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Von Neumann algebras and Boolean valued analysis

By Gaisi TAKEUTI

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The aim of this paper is to apply Boolean valued analysis developed in [**5**], [**6**], [**7**], [**8**] to von Neumann algebras. Let $\mathcal{M}$ be a von Neumann algebra, $\mathcal{Z}$ its center, and $\mathcal{B}$ the complete Boolean algebra of all projections in $\mathcal{Z}$. Then $\mathcal{B}$-valued analysis gives a machinery to transfer theorems on factors into theorems on von Neumann algebras whose centers are $\mathcal{Z}$. Our machinery is a kind of a dictionary which gives a translation of the notions on factors into the notions on von Neumann algebras with the center $\mathcal{Z}$. E. g., the translations of type I, type II_1, type II_∞ and type III are exactly type I, type II_1, type II_∞ and type III themselves. However, the translations of trace and weight become generalized center valued trace and generalized center valued weight respectively. This machinery immediately reduces many properties of center valued traces or weights to properties of traces or weights on factors. In § 5, we state direct translations of Murray and von Neumann's theorem on factors of type II_∞ and Takesaki's theorem on factors of type III in order to show how our machinery works.

Our theory is closely related to von Neumann's reduction theory. We prove that every von Neumann algebra is a factor in a Boolean valued sense while the reduction theory proves that every von Neumann algebra with a countability condition is a direct integral of factors. Thus a systematic interpretation of Boolean valued notion provides us with a machinery of automatic translations of notions on factors into notions on von Neumann algebras with the center $\mathcal{Z}$, while the reduction theory reduces many problems on von Neumann algebras to problems on factors.

It is not difficult to eliminate $\mathcal{B}$-valued analysis in our machinery. In order to do so, we have to introduce the following $\bar{\mathcal{Z}}$-valued Hilbert space, where $\bar{\mathcal{Z}}$ is the unbounded extension of $\mathcal{Z}$. If we express $\mathcal{Z}$ as $L^\infty(\Omega, \mu)$, then $\bar{\mathcal{Z}}$ is the set of all measurable functions. $\hat{\mathcal{H}}$ is called a $\bar{\mathcal{Z}}$-valued Hilbert space if $\hat{\mathcal{H}}$ is a $\bar{\mathcal{Z}}$-module with the inner product satisfying the following properties:

1) $\zeta, \eta \in \hat{\mathcal{H}} \Rightarrow (\zeta | \eta) \in \bar{\mathcal{Z}}$

2) Let $\zeta, \eta, \xi \in \hat{\mathcal{H}}$ and $f, g \in \bar{\mathcal{Z}}$. Then the following hold.

2.1) $(f\zeta | \eta) = f \cdot (\zeta | \eta)$ a. e.

2.2) $(\zeta|\eta)=\overline{(\eta|\zeta)}$ a.e.

2.3) $(f\zeta+g\eta|\xi)=f\cdot(\zeta|\xi)+g\cdot(\eta|\xi)$ a.e.

3) $(\zeta|\zeta)\geqq 0$ a.e., and

$$\|\zeta\|=0 \text{ a.e., iff } \zeta=0, \text{ where } \|\zeta\|=\sqrt{(\zeta|\zeta)}.$$

4) $\hat{\mathcal{H}}$ is complete in the following sense.

If $\lim_{\substack{i\to\infty\\ j\to\infty}}\|\zeta_i-\zeta_j\|=0$ a.e., then there exists $\zeta\in\hat{\mathcal{H}}$ such that $\lim_{i\to\infty}\|\zeta_i-\zeta\|=0$ a.e.

However, it is our belief that $\mathcal{B}$-valued analysis makes our machinery much simpler. Besides, there is a possibility that some problem on von Neumann algebras can be proved independent from set theory by our method, though we have not found any good candidate of this sort of problems.

Let us summarize the basic facts on Boolean valued universe of sets herewith.

Let $\mathcal{B}$ be a complete Boolean algebra. Then $V^{(\mathcal{B})}$ the universe of $\mathcal{B}$-valued sets satisfies the following properties.

1. Let u and v be members of $V^{(\mathcal{B})}$. Then $u\in v$ or $u=v$ has a truth value in $\mathcal{B}$. If the truth value of $u\in v$ or $u=v$ is $b\in\mathcal{B}$, then we write $[\![u\in v]\!]=b$ or $[\![u=v]\!]=b$ respectively. We can assign truth values $[\![\varphi]\!]$ in $\mathcal{B}$ to all set theoretic statement φ. If $[\![\varphi]\!]=1$, then we say 'φ is true' or 'φ holds'. If $[\![\varphi]\!]=0$, then we say 'φ is false' or 'φ does not hold'. If $[\![\varphi]\!]=b$ and $0<b<1$, then φ takes an intermediate truth value and we say 'φ holds as much as b'.

2. $V^{(\mathcal{B})}$ is a model of ZFC. Let φ be a theorem in modern mathematics i.e., a theorem in ZFC. Then $[\![\varphi]\!]=1$ is also a theorem in ZFC i.e., $[\![\varphi]\!]=1$ is a different theorem in modern mathematics. Therefore, this procedure provides us with a machinery to produce a new theorem from an old theorem φ.

3. Since $V^{(\mathcal{B})}$ satisfies ZFC, we can construct real numbers in $V^{(\mathcal{B})}$ by Dedekind's cuts.

If $\mathcal{B}$ is a measure algebra, then the real numbers in $V^{(\mathcal{B})}$ can be interpreted as the real valued measurable functions. Therefore, the procedure in 2 in this case provides us with a machinery to transform theorems on real numbers into theorems on real valued measurable functions.

4. Let $\mathcal{B}$ be a complete Boolean algebra of projections on a Hilbert space $\mathcal{H}$. Let x be a self-adjoint operator on $\mathcal{H}$ and $x=\int\lambda dE_\lambda$ its spectral decomposition. Then x is said to be in $(\mathcal{B})$ if for every λ, E_λ is a member of $\mathcal{B}$. Then the real numbers in $V^{(\mathcal{B})}$ can be interpreted as the self-adjoint operators in $(\mathcal{B})$. In this case, the procedure in 2 provides us with a machinery to transform theorems on real numbers into theorems on self-adjoint operators in $(\mathcal{B})$.

5. Let $\mathcal{B}$ be a measure algebra. Then a sequence of real numbers in $V^{(\mathcal{B})}$

corresponds to a sequence of measurable functions. A Cauchy sequence of real numbers in $V^{(\mathcal{B})}$ corresponds to a sequence of measurable functions which forms a pointwise Cauchy sequence almost everywhere (abbreviated by a. e.).

The convergence of a sequence of real numbers in $V^{(\mathcal{B})}$ corresponds to a pointwise convergence of a sequence of measurable functions a. e.

§1. The construction of $\tilde{\mathcal{H}}$ from $\mathcal{H}$.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, $\mathcal{Z}_0$ a von Neumann algebra included in its center, and $\mathcal{B}$ a complete Boolean algebra of all projections in $\mathcal{Z}_0$.

We construct $V^{(\mathcal{B})}$ the $\mathcal{B}$-valued model of ZFC. Let x be a self-adjoint operator on $\mathcal{H}$ and $\int \lambda dE_\lambda$ its spectral decomposition. The operator x is said to be in $(\mathcal{B})$ if for every real λ, $E_\lambda \in \mathcal{B}$. The reals in $V^{(\mathcal{B})}$ is denoted by $\boldsymbol{R}^{\mathcal{B}}$. $\boldsymbol{R}^{\mathcal{B}}$ corresponds to the set of all self-adjoint operators in $(\mathcal{B})$. A normal operator $x+iy$ is said to be in $(\mathcal{B})$ if both x and y are in $(\mathcal{B})$. The complex numbers in $V^{(\mathcal{B})}$ is denoted by $\boldsymbol{C}^{\mathcal{B}}$. $\boldsymbol{C}^{\mathcal{B}}$ corresponds to the set of all normal operators in $(\mathcal{B})$. If a is a complex number, $\check{a}$ is a member of $\boldsymbol{C}^{\mathcal{B}}$ and corresponds to aI where I is the identity operator.

Let x be a normal operator in $(\mathcal{B})$ and a be a real. Then $\|x\| \leqq a$ is equivalent to $|x| \leqq \check{a}$ in $V^{(\mathcal{B})}$, where $|x|$ is the absolute value of complex number x in $V^{(\mathcal{B})}$ and $\leqq$ is the order relation of reals in $V^{(\mathcal{B})}$. Then $\mathcal{Z}_0$ corresponds to the complex numbers in $V^{(\mathcal{B})}$ whose absolute value is less than some $\check{a}$.

Let Ω be the spectrum of $\mathcal{Z}_0$. As usual we define the topology of Ω as the weak topology of the dual space of $\mathcal{Z}_0$. Then Ω is compact. By Zorn's lemma, there exists a maximal subset $\{\zeta_\alpha\}$ of $\mathcal{H}$ satisfying the following conditions:

1. $[\mathcal{Z}_0'\zeta_\alpha] \perp [\mathcal{Z}_0'\zeta_\beta]$ if $\alpha \neq \beta$, where $\mathcal{Z}_0'$ is the commutant of $\mathcal{Z}_0$ and $[\mathcal{Z}_0'\zeta_\alpha]$ is the closed linear space generated by $\mathcal{Z}_0'\zeta_\alpha$.
2. $\bigcup_\alpha [\mathcal{Z}_0'\zeta_\alpha]$ generates $\mathcal{H}$.

