive function is P_t-α-supermedian. Conversely, if $f=U_\alpha g$, with $g\in L^\infty(m)$ and if g' is any representative of g, $V_\alpha g'$ is a representative of f. (Therefore, any bounded α-supermedian function of $L^\infty(m)$ has an α-excessive representative. Using the right continuity of the associated supermartingale, one checks easily that two excessive functions equal m a. e. are equal quasi-everywhere). From this, it follows that $\mathcal{U}$ is the image of $\mathcal{V}$ in $L^\infty(m)$ and that the quasi uniform convergence implies the almost uniform convergence in $L^\infty(m)$ (by Lemma 3). The regularity condition is then a direct consequence of Theorem 1 and 2. $\mathcal{C}$ appears to be the image of B in $L^\infty(m)$.

Sufficiency. Let f be a function in $\mathcal{C}$ and f_n a sequence of functions in $\mathcal{C}_K$ converging towards f almost uniformly. Let g_n be a sequence of bounded α-supermedian functions dominating $|f-f_n|$ and decreasing to 0. By extracting a subsequence, we may assume that $\sum_n 2^n m(g_n)$ converges and since $|f_{n+1}-f_n|\leqq 2g_n$,

$\sum_n \mathrm{Cap}_\alpha(\{|f_{n+1}-f_n|>2^{-n}\})$ converges. This proves that f_n is converging quasi-uniformly, and therefore each element of $\mathcal{C}$ has a quasi-continuous representative $\tilde{f}$. Since the quasi-uniform convergence of continuous functions implies their almost uniform convergence, it is unique. One checks also that the almost uniform convergence in $\mathcal{C}$ implies the quasi-uniform convergence of the representatives (cf. [**3**] Theorem 3-1-4). Each α-supermedian function ϕ has a quasi-everywhere refinement $\tilde{\phi}=\lim_{\beta\uparrow\infty}\uparrow\beta\tilde{U}_{\alpha+\beta}\phi$ (not necessarily quasi-continuous). This applies in particular to the capacitary potentials $p_\alpha^G=\inf(f\ \alpha\text{-supermedian},\ f\geqq 1_G$ m a. e.), G being an open set. $\tilde{p}_\alpha^G=1$ quasi-everywhere on G (since for any $g\in\mathcal{C}_K$, $g\leqq 1_G$, $\beta\tilde{U}_{\beta+\alpha}p_\alpha^G\geqq\beta\tilde{U}_{\beta+\alpha}g$ which converges towards g almost uniformly and therefore quasi-uniformly). The construction given in [**3**], Chapter 6 can now be used with some obvious modifications to construct the process. Two processes associated with the same semigroup of operators P_t are m-equivalent (cf. [**3**] § 4-3). In particular, for any initial distribution μ charging no set of zero capacity, the law P_μ is the same for both processes.

REMARK 1. In the symmetric case, our regularity assumption is implied by the regularity of the Dirichlet space, and the associated capacities are equivalent. But it seems to be weaker. (The existence of the Hunt process does not imply the regularity of the Dirichlet space.)

REMARK 2. One checks easily that the regularity hypothesis (R) is equivalent to the following:

(R′) For any f in $\mathcal{C}_K$, P_tf has a quasi continuous representative $\widetilde{P_tf}$ (with respect to the capacity defined by the supermedian functions) and $f=\lim_{t\downarrow 0}\widetilde{P_tf}$ quasi-everywhere. The capacity Cap_α defined by the supermedian functions coincides with the capacity associated with the process with initial distribution m.

References

[1] N. Bourbaki, Integration, Chap. 1-4, 2nd Ed., Hermann, Paris, 1965.
[2] C. Dellacherie and P.A. Meyer, Probabilities and potentials, Hermann-North Holland, 1978.
[3] M. Fukushima, Dirichlet forms and Markov processes, North Holland-Kodansha, 1980.
[4] Y. Le Jan, Sur les resolvantes des processus de Hunt, C.R. Acad. Sci. Paris, **291** (1980), p. 313.
[5] Y. Le Jan, Quasi-continuous functions associated to a Hunt process, Proc. Amer. Math. Soc., to appear.

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