Let b_α be the projection to $[\mathcal{Z}_0'\zeta_\alpha]$. Then $b_\alpha \in \mathcal{B}$ and $\{b_\alpha\}$ is a partition of unity in $\mathcal{B}$. By Proposition 2.5-2.7 in [5], the properties of $V^{(\mathcal{B})}$ can be reduced to the properties of $V^{(\mathcal{B}_\alpha)}$, where $\mathcal{B}_\alpha$ is the complete Boolean algebra of projections in $\mathcal{B}$ which are below b_α. Therefore, we can assume that without loss of generality that there exists

$$\zeta_0 \in \mathcal{H} \quad \text{such that} \quad [\mathcal{Z}_0'\zeta_0] = \mathcal{H}\,.$$

The set of all complex valued continuous functions on Ω is denoted by $L_\infty(\Omega)$. Then for every $f \in L_\infty(\Omega)$ there exists a unique T_f in $\mathcal{Z}_0$ such that for every $\omega \in \Omega$,

$$\omega(T_f) = f(\omega)\,.$$

Furthermore, for $f\in L_\infty(\Omega)$ and for $\zeta,\ \eta\in\mathcal{H}$, $(T_f\zeta\,|\,\eta)$ is a linear form of f in $L_\infty(\Omega)$. Therefore, there exists a Radon measure $\nu_{\zeta,\eta}$ such that

$$\int_\Omega f(\omega)d\nu_{\zeta,\eta}(\omega)=(T_f\zeta\,|\,\eta)\,.$$

Let ν be ν_{ζ_0,ζ_0}. Then ν is a bounded basic measure. Therefore, there exists a measurable function $h_{\zeta,\eta}$ with respect to ν such that

$$d\nu_{\zeta,\eta}(\omega)=h_{\zeta,\eta}(\omega)d\nu(\omega)\,.$$

The measure algebra with respect to ν is nothing but $\mathcal{B}$ itself. Therefore, the measurable functions correspond to $\boldsymbol{C}^{\mathcal{B}}$.

Now we are going to construct $\mathcal{B}$-valued Hilbert space $\tilde{\mathcal{H}}$ and $\mathcal{B}$-valued von Neumann algebra $\tilde{\mathcal{M}}$ on $\tilde{\mathcal{H}}$. Let $\tilde{\mathcal{H}}$ be the set of all $\langle f,\check{\zeta}\rangle$ where $f\in\boldsymbol{C}^{\mathcal{B}}$ and $\zeta\in\mathcal{H}$. $\langle f,\check{\zeta}\rangle$ is denoted by $f\check{\zeta}$. We define the inner product $(f\check{\zeta}\,|\,g\check{\eta})$ by

$$(f\check{\zeta}\,|\,g\check{\eta})=f\bar{g}\,h_{\zeta,\eta}\,.$$

Since $f,\ \bar{g},\ h_{\zeta,\eta}$ are members of $\boldsymbol{C}^{\mathcal{B}}$, $f\bar{g}\,h_{\zeta,\eta}$ i.e., the value of the inner product is a complex number in $V^{(\mathcal{B})}$. We define $f\check{\zeta}=0$ as an element of $\tilde{\mathcal{H}}$ if $(f\check{\zeta}\,|\,f\check{\zeta})=0$ i.e., $|f|^2h_{\zeta,\zeta}=0$ a.e. Therefore, we have

$$\begin{aligned}[\![\check{\zeta}=0]\!]&=[\![(\check{\zeta}\,|\,\check{\zeta})=0]\!]\\&=\{\omega\,|\,h_{\zeta,\zeta}(\omega)=0\}/\mathcal{N}\,,\end{aligned}$$

where $\mathcal{N}$ denote the ideal of all sets with measure 0. Let A be a measurable subset of Ω and χ_A its characteristic function. Then we have

$$\chi_A(\omega)h_{\zeta,\zeta}(\omega)=0\qquad\text{iff}\quad A\subseteqq\{\omega\,|\,h_{\zeta,\zeta}(\omega)=0\}\,.$$

Therefore we have

$$[\![\check{\zeta}=0]\!]=\bigvee\{A\,|\,\chi_A(\omega)h_{\zeta,\zeta}(\omega)=0\}/\mathcal{N}\,.$$

Let b be T_{χ_A}. Then $b\in\mathcal{B}$ and

$$\begin{aligned}b\zeta=0\quad&\text{iff}\quad\int_\Omega\chi_A(\omega)h_{\zeta,\zeta}(\omega)d\nu(\omega)=0\\&\text{iff}\quad\chi_A(\omega)h_{\zeta,\zeta}(\omega)=0\qquad\text{a.e.}\end{aligned}$$

Therefore, we have

$$[\![\check{\zeta}=0]\!]=\bigvee\{b\,|\,b\zeta=0\}\,.$$

We also have

$$[\![f\check{\zeta}=0]\!]=[\![f=0]\!]\vee[\![\check{\zeta}=0]\!]\,.$$

Now we would like to define $+$ on $\tilde{\mathcal{H}}$. Let $f\check{\zeta}$ and $g\check{\eta}$ be two members of $\tilde{\mathcal{H}}$. Define h by $h=1+|f|+|g|$. The f/h and g/h are bounded and measurable,

therefore, $T_{f/h}$ and $T_{g/h}$ in the natural extension of T to $L^{\infty}(\Omega, \nu)$ are in $\mathcal{Z}_0$. We define ξ by

$$\xi = T_{f/h}\zeta + T_{g/h}\eta .$$

Now $f\check{\zeta}+g\check{\eta}$ is defined by

$$f\check{\zeta}+g\check{\eta}=h\check{\xi} .$$

From the properties of $h_{\zeta,\eta}$, it is easily seen that $\tilde{\mathcal{H}}$ is a prehilbert space in $V^{(\mathcal{B})}$. Now we are going to show that $\tilde{\mathcal{H}}$ is indeed a Hilbert space in $V^{(\mathcal{B})}$.

LEMMA. *Let $\zeta \in \mathcal{H}$ and f be measurable and*

$$\int_{\Omega} |f(\omega)|^2 h_{\zeta,\zeta}(\omega) d\nu(\omega) < \infty .$$

Then there exists an $\eta \in \mathcal{H}$ such that

$$f\check{\zeta}=\check{\eta} .$$

More precisely for every $\xi \in \mathcal{H}$, the following holds

$$f(\omega)h_{\zeta,\xi}(\omega)=h_{\eta,\xi}(\omega) \quad a.e.$$

PROOF. Let $f_n(\omega)$ satisfy the following

$$f_n(\omega)=\begin{cases} f(\omega) & \text{if} \quad |f(\omega)| \leqq n \\ n & \text{otherwise} . \end{cases}$$

Let η_n be $T_{f_n\zeta}$. Then we have

$$\|\eta_n - \eta_m\|^2 = \int_{\Omega} |f_n(\omega) - f_m(\omega)|^2 h_{\zeta,\zeta}(\omega) d\nu(\omega) \to 0$$

if $n, m \to \infty$. Therefore, there exists $\eta \in \mathcal{H}$ such that $\eta = \lim_{n\to\infty} \eta_n$. Then we have for every bounded measurable function g

$$\begin{aligned} \int_{\Omega} g(\omega) h_{\eta,\xi}(\omega) d\nu(\omega) &= (T_g \eta \mid \xi) \\ &= \lim_{n\to\infty} (T_g \eta_n \mid \xi) \\ &= \lim_{n\to\infty} \int_{\Omega} g(\omega) f_n(\omega) h_{\zeta,\xi}(\omega) d\nu(\omega) \\ &= \int_{\Omega} g(\omega) f(\omega) h_{\zeta,\xi}(\omega) d\nu(\omega) . \end{aligned}$$

The last equality holds by Lebesgue's convergence theorem. Therefore, we have

$$h_{\eta,\xi}(\omega) = f(\omega) h_{\zeta,\xi}(\omega) \quad \text{a. e.}$$

THEOREM 1. *$\tilde{\mathcal{H}}$ is a Hilbert space in $V^{(\mathcal{B})}$.*

PROOF. Let $\{f_i\check{\zeta}_i\}$ be a Cauchy sequence. We apply Schmidt's orthogonalization to $\{\check{\zeta}_i\}$ in $V^{(\mathcal{B})}$. Let $\eta_1=\zeta_1$. Since

$$|h_{\zeta_1,\zeta_2}(\omega)|^2 \leqq |h_{\zeta_1,\zeta_1}(\omega)|\,|h_{\zeta_2,\zeta_2}(\omega)| \quad \text{a. e.,}$$

there exists a measurable function $f_{1,2}(\omega)$ such that

$$h_{\zeta_1,\zeta_1}(\omega)=f_{1,2}(\omega)h_{\zeta_1,\zeta_1}(\omega) \quad \text{a. e.}$$

We define η_2 by the following equation

$$\eta_2=\frac{\zeta_2-f_{1,2}\eta_1}{1+|f_{1,2}|}$$

where $g\eta$ is an abbreviation of $T_g\eta$ for any bounded measurable function g. In general, there exists a measurable function $f_{i,n}(\omega)$ $(i<n)$ satisfying

$$h_{\eta_i,\zeta_n}(\omega)=f_{i,n}(\omega)h_{\eta_i,\eta_i}(\omega) \quad \text{a. e.}$$

We define η_n by the following equation

$$\eta_n=\frac{\zeta_n-f_{1,n}\eta_1-\cdots-f_{n-1,n}\eta_{n-1}}{1+|f_{1,n}|+\cdots+|f_{n-1,n}|}.$$

Then $h_{\eta_i,\eta_j}(\omega)=0$ a. e. for $i\neq j$ and ζ_n can be expressed by the following form

$$\zeta_n=\sum_{i=1}^{n} g_i\eta_i\,.$$

Therefore, the Cauchy sequence $\{f_i\check{\zeta}_i\}$ in $\tilde{\mathcal{H}}$ can be expressed as $\sum_{i=1}^{n} f_i^n\check{\eta}_i$, where we may assume that $f_i^n(\omega)=0$ whenever $h_{\eta_i,\eta_i}(\omega)=0$. Now $\sum_{i=1}^{n} f_i^n\check{\eta}_i$ is a Cauchy sequence implies

$$[\![\forall\varepsilon>0\exists N\forall m,\ n>N\,\|\sum_{i=1}^{m} f_i^m\check{\eta}_i-\sum_{i=1}^{n} f_i^n\check{\eta}_i\|^2<\varepsilon]\!]=1\,.$$

Therefore, we have

$$\bigwedge_{\varepsilon>0}\bigvee_{N}\bigwedge_{m,n>N}\{\omega\,|\,\sum_{i=1}^{m}|f_i^m(\omega)-f_i^n(\omega)|^2h_{\eta_i,\eta_i}(\omega)<\varepsilon\}/\mathfrak{N}=1\,.$$

Since $\bigwedge_{\varepsilon>0}$, $\bigvee_{N}$, $\bigwedge_{m,n>N}$ are countable operations, we have

$$\forall\varepsilon>0\exists N\forall n,\ m>N\sum_{i=1}^{m}|f_i^m(\omega)-f_i^n(\omega)|^2h_{\eta_i,\eta_i}(\omega)<\varepsilon \quad \text{a. e.}$$

Since $f_i^n(\omega)=0$ whenever $h_{\eta_i,\eta_i}(\omega)=0$, we have $\{f_i^n(\omega)\}$ is a Cauchy sequence a. e., for a fixed i. Therefore, there exists a measurable function $f_i(\omega)$ such that $\lim_{n\to\infty} f_i^n(\omega)=f_i(\omega)$ a. e. Since $\sum_{i=1}^{n} f_i^n\check{\eta}_i$ is a Cauchy sequence in $\tilde{\mathcal{H}}$, $\|\sum_{i=1}^{n} f_i^n\check{\eta}_i\|^2$

is a Cauchy sequence of reals in $V^{(\mathcal{B})}$. Therefore, there exists a non-negative measurable function $g(\omega)$ such that

$$\lim_{n\to\infty}\sum_{i=1}^{n}|f_i^n(\omega)|^2 h_{\eta_i,\eta_i}(\omega)=g^2(\omega) \quad \text{a. e.}$$

Now we have for every n, m with $n<m$

$$\sum_{i=n+1}^{m}|f_i^m(\omega)|^2 h_{\eta_i,\eta_i}(\omega)\leqq\|\sum_{i=1}^{m}f_i^m\check{\eta}_i-\sum_{i=1}^{n}f_i^n\check{\eta}_i\|^2,$$

where the right hand side of inequality is a real in $V^{(\mathcal{B})}$ and it expresses a measurable function. Therefore, we have

$$\limsup_{n\to\infty}\sup_{m>n}\sum_{i=n+1}^{m}|f_i^m(\omega)|^2 h_{\eta_i,\eta_i}(\omega)=0 \quad \text{a. e.}$$

and

$$g^2(\omega)=\sum_{i=1}^{\infty}|f_i(\omega)|^2 h_{\eta_i,\eta_i}(\omega) \quad \text{a. e.}$$

Now there exists $\xi_n\in\mathcal{H}$ such that

$$\frac{1}{1+g}\sum_{i=1}^{n}f_i\check{\eta}_i=\check{\xi}_n.$$

For $j\geqq i$, we have

$$\|\xi_j-\xi_i\|^2=\int_\Omega\frac{1}{(1+g(\omega))^2}\sum_{k=i+1}^{j}|f_k(\omega)|^2 h_{\eta_k,\eta_k}(\omega)d\nu(\omega).$$

Therefore, $\{\xi_i\}$ is a Cauchy sequence in $\mathcal{H}$ and there exists $\xi\in\mathcal{H}$ such that $\lim_{i\to\infty}\xi_i=\xi$. Then for every bounded measurable function $g_0(\omega)$ and for every $\zeta\in\mathcal{H}$, we have

$$\begin{aligned}\int_\Omega g_0(\omega)h_{\xi,\zeta}(\omega)d\nu(\omega)&=(T_{g_0}\xi|\zeta)\\&=\lim_{n\to\infty}(T_{g_0}\xi_n|\zeta)\\&=\lim_{n\to\infty}\int_\Omega\frac{g_0(\omega)}{1+g(\omega)}\sum_{i=1}^{n}f_i(\omega)h_{\eta_i,\zeta}(\omega)d\nu(\omega).\end{aligned}$$

Since

$$\frac{1}{1+g(\omega)}|\sum_{i=1}^{n}f_i(\omega)h_{\eta_i,\zeta}(\omega)|\leqq\frac{1}{(1+g(\omega))^2}\sum_{i=1}^{n}|f_i(\omega)|^2 h_{\eta_i,\eta_i}(\omega)+h_{\zeta,\zeta}(\omega),$$

we have by Lebesgue's convergence theorem

$$\int_\Omega g_0(\omega)h_{\xi,\zeta}(\omega)d\nu(\omega)=\int_\Omega\frac{g_0(\omega)}{1+g(\omega)}\sum_{i=1}^{\infty}f_i(\omega)h_{\eta_i,\zeta}(\omega)d\nu(\omega).$$

Therefore, we have

$$h_{\xi,\zeta}(\omega)=\frac{1}{1+g(\omega)}\sum_{i=1}^{\infty}f_i(\omega)h_{\eta_i,\zeta}(\omega) \quad \text{a. e.}$$

Therefore, we have

$$h_{\xi-\xi_n,\xi-\xi_n}(\omega)=\frac{1}{(1+g(\omega))^2}\sum_{i=n+1}^{\infty}|f_i(\omega)|^2h_{\eta_i,\eta_i}(\omega) \quad \text{a. e.}$$

Therefore, $\check{\xi}_n$ converges to $\check{\xi}$ in $V^{(\mathcal{B})}$. This shows that $\sum_{i=1}^{n}f_i\check{\eta}_i$ converges to $(1+g)\check{\xi}$. Now consider the inequality

$$\|\sum_{i=1}^{m}f_i^m\check{\eta}_i-\sum_{i=1}^{\infty}f_i\check{\eta}_i\|\leqq\sum_{i=1}^{n}\|f_i^m-f_i\|\|\check{\eta}_i\|$$

$$+\|\sum_{i=n+1}^{m}f_i^m\check{\eta}_i\|+\|\sum_{i=n+1}^{\infty}f_i\check{\eta}_i\|^2.$$

For every $\varepsilon>0$, there exists n such that the second term and the third term in the right hand side of the inequality is less than $1/3\,\varepsilon$. After such n is fixed, we can take some $m\geqq n$ such that the first term in the right-hand side of the inequality is also less than $1/3\,\varepsilon$. This completes the proof.

REMARK. Let $\check{\zeta}_1, \check{\zeta}_2, \cdots\to 0$. Then $\lim_{i\to\infty}h_{\zeta_i,\zeta_i}(\omega)=0$ a. e. Let

$$g(\omega)=\frac{1}{1+\sup_i h_{\zeta_i,\zeta_i}(\omega)}.$$

Then g is positive, bounded, measurable and $\lim_{i\to\infty}T_g\zeta_i=0$. In general let $\lim_{i\to\infty}\check{\zeta}_i=\check{\zeta}$. Then there exists a positive bounded measurable function g such that

$$\lim_{i\to\infty}T_g\zeta_i=T_g\zeta.$$

REMARK. Our construction of $\tilde{\mathcal{H}}$ depends on the choice of $\{\zeta_\alpha\}$ such that $\{b_\alpha\}$, the projections to the space $[\mathcal{Z}_0'\zeta_\alpha]$, becomes a partition of unity in $\mathcal{B}$. Now take another choice of $\{\eta_\beta\}$. Let $\{b'_\beta\}$ be the partition of unity obtained from $\{\eta_\beta\}$. Then $\{b_\alpha b'_\beta\}$ is again a partition of unity and $[\mathcal{Z}_0'b'_\beta\zeta_\alpha]=[\mathcal{Z}_0'b_\alpha\eta_\beta]$. Therefore, we may assume that $[\mathcal{Z}_0'\zeta]=[\mathcal{Z}_0'\eta]=\mathcal{H}$. Now μ and ν are basic measures on Ω obtained from ζ and η respectively. Then there exist positive measurable functions such that $d\mu(\omega)=h_1(\omega)d\nu(\omega)$ and $d\nu(\omega)=h_2(\omega)d\mu(\omega)$ and $h_1(\omega)h_2(\omega)=1$. Let $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ be Hilbert spaces in $V^{(\mathcal{B})}$ obtained from μ and ν respectively. Then the isomorphism between $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ is obtained by assignment of $\sqrt{h_2}\check{\zeta}$ in $\tilde{\mathcal{H}}_2$ to $\check{\zeta}$ in $\tilde{\mathcal{H}}_1$. Since $\sqrt{h_2}$ is a positive real in $V^{(\mathcal{B})}$, $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ are the same up to a positive constant multiplication.

§ 2. The construction of $\tilde{\mathcal{M}}$ from $\mathcal{M}$.

Let $\mathcal{H}$, $\mathcal{M}$, $\mathcal{Z}_0$, $\mathcal{B}$, Ω, and $\tilde{\mathcal{H}}$ be the same as in § 1. We are going to con-

struct a von Neumann algebra $\tilde{\mathcal{M}}$ on $\tilde{\mathcal{H}}$ in $V^{(\mathcal{B})}$ which corresponds to $\mathcal{M}$.

Let $x\in\mathcal{Z}_0'$. We define $\tilde{x}: \tilde{\mathcal{H}}\to\tilde{\mathcal{H}}$ by the equation $\tilde{x}(f\check{\eta})=f(x(\eta))\check{}$. Since $b\zeta=b\eta$ implies $bx(\zeta)=bx(\eta)$ for every b in $\mathcal{B}$, we have

$$[\![\check{\zeta}=\check{\eta}]\!]\leqq[\![\tilde{x}(\check{\zeta})=\tilde{x}(\check{\eta})]\!]\,.$$

Therefore, $\tilde{x}$ is a function in $V^{(\mathcal{B})}$.

PROPOSITION 1. *If* $\|x\|=M$, *then* $\|\tilde{x}\|\leqq\check{M}$.

PROOF. Let $g(\omega)$ be a non-negative measurable function on Ω. Then we have

$$\int_\Omega g^2(\omega)h_{x\zeta,x\zeta}(\omega)d\nu(\omega)=(xT_g\zeta\,|\,xT_g\zeta)$$

$$\leqq M^2\int_\Omega g^2(\omega)h_{\zeta,\zeta}(\omega)d\nu(\omega)\,.$$

Therefore, we have $h_{x\zeta,x\zeta}(\omega)\leqq M^2h_{\zeta,\zeta}(\omega)$ a.e. This implies $\|\tilde{x}\|\leqq\check{M}$ in $V^{(\mathcal{B})}$.

PROPOSITION 2. $\|x\|=M$ *iff* $M=\inf\{K\,|\,\|\tilde{x}\|\leqq\check{K}\}$.

PROOF. Suppose $\|\tilde{x}\|\leqq\check{K}$. Then we have

$$h_{x\zeta,x\zeta}(\omega)\leqq K^2h_{\zeta,\zeta}(\omega)\quad\text{a.e.}$$

Therefore, we have $\|x\zeta\|\leqq K\|\zeta\|$. The proposition immediately follows from this and the previous proposition.

PROPOSITION 3. *Let* $\hat{x}$ *be a bounded linear operator on* $\tilde{\mathcal{H}}$ *in* $V^{(\mathcal{B})}$ *and* $\|\hat{x}\|\leqq\check{M}$. *Then there exists* $y\in\mathcal{Z}_0'$ *such that* $\|y\|\leqq M$ *and* $\hat{x}=\tilde{y}$.

PROOF. Let $\zeta\in\mathcal{H}$. Since $\hat{x}(\check{\zeta})\in\tilde{\mathcal{H}}$, $\hat{x}(\check{\zeta})$ is of the form $f\check{\eta}$. Then we have

$$|f(\omega)|^2h_{\eta,\eta}(\omega)\leqq M^2h_{\zeta,\zeta}(\omega)\quad\text{a.e.}$$

Therefore, there exists $\xi\in\mathcal{H}$ such that $f\check{\eta}=\check{\xi}$. We define y by $y(\zeta)=\xi$. Then the proposition is obvious.

PROPOSITION 4. *Let* $\hat{x}$ *be a bounded linear operator of* $\tilde{\mathcal{H}}$ *in* $V^{(\mathcal{B})}$. *Then there exist* $y\in\mathcal{Z}_0'$ *and* $f\in\boldsymbol{C}^{\mathcal{B}}$ *such that* $\hat{x}=f\tilde{y}$.

PROOF. Let $\hat{y}=\hat{x}/(1+\|\hat{x}\|)$. Then there exists $y\in\mathcal{Z}_0'$ such that $\hat{y}=\tilde{y}$. Then $\hat{x}=(1+\|\hat{x}\|)\tilde{y}$, where $1+\|\hat{x}\|\in\boldsymbol{C}^{\mathcal{B}}$.

PROPOSITION 5. $(\tilde{x})^*=x^*$, *where* $*$ *in the left-hand side of the equality is the adjoint in* $\tilde{\mathcal{H}}$ *and* $*$ *in the right-hand side of the equality is the adjoint in* $\mathcal{H}$.

PROOF. $(\tilde{x}\check{\zeta}\,|\,\check{\eta})$ is $h_{x\zeta,\eta}(\omega)$. Since we have

$$h_{x\zeta,\eta}=\overline{h_{\eta,x\zeta}}=\overline{h_{x^*\eta,\zeta}}=h_{\zeta,x^*\eta}\,,$$

we have $(\tilde{x}\check{\zeta}\,|\,\check{\eta})=(\check{\zeta}\,|\,\tilde{x}^*\check{\eta})$.

PROPOSITION 6. $x\geqq0$ *iff* $\tilde{x}\geqq0$.

PROOF. Let $x\geqq0$ and $f\geqq0$. Then we have

$$(T_f x\zeta|\zeta)=(xT_{\sqrt{f}}\zeta|T_{\sqrt{f}}\zeta)\geqq 0\,.$$

Therefore, we have

$$\int_\Omega f(\omega)h_{x\zeta,\zeta}(\omega)d\nu(\omega)\geqq 0\,.$$

Hence, we have $h_{x\zeta,\zeta}(\omega)\geqq 0$ a.e., i.e., $\tilde{x}\geqq 0$. Conversely suppose $\tilde{x}\geqq 0$. Then we have $(\tilde{x}\check{\zeta}|\check{\zeta})=h_{x\zeta,\zeta}\geqq 0$. Hence, we have $(x\zeta|\zeta)\geqq 0$ and $x\geqq 0$.

Now we define $\tilde{\mathcal{M}}$ in $V^{(\mathcal{B})}$ by the following equation.

$$\tilde{\mathcal{M}}=\{f\tilde{x}\mid x\in\mathcal{M} \text{ and } f\in \boldsymbol{C}^{\mathcal{B}}\}.$$

PROPOSITION 7. $\tilde{\mathcal{M}}'=(\tilde{\mathcal{M}})'$.

PROOF. We may assume that every member $f\tilde{x}$ in $\tilde{\mathcal{M}}$ satisfies $[\![f=0]\!]=[\![f\tilde{x}=0]\!]$. Now let $f\tilde{x}$, $g\tilde{y}$ in $\tilde{\mathcal{M}}$. Then we have

$$(f\tilde{x})(g\tilde{y})=(g\tilde{y})(f\tilde{x}) \qquad \text{if} \quad xy=yx\,.$$

Now the proposition is obvious.

PROPOSITION 8. *$\tilde{\mathcal{M}}$ is a von Neumann algebra on $\tilde{\mathcal{H}}$.*

PROOF. By Proposition 5, $\tilde{\mathcal{M}}$ is a $*$-algebra. We also have $(\tilde{\mathcal{M}})''=\tilde{\mathcal{M}}$ by Proposition 7.

We define $(\mathcal{M}_+)^\sim$ in $V^{(\mathcal{B})}$ by the following equation.

$$(\mathcal{M}_+)^\sim=\{f\tilde{x}\mid x\in\mathcal{M}_+ \text{ and } f\geqq 0\}.$$

PROPOSITION 9. $(\mathcal{M}_+)^\sim=\tilde{\mathcal{M}}_+$.

PROOF. $(\mathcal{M}_+)^\sim\subseteqq\tilde{\mathcal{M}}_+$ follows from Proposition 6. Suppose $f\tilde{x}\in\tilde{\mathcal{M}}_+$. Without loss of generality, we may assume $f\geqq 0$ and $[\![f=0]\!]=[\![\tilde{x}=0]\!]$. Then $x\in\mathcal{M}_+$ follows from Proposition 6.

PROPOSITION 10. *Let $\hat{\mathcal{M}}$ be a von Neumann algebra on $\tilde{\mathcal{H}}$. Then there exists a von Neumann algebra $\mathcal{M}$ on $\mathcal{H}$ such that $\hat{\mathcal{M}}=\tilde{\mathcal{M}}$.*

PROOF. Let $\mathcal{M}=\{x\mid\tilde{x}\in\hat{\mathcal{M}}\}$. Then $\hat{\mathcal{M}}=\tilde{\mathcal{M}}$.

PROPOSITION 11. *If $\mathcal{Z}_0$ is the center of $\mathcal{M}$, then $\tilde{\mathcal{M}}$ is a factor.*

PROOF. This is obvious since the center of $\tilde{\mathcal{M}}$ is $\boldsymbol{C}^{\mathcal{B}}$.

PROPOSITION 12. *If p is a projection in $\mathcal{M}$, then $\tilde{p}$ is a projection in $\tilde{\mathcal{M}}$. If $\hat{p}$ is a projection in $\tilde{\mathcal{M}}$, then there exists a projection p in $\mathcal{M}$ such that $\tilde{p}=\hat{p}$.*

PROOF. The first part is obvious. For the second part, we have x in $\mathcal{M}$ with $\tilde{x}=\hat{p}$ by Proposition 3. Since $(\tilde{x})^*=\tilde{x}$ and $\tilde{x}\cdot\tilde{x}=\tilde{x}$, x is a projection.

In the same way as in Proposition 12, we can show that a unitary operator in $\tilde{\mathcal{M}}$ is nothing but $\tilde{u}$ where u is a unitary operator in $\mathcal{M}$.

Let p and q be two projections in $\mathcal{M}$. If $p\sim q$, then there exists $u\in\mathcal{M}$ such that $p=u^*u$ and $q=uu^*$. Then obviously $\tilde{p}\sim\tilde{q}$. On the other hand, suppose $\tilde{p}\sim\tilde{q}$. Then there exists $\hat{u}\in\tilde{\mathcal{M}}$ such that $\hat{u}^*\hat{u}=\tilde{p}$ and $\hat{u}\hat{u}^*=\tilde{q}$. Since $\|\hat{u}\|\leqq 1$, $\hat{u}$ must be of the form $\tilde{u}$. From this immediately follows $p\sim q$. Therefore,

the relation $\sim$ has the same meaning in $\mathcal{M}$ and $\tilde{\mathcal{M}}$ i.e., $\sim$ is absolute.

Let p and q be still projections in $\mathcal{M}$. Since $p \leqq q$ is equivalent to $pq=p$, $p \leqq q$ is equivalent to $\tilde{p} \leqq \tilde{q}$ in $\tilde{\mathcal{M}}$ i.e., $p \leqq q$ is absolute. Therefore, $p \lesssim q$ is also absolute.

PROPOSITION 13. *Let p be a projection in $\mathcal{M}$. p is finite in $\mathcal{M}$ iff $\tilde{p}$ is finite in $\tilde{\mathcal{M}}$.*

PROOF. '$\tilde{p}$ is finite in $\tilde{\mathcal{M}}$' is an abbreviation of $[\![\tilde{p}$ is finite in $\tilde{\mathcal{M}}]\!]=1$. Now we have

$$\begin{aligned} [\![\tilde{p} \text{ is finite in } \tilde{\mathcal{M}}]\!]=1 \quad &\text{iff} \quad [\![\forall \hat{q} \in \tilde{\mathcal{M}}(\tilde{p} \sim \hat{q} \leqq \tilde{p} \to \tilde{p}=\hat{q})]\!]=1 \\ &\text{iff} \quad \forall q \in \mathcal{M}([\![\tilde{p} \sim \tilde{q} \leqq \tilde{p} \to \tilde{p}=\tilde{q}]\!]=1)\,. \end{aligned}$$

Since $[\![\tilde{p} \sim \tilde{p} \leqq \tilde{p}]\!]=1$, $\forall q \in \mathcal{M}([\![\tilde{p} \sim \tilde{q} \leqq \tilde{p}]\!] \leqq [\![\tilde{p}=\tilde{q}]\!])$ is equivalent to

$$\forall q \in \mathcal{M}([\![\tilde{p} \sim \tilde{q} \leqq \tilde{p}]\!]=1 \text{ implies } [\![\tilde{p}=\tilde{q}]\!]=1), \quad \text{i.e.,} \quad \forall q \in \mathcal{M}(p \sim q \leqq p \to p=q)\,.$$

In the same way, we can prove the following proposition. From now on, p and q ranges over projections.

PROPOSITION 14. *p is purely infinite in $\mathcal{M}$ iff $\tilde{p}$ is purely infinite in $\tilde{\mathcal{M}}$. p is abelian in $\mathcal{M}$ iff $\tilde{p}$ is abelian in $\tilde{\mathcal{M}}$.*

REMARK. 'p is infinite in $\mathcal{M}$' is not equivalent to '$\tilde{p}$ is infinite in $\tilde{\mathcal{M}}$'. More precisely

$$\begin{aligned} \text{'}p \text{ is infinite in } \mathcal{M}\text{'} \quad &\text{iff} \quad \neg \text{ '}p \text{ is finite in } \mathcal{M}\text{'} \\ &\text{iff} \quad \neg([\![\tilde{p} \text{ is finite in } \tilde{\mathcal{M}}]\!]=1) \\ &\text{iff} \quad [\![\tilde{p} \text{ is infinite in } \tilde{\mathcal{M}}]\!]>0\,. \end{aligned}$$

Now let $b \in \mathcal{B}$. Then $[\![\tilde{b}=1]\!]=b$. Let $\mathcal{Z}_0$ be the center of $\mathcal{M}$. Then '$\tilde{\mathcal{M}}$ is infinite' is equivalent to '$\mathcal{M}$ is properly infinite'. This is shown automatically as follows.

$$\begin{aligned} \text{'}\mathcal{M} \text{ is infinite'} \quad &\text{iff} \quad [\![1 \text{ is infinite in } \tilde{\mathcal{M}}]\!]=1 \\ &\text{iff} \quad \forall b \in \mathcal{B}(b \leqq [\![1 \text{ is infinite in } \tilde{\mathcal{M}}]\!]) \\ &\text{iff} \quad \forall b \in \mathcal{B}([\![\tilde{b}=1 \to 1 \text{ is infinite in } \tilde{\mathcal{M}}]\!]=1) \\ &\text{iff} \quad \forall b \in \mathcal{B}([\![\tilde{b}=1 \to \tilde{b} \text{ is infinite in } \tilde{\mathcal{M}}]\!]=1) \\ &\text{iff} \quad \forall b \in \mathcal{B}([\![\tilde{b} \text{ is finite in } \tilde{\mathcal{M}} \to \tilde{b}=0]\!]=1) \\ &\text{iff} \quad \forall b \in \mathcal{B}([\![\tilde{b} \text{ is finite in } \tilde{\mathcal{M}}]\!]=1 \to [\![\tilde{b}=0]\!]=1) \\ &\text{iff} \quad \forall b \in \mathcal{B}(b \text{ is finite in } \mathcal{M} \to b=0) \\ &\text{iff} \quad \mathcal{M} \text{ is properly infinite.} \end{aligned}$$

Here we used the fact that if $[\![\tilde{b} \text{ is finite in } \tilde{\mathcal{M}}]\!]=b_0$ and $b_0 \leqq [\![\tilde{b}=0]\!]$, then $[\![\tilde{b}_1$ is finite in $\tilde{\mathcal{M}}]\!]=1$ and $[\![\tilde{b}_1=0]\!]=1$, where $b_1=b(1-b_0)$.

Now let $\mathcal{Z}$ be the center of $\mathcal{M}$. Then $\mathcal{Z}_0 \subseteqq \mathcal{Z}$ and $\tilde{\mathcal{Z}}$ is the center of $\tilde{\mathcal{M}}$.

PROPOSITION 15. *$\tilde{\mathcal{M}}$ is of type I in $V^{(\mathcal{B})}$ iff $\mathcal{M}$ is of type I.*

PROOF. First we assume that $\tilde{\mathcal{M}}$ is of type I. Let $p\neq 0$. Then $[\![\tilde{p}\neq 0]\!]>0$. Let $b=[\![\tilde{p}\neq 0]\!]$ and $p_0=p+(1-b)$. Then $[\![\tilde{p}_0\neq 0]\!]=1$. Therefore, we have

$$[\![\exists\hat{q}\in\tilde{\mathcal{M}}(\hat{q}\neq 0\wedge(\hat{q}\text{ is abelian})\wedge\hat{q}\leqq\tilde{p}_0]\!]=I\,,$$

and

$$\bigvee_{q\in\mathcal{M}}[\![\tilde{q}\neq 0\wedge(\tilde{q}\text{ is abelian})\wedge\tilde{q}\leqq\tilde{p}_0]\!]=I\,.$$

From this follows that there exists $q\in\mathcal{M}$ satisfying

$$[\![\tilde{q}\neq 0\wedge(\tilde{q}\text{ is abelian})\wedge\tilde{q}\leqq\tilde{p}_0]\!]\wedge b>0\,.$$

Let $b_0=[\![\tilde{q}\neq 0\wedge(\tilde{q}\text{ is abelian})\wedge\tilde{q}\leqq\tilde{p}_0]\!]\wedge b$ and $q_0=b_0q$. Then $q_0\neq 0$ and q_0 is abelian and $q_0\leqq p$. This proves that $\mathcal{M}$ is of type I. Now let us assume that $\mathcal{M}$ is of type I. In order to prove that $\tilde{\mathcal{M}}$ is of type I, it suffices to show that for every $p\in\mathcal{Z}$,

$$[\![\tilde{p}\neq 0\rightarrow\exists\hat{q}\in\tilde{\mathcal{M}}(\hat{q}\neq 0\wedge(\hat{q}\text{ is abelian})\wedge\hat{q}\leqq\tilde{p}]\!]=1\,.$$

Let $b=[\![\tilde{p}\neq 0]\!]$ and $p_1=p+(1-b)$. Then $[\![\tilde{p}_1\neq 0]\!]=1$ and it suffices to show

$$[\![\exists\hat{q}\in\tilde{\mathcal{M}}(\hat{q}\neq 0\wedge(\hat{q}\text{ is abelian})\wedge\hat{q}\leqq\tilde{p}_1]\!]=1\,.$$

Now let $b_1>0$. Then $b_1p_1\neq 0$. Since $\mathcal{M}$ is of type I, there exists $q\in\mathcal{M}$ such that $q\neq 0$ and q is abelian and $q\leqq b_1p_1$. This implies

$$[\![\tilde{q}\neq 0\wedge(\tilde{q}\text{ is abelian})\wedge\tilde{q}\leqq\tilde{p}_1]\!]\wedge b_1>0\,.$$

This completes the proof.

PROPOSITION 16. *$\tilde{\mathcal{M}}$ is of type II in $V^{(\mathcal{B})}$ iff $\mathcal{M}$ is of type II.*

PROOF. $\mathcal{M}$ is of type II iff $\mathcal{M}$ satisfies C_1 and C_2, where C_1 is $\forall p\in\mathcal{M}(p$ is abelian$\rightarrow p=0)$ and C_2 is $\forall p\in\mathcal{Z}(p\neq 0\rightarrow\exists q\in\mathcal{M}(q\neq 0\wedge(q$ is finite$)\wedge q\leqq p))$. In the same way as in Proposition 15, we can prove that $\tilde{\mathcal{M}}$ satisfies C_2 in $V^{(\mathcal{B})}$ iff $\mathcal{M}$ satisfies C_2. Now we are going to show that C_1 is also absolute. First we have

$$\begin{aligned}&[\![\forall\hat{p}\in\tilde{\mathcal{M}}(\hat{p}\text{ is abelian}\rightarrow\hat{p}=0)]\!]=1\\ &\qquad\text{iff}\quad\forall p\in\mathcal{M}[\![\tilde{p}\text{ is abelian}\rightarrow\tilde{p}=0]\!]=1\\ &\qquad\text{iff}\quad\forall p\in\mathcal{M}([\![\tilde{p}\text{ is abelian}]\!]\leqq[\![\tilde{p}=0]\!])\,.\end{aligned}$$

Now assume that $\mathcal{M}$ satisfies C_1. Let $b=[\![\tilde{p}$ is abelian$]\!]$. Then bp is abelian. Hence, we have $bp=0$ i.e., $b\leqq[\![\tilde{p}=0]\!]$. Therefore, $\tilde{\mathcal{M}}$ satisfies C_1 in $V^{(\mathcal{B})}$. Now assume that $\tilde{\mathcal{M}}$ satisfies C_1 in $V^{(\mathcal{B})}$ and $p\in\mathcal{M}$ is abelian. Then $1=[\![\tilde{p}$ is abelian$]\!]$ $\leqq[\![\tilde{p}=0]\!]$. Therefore, $p=0$ and $\mathcal{M}$ satisfies C_1.

COROLLARY. *$\tilde{\mathcal{M}}$ is of type II_1 in $V^{(\mathcal{B})}$ iff $\mathcal{M}$ is of type II_1. $\tilde{\mathcal{M}}$ is of type II_∞ in $V^{(\mathcal{B})}$ iff $\mathcal{M}$ is of type II_∞.*

PROOF. The first part is immediate from Propositions 13 and 16. In order to prove the second part, it suffices to show that $\forall p\in\mathcal{Z}(p$ is finite$\rightarrow p=0)$ is

absolute. Let us call this property by C. Then '$\tilde{\mathcal{M}}$ satisfies C' is equivalent to $\forall p \in \mathcal{Z}([\![\tilde{p} \text{ is finite}]\!] \leqq [\![\tilde{p}=0]\!])$. Now assume that $\mathcal{M}$ satisfies C and $[\![\tilde{p} \text{ is finite}]\!] > 0$. Let $b=[\![\tilde{p} \text{ is finite}]\!]$ and $p_1=bp$. Then $[\![\tilde{p}_1 \text{ is finite}]\!]=1$. Therefore, p_1 is finite and $p_1=0$ i.e., $[\![\tilde{p}=0]\!] \geqq b$. Hence, $\tilde{\mathcal{M}}$ satisfies C. Now assume that $\tilde{\mathcal{M}}$ satisfies C and p is finite. Then $[\![\tilde{p} \text{ is finite}]\!]=1$ and $[\![\tilde{p}=0]\!]=1$. Therefore, $p=0$ and $\mathcal{M}$ satisfies C.

PROPOSITION 17. *$\tilde{\mathcal{M}}$ is of type III in $V^{(\mathcal{B})}$ iff $\mathcal{M}$ is of type III.*

PROOF. This is immediate from Proposition 14.

§ 3. The construction of $\mathcal{H}$ from $\hat{\mathcal{H}}$.

Let $\hat{\mathcal{H}}$ be a Hilbert space in $V^{(\mathcal{B})}$, where $\mathcal{B}$ is a complete Boolean algebra of all projections in an abelian von Neumann algebra $\mathcal{Z}$. We are going to show that $\hat{\mathcal{H}}$ is of the form $\tilde{\mathcal{H}}$ i.e., there exists a Hilbert space $\mathcal{H}$ such that $\mathcal{Z}$ is a von Neumann algebra on $\mathcal{H}$ and $\hat{\mathcal{H}}$ is isomorphic to $\tilde{\mathcal{H}}$ in $V^{(\mathcal{B})}$.

By taking an adequate partition $\{b_\alpha\}$ of unity in $\mathcal{B}$, we can assume that there exist compact spaces Ω_α and positive Radon measures μ_α on Ω_α such that Ω_α is the spectrum of $\mathcal{Z}_\alpha=b_\alpha\mathcal{Z}$, μ_α is a basic measure of Ω_α, $\mu_\alpha(\Omega_\alpha)=1$, and $\mathcal{Z}_\alpha$ can be identified as $L^\infty(\Omega_\alpha, \mu_\alpha)$. So we now assume that μ is a positive Radon measure on a compact space Ω, $\mu(\Omega)=1$, and $\mathcal{Z}$ can be identified as $L^\infty(\Omega, \mu)$. Let $\zeta, \eta \in \hat{\mathcal{H}}$. Then $(\xi|\mu)$ in $V^{(\mathcal{B})}$ can be identified as a measurable function on Ω. This function is denoted by $(\zeta|\eta)(\omega)$. Let $\|\zeta\|(\omega)=\sqrt{(\zeta|\zeta)(\omega)}$. Now let

$$\mathcal{H}=\left\{\zeta \in \hat{\mathcal{H}} \,\middle|\, \int_\Omega \|\zeta\|^2(\omega)d\mu(\omega)<\infty\right\}.$$

For $\zeta, \eta \in \mathcal{H}$ and $\alpha \in \boldsymbol{C}$, we define $\alpha\zeta=\check{\alpha}\zeta$ and $(\zeta|\eta)=\int_\Omega(\zeta|\eta)(\omega)d\mu(\omega)$. Then $\mathcal{H}$ becomes a prehilbert space. We show that $\mathcal{H}$ is complete. The proof goes in the same way as in the proof of Riesz-Fischer's theorem. Obviously we have $\|\zeta+\eta\|(\omega) \leqq \|\zeta\|(\omega)+\|\eta\|(\omega)$ a.e. for every $\zeta, \eta \in \mathcal{H}$. Let $\{\zeta_i\}$ be a Cauchy sequence in $\mathcal{H}$. It suffices to show that a subsequence converges. Therefore, we may assume that $\|\zeta_i-\zeta_{i+1}\|<2^{-i}$. Let $\sigma_k(\omega)=\|\zeta_1-\zeta_2\|(\omega)+\cdots+\|\zeta_{k-1}-\zeta_k\|(\omega)$ and $\|\sigma_k\|^2=\int_\Omega \sigma_k^2(\omega)d\mu(\omega)$. Then $\|\sigma_k\| \leqq 1$. Let $\sigma(\omega)=\lim_{k\to\infty}\sigma_k(\omega)$. Since $\{\sigma_k(\omega)\}$ is an increasing sequence, we have

$$\|\sigma\|=\lim_{k\to\infty}\|\sigma_k\| \leqq 1 \quad \text{and} \quad \sigma(\omega)<\infty \quad \text{a.e.}$$

Therefore, $\lim_{\substack{i\to\infty \\ j\to\infty}}\|\zeta_i-\zeta_j\|(\omega)=0$ a.e. and $\{\zeta_i\}$ is a Cauchy sequence in $\hat{\mathcal{H}}$. Hence, there exists ζ in $\hat{\mathcal{H}}$ such that $\lim_{i\to\infty}\zeta_i=\zeta$ in $\hat{\mathcal{H}}$. Then $\lim_{i\to\infty}\|\zeta_i\|(\omega)=\|\zeta\|(\omega)$ a.e.

Since $\|\zeta\|(\omega)\leqq\sigma(\omega)+\|\zeta_1\|(\omega)$, we have $\|\zeta\|=\lim_{i\to\infty}\|\zeta_i\|<\infty$ and $\zeta\in\mathcal{H}$. Hence we have $\lim_{i\to\infty}\|\zeta-\zeta_i\|=0$.

Now let $\hat{x}\in\mathcal{L}(\hat{\mathcal{H}})$ in $V^{(\mathcal{B})}$ and $\|\hat{x}\|_{\mathcal{B}}\leqq\check{M}$, where $\|\ \|_{\mathcal{B}}$ is the norm in $V^{(\mathcal{B})}$ and M is a positive real number. Then for every $\zeta\in\mathcal{H}$, we have

$$\|\hat{x}(\zeta)\|^2_{\mathcal{B}}(\omega)\leqq M^2\|\zeta\|^2(\omega)\quad \text{a.e.}$$

Therefore, $\hat{x}(\zeta)\in\mathcal{H}$. Now we define x by

$$x(\zeta)=\hat{x}(\zeta)\,.$$

Then $x\in\mathcal{L}(\mathcal{H})$ and $\|x\|\leqq M$. Especially every member of $\mathcal{Z}$ is a member of $\mathcal{L}(\mathcal{H})$. Suppose $\{z_i\}$ is a subset of $\mathcal{Z}$, $z\in\mathcal{L}(\mathcal{H})$ and $\lim_{i\to\infty}z_i=z$ in the strong topology of $\mathcal{L}(\mathcal{H})$. Let $\zeta_0\in\mathcal{H}$ satisfy $\|\zeta_0\|_{\mathcal{B}}=\check{1}$. Moreover let $g(\omega)\geqq 0$ and $g\in L^\infty(\Omega,\mu)$. We define $\zeta=\sqrt{g}\zeta_0$ and $F(g)=(z\zeta|\zeta)$. Then

$$|F(g)|=|(z\zeta|\zeta)|\leqq\|z\|\|\zeta\|\|\zeta\|=\|z\|\|g\|_1\,.$$

Also we have

$$F(g)=\lim_{i\to\infty}(z_i\zeta|\zeta)=\lim_{i\to\infty}\int_\Omega f_i(\omega)g(\omega)d\mu(\omega)\,,$$

where $f_i(\omega)=(z_i\zeta_0|\zeta_0)(\omega)$. Now we extend the definition of F over $L^1(\Omega,\mu)$ by the linearity. Then F is a linear functional on $L^1(\Omega,\mu)$ and $\|F\|_\infty\leqq\|z\|$. Therefore, $F\in L^\infty(\Omega,\mu)$ and there exists $f\in L^\infty(\Omega,\mu)$ such that

$$F(g)=\int_\Omega f(\omega)g(\omega)d\mu(\omega)\,.$$

Also we have $\|f-f_i\|_\infty\leqq\|z-z_i\|$ and $\lim_{i\to\infty}f_i=f$ in $L^\infty(\Omega,\mu)$. Then we have

$$\begin{aligned}(z\zeta|\eta)&=\lim_{i\to\infty}(z_i\zeta|\eta)\\&=\lim_{i\to\infty}\int_\Omega f_i(\omega)(\zeta|\eta)(\omega)d\mu(\omega)\\&=\int_\Omega f(\omega)(\zeta|\eta)(\omega)d\mu(\omega)\,.\end{aligned}$$

Therefore, $z=T_f$ and $z\in\mathcal{Z}$.

Let $x\in\mathcal{L}(\mathcal{H})$ be a member of $\mathcal{Z}'$ in $\mathcal{L}(\mathcal{H})$. Then for every $z\in\mathcal{Z}$ and for every $\zeta\in\mathcal{H}$

$$xz\zeta=zx\zeta\,.$$

This implies that x can be considered as an operator in $\mathcal{L}(\hat{\mathcal{H}})$ in $V^{(\mathcal{B})}$.

Let $\eta\in\mathcal{H}$. Then there exists $\{b_n\}$ in $\mathcal{B}$ such that $b_n\to 1$ and $\|b_n\eta\|(\omega)\leqq n$. Then there exists $\hat{x}$ in $\mathcal{L}(\hat{\mathcal{H}})$ in $V^{(\mathcal{B})}$ such that $\hat{x}(\zeta_0)=b_n\eta$ and for every $\zeta\perp\zeta_0$, $\hat{x}(\zeta)=0$, where $\zeta_0\in\mathcal{H}$ satisfies $\|\zeta_0\|(\omega)=1$. Then $\|x\|_{\mathcal{B}}=\|b_n\eta\|\leqq n$. Therefore, $\hat{x}$

is represented by an operator in $\mathcal{Z}'$ in $\mathcal{L}(\mathcal{H})$. Since $\lim_{n\to\infty} b_n\eta=\eta$ in $\mathcal{H}$, we have $[\mathcal{Z}'\zeta_0]=\mathcal{H}$.

Now we construct the basic measure ν and $h_{\zeta,\eta}$ from $\mathcal{Z}$, $\mathcal{Z}'$, ζ_0, and $\mathcal{H}$. We are going to show that $\nu=\mu$ and $h_{\zeta,\eta}(\omega)=(\zeta|\eta)(\omega)$ i.e., $\tilde{\mathcal{H}}=\hat{\mathcal{H}}$. For every $f\in L^\infty(\Omega,\mu)$, we have

$$(T_f\zeta|\eta)(\omega)=f(\omega)(\zeta|\eta)(\omega) \quad \text{a.e.}$$

Therefore, we have

$$\nu_{\zeta,\eta}(f)=(T_f\zeta|\eta)=\int_\Omega (T_f\zeta|\eta)(\omega)d\mu(\omega)$$
$$=\int_\Omega f(\omega)(\zeta|\eta)(\omega)d\mu(\omega)\,.$$

Especially we have $\nu(f)=\int_\Omega f(\omega)d\mu(\omega)$ and $\mu=\nu$. Therefore, we have

$$\nu_{\zeta,\eta}(f)=\int_\Omega f(\omega)h_{\zeta,\eta}(\omega)d\mu(\omega)=\int_\Omega f(\omega)h_{\zeta,\eta}(\omega)d\mu(\omega)\,.$$

Therefore, we have $h_{\zeta,\eta}(\omega)=(\zeta|\eta)(\omega)$ a.e.

THEOREM 2. *Let $\{x_i\}$ in $\mathcal{Z}'$. Then $\lim_{i\to\infty} x_i=0$ in the strong topology of $\mathcal{L}(\mathcal{H})$ iff there exists M such that $\|\tilde{x}_i\|_{\mathcal{B}}\leqq\check{M}$ for every i and $\lim_{i\to\infty}\tilde{x}_i=0$ in the strong topology of $\mathcal{L}(\tilde{\mathcal{H}})$.*

PROOF. Suppose $\lim_{i\to\infty}\tilde{x}_i=0$. For every $\zeta\in\mathcal{H}$, we have

$$\lim_{i\to\infty}\|\tilde{x}_i\zeta\|(\omega)=0 \quad \text{a.e.}$$

If $\|x_i\|_{\mathcal{B}}\leqq\check{M}$ for every i, then $\lim_{i\to\infty}\int_\Omega\|x_i\zeta\|(\omega)d\mu(\omega)=0$ i.e., $\lim_{i\to\infty}\|x_i\zeta\|=0$. Conversely let $\lim_{i\to\infty} x_i=0$. Then there exists a bound M for all $\|x_i\|$ i.e., $\|\tilde{x}_i\|_{\mathcal{B}}\leqq\check{M}$. It suffices to show that $\lim_{i\to\infty}\tilde{x}_i\zeta=0$ for all ζ with $(\zeta|\zeta)(\omega)=1$ a.e. Let $g\geqq 0$ and $g\in L^1(\Omega,\mu)$ and $\eta=\sqrt{g}\zeta$. Then $\eta\in\mathcal{H}$ and we have

$$\|x_i\eta\|^2\leqq\|x_i\|^2\|\eta\|^2=\|x_i\|^2\|g\|_1\,.$$

Let $F_i(g)=\int_\Omega g(\omega)\|\tilde{x}_i\zeta\|^2(\omega)d\mu(\omega)$. Then $\|F_i\|_\infty\leqq\|x_i\|^2$. Therefore, $\lim_{i\to\infty}F_i=0$ in $L^\infty(\Omega,\mu)$. Therefore, $\lim_{i\to\infty}\|\tilde{x}_i\zeta\|^2(\omega)=0$ a.e. i.e., $\lim_{i\to\infty}\tilde{x}_i\zeta=0$ in $\tilde{\mathcal{H}}$.

REMARK. Let $\hat{\mathcal{M}}$ be a von Neumann algebra on $\hat{\mathcal{H}}$ in $V^{(\mathcal{B})}$. Though $\mathcal{H}$ depends on the choice of $\{b_\alpha\}$, $\{\mu_\alpha\}$, it is easily seen that $\mathcal{M}$ on $\mathcal{H}$ corresponding to $\hat{\mathcal{M}}$ is uniquely determined up to isomorphism.

§ 4. Traces and weights.

First we have to consider $[0, \infty]$ in $V^{(\mathscr{B})}$ which is denoted by $[0, \infty]^{\mathscr{B}}$. If we treat $\mathscr{B}$ as a measure algebra, then a member of $[0, \infty]^{\mathscr{B}}$ is naturally represented by a measurable function from Ω into $[0, \infty]$. If we treat $\mathscr{B}$ as a complete Boolean algebra of projections, then a member of $[0, \infty]^{\mathscr{B}}$ is naturally represented by a resolution of the identity $E(\lambda)$ where $-\infty<\lambda\leqq+\infty$. More precisely $E(\lambda)$ satisfy the following conditions.

1. For every $\lambda\in(-\infty, +\infty]$, $E(\lambda)$ is a member of $\mathscr{B}$.
2. $E(\lambda)=0$ if $\lambda<0$.
3. $E(+\infty)=1$.
4. $\lambda_1<\lambda_2 \Rightarrow E(\lambda_1)\leqq E(\lambda_2)$.
5. $\lim_{\lambda\to\lambda_0+0} E(\lambda)=E(\lambda_0)$, where the limit denotes the limit in the strong topology.

When we consider $\mathscr{B}$ as a measure algebra, we choose a partition of unity $\{b_\alpha\}$ in $\mathscr{B}$ and consider $\mathscr{B}_\alpha=b_\alpha\mathscr{B}$ as a measure algebra on $(\Omega_\alpha, \mu_\alpha)$. In this case, our measure space Ω is the sum of all Ω_α i.e., a measurable set in Ω is of the form $\bigcup_\alpha A_\alpha$, where A_α is a measurable set in Ω_α, a set with measure zero in Ω is of the form $\bigcup_\alpha N_\alpha$, where N_α is a set with measure zero in Ω_α, and the measure μ on Ω is defined by $\mu(\bigcup_\alpha A_\alpha)=\sum_\alpha \mu_\alpha(A_\alpha)$. This measure space Ω has a disadvantage that there exists a measurable set A in Ω such that $\mu(A)=\infty$ but A is not the sum of countable measurable sets with a finite measure. However, this does not give us much difficulty since the notions we mainly use are pointwise convergence a.e., $L^\infty(\Omega, \mu)$ etc. and the integration is used locally i.e., in some Ω_α. The representation (Ω, μ) depends on the choice of $(\Omega_\alpha, \mu_\alpha)$. However, we can always go to the representation using resolutions of the identity and our theory is independent of the choice of (Ω, μ). We denote $[0, \infty]^{\mathscr{B}}$ by $\hat{\mathfrak{Z}}_{0+}$ as is commonly used. Also we denote $\boldsymbol{C}^{\mathscr{B}}$ by $\bar{\mathfrak{Z}}_0$, which is represented by the measurable functions. $(0, \infty)^{\mathscr{B}}$ is denoted by $\bar{\mathfrak{Z}}_0^{>0}$, which is represented by the measurable function into $(0, \infty)$. $[0, \infty)^{\mathscr{B}}$ is denoted by $\bar{\mathfrak{Z}}_{0+}$.

Now let $\hat{\tau}$ be a trace on $\tilde{\mathscr{M}}$ in $V^{(\mathscr{B})}$. Then the following conditions hold.

1) $x, y\in\mathscr{M}_+ \Rightarrow \hat{\tau}(\tilde{x}+\tilde{y})=\hat{\tau}(\tilde{x})+\hat{\tau}(\tilde{y})$
2) $f\in\bar{\mathfrak{Z}}_{0+}$, $x\in\mathscr{M}_+ \Rightarrow \hat{\tau}(f\tilde{x})=f\hat{\tau}(\tilde{x})$
3) $x\in\mathscr{M} \Rightarrow \hat{\tau}(\tilde{x}^*\tilde{x})=\hat{\tau}(\tilde{x}\tilde{x}^*)$.

We define $T: \mathscr{M}_+\to\hat{\mathfrak{Z}}_{0+}$ by the following equation

$$T(x)=\hat{\tau}(\tilde{x}).$$

Then T satisfies the following conditions.

1) $x, y\in\mathscr{M}_+ \Rightarrow T(x+y)=T(x)+T(y)$

2) $a \in \mathcal{Z}_{0+},\ x \in \mathcal{M}_+ \Rightarrow T(ax)=aT(x)$

3) $x \in \mathcal{M} \Rightarrow T(x^*x)=T(xx^*)$.

A mapping $T: \mathcal{M}_+ \to \hat{\mathcal{Z}}_{0+}$ is said to be a generalized $\mathcal{Z}_0$-valued trace on $\mathcal{M}$ if these three conditions are satisfied.

Now let T be a generalized $\mathcal{Z}_0$-valued trace on $\mathcal{M}$. Then we define $\hat{\tau}$ on $\tilde{\mathcal{M}}_+$ by the following equation

$$\hat{\tau}(f\tilde{x})=fT(x) \quad \text{for} \quad f \in \bar{\mathcal{Z}}_{0+} \quad \text{and} \quad x \in \mathcal{M}_+ .$$

If $f\tilde{x}=g\tilde{y}$, then $\frac{f}{h}x=\frac{g}{h}y$, where $h=1+f+g$. Since $\frac{f}{h}$ and $\frac{g}{h}$ are members of $\mathcal{Z}_{0+}$, we have $\frac{f}{h}T(x)=\frac{g}{h}T(y)$ and $fT(x)=gT(y)$. This together with condition 2) on T guarantees that $\hat{\tau}$ is a function from $\tilde{\mathcal{M}}_+$ into $[0, \infty]^{\mathcal{B}}$ in $V^{(\mathcal{B})}$. Therefore, the traces on $\tilde{\mathcal{M}}$ in $V^{(\mathcal{B})}$ correspond to the generalized $\mathcal{Z}_0$-valued traces on $\mathcal{M}$.

Now suppose that $\hat{\tau}$ is faithful i.e., $[\![\hat{\tau}(\tilde{x}^*\tilde{x})=0 \to \tilde{x}=0]\!]=1$ and $T(x^*x)=0$. Then $[\![\hat{\tau}(\tilde{x}^*\tilde{x})=0]\!]=1$. Therefore, $[\![\tilde{x}=0]\!]=1$ i.e., $x=0$. Hence T is faithful. On the other hand, suppose that T is faithful and $[\![\hat{\tau}(\tilde{x}^*\tilde{x})=0]\!]=b$. Then $[\![\hat{\tau}(\tilde{y}^*\tilde{y})=0]\!]=1$ i.e., $T(y^*y)=0$, where $y=bx$. Therefore, $y=0$ i.e., $[\![b\tilde{x}=0]\!]=1$ and $[\![\tilde{x}=0]\!] \geqq b$. Therefore, $[\![\hat{\tau}(\tilde{x}^*\tilde{x})=0 \to \tilde{x}=0]\!]=1$ and $\hat{\tau}$ is faithful. Therefore, the faithfulness of T and the faithfulness of $\hat{\tau}$ are equivalent.

Now we are going to prove that the normality of T and the normality of $\hat{\tau}$ are equivalent.

LEMMA. *Let* $x: I \to \mathcal{M}_+$. *Then* $y=\sup_{i \in I} x(i)$ *in* $\mathcal{M}_+$ *iff* $\tilde{y}=\sup_{i \in I} \tilde{x}(i)$ *in* $\tilde{\mathcal{M}}_+$.

PROOF. First let $y=\sup_{i \in I} x(i)$ in $\mathcal{M}_+$. Then we have $\tilde{y} \geqq \sup_{i \in I} \tilde{x}(i)$ in $\tilde{\mathcal{M}}_+$. Suppose that for every $i \in I$, $(x(i))^{\sim} \leqq f\tilde{z} \leqq \tilde{y}$, where $f \geqq 0$ and $z \geqq 0$. Then we have

$$0 \leqq \|f\tilde{z}\|_{\mathcal{B}} \leqq \|y\| \quad \text{a.e.}$$

Therefore, there exists $z_1 \in \mathcal{M}$ such that $\tilde{z}_1=f \cdot \tilde{z}$. Hence, we have $\sup_{i \in I} x(i) \leqq z_1 \leqq y$ and $y=z_1$. Now let $\tilde{y}=\sup_{i \in I} \tilde{x}(i)$ in $\tilde{\mathcal{M}}_+$. Then we have $\sup_{i \in I} x(i) \leqq y$ in $\mathcal{M}_+$. Since $\sup_{i \in I} x(i)=z$ implies $\tilde{z}=\sup_{i \in I} \tilde{x}(i)$, we have $\tilde{y}=\tilde{z}$ i.e., $y=z$.

Now let $\hat{\tau}$ be normal in $V^{(\mathcal{B})}$ and $y=\sup_{i \in I} x(i)$ in $\mathcal{M}_+$. Then $\tilde{y}=\sup_{i \in I} \tilde{x}(i)$ and $\hat{\tau}(\tilde{y})=\sup_{i \in I} \hat{\tau}(\tilde{x}(i))$. Since $\sup_{i \in I} \hat{\tau}(\tilde{x}(i))$ is decided by the order relation on $[0, \infty]^{\mathcal{B}}$ and $\sup_{i \in I} T(x(i))$ is decided by the order relation on $\hat{\mathcal{Z}}_+$ and the order relation of $[0, \infty]^{\mathcal{B}}$ corresponds to the order relation on $\hat{\mathcal{Z}}_+$, $\sup_{i \in I} T(x(i))$ corresponds to $\sup_{i \in I} \hat{\tau}(\tilde{x}(i))$. Therefore, we have $T(y)=\sup_{i \in I} T(x(i))$ i.e., T is normal. Now let T be normal and $\hat{x}: \hat{I} \to \tilde{\mathcal{M}}_+$ and $\hat{y}=\sup_{i \in I} \hat{x}(i)$ in $\tilde{\mathcal{M}}_+$. Then we define $J=\{i \mid i \in \mathcal{D}(\hat{I})\}$

and we may assume that $\hat{x}: \check{J} \to \mathcal{M}_+$ and $\hat{y} = \sup_{i \in J} \hat{x}(i)$. Let $\hat{y} = f\tilde{y}$, where $f \geqq 0$. There exists $y_0 \in \mathcal{M}_+$ such that $\tilde{y}_0 = \frac{1}{1+f}\hat{y}$. Since for every $i \in J$, $0 \leqq \frac{1}{1+f}\hat{x}(i) \leqq \tilde{y}$, there exists $x_i \in \mathcal{M}_+$ such that $\frac{1}{1+f}\hat{x}(i) = \tilde{x}_i$. Then it suffices to show $\hat{\tau}(\tilde{y}_0) = \sup_{i \in J} \hat{\tau}(x_i)$. However, this follows from $T(y_0) = \sup_{i \in J} T(x_i)$ and the proof of the normality of $\hat{\tau}$ is established.

Now we are going to show that semi-finiteness of $\hat{\tau}$ and the semifiniteness of T are equivalent. First for every $x \in \mathcal{M}$ we define the $\mathcal{Z}_0$-support of x, denoted by $z(x)$, to be the smallest b in $\mathcal{B}$ satisfying $bx = x$. Obviously $z(x) = [\![\tilde{x} \neq 0]\!]$. Let $S = \{y \in \mathcal{M}_+ \mid T(y) \in \mathcal{Z}_{0+}\}$. The semi-finiteness of T is defined to be

$$\forall x \in \mathcal{M}_+ \ z(x) = \sup\{z(y) \mid y \in S \wedge y \leqq x\}.$$

Now '$\hat{\tau}$ is semi-finite' is equivalent to

$$[\![\forall \hat{x} \in \tilde{\mathcal{M}}_+(\hat{x} \neq 0 \to \exists \hat{y} \in \tilde{\mathcal{M}}_+(\hat{y} \neq 0 \wedge \hat{\tau}(\hat{y}) < \infty \wedge \hat{y} \leqq \hat{x}))]\!] = 1,$$

i. e., for every $\hat{x} \in \tilde{\mathcal{M}}_+$,

$$[\![\hat{x} \neq 0 \to \exists \hat{y} \in \tilde{\mathcal{M}}_+(\hat{y} \neq 0 \wedge \hat{\tau}(\hat{y}) < \infty \wedge \hat{y} \leqq \hat{x})]\!] = 1.$$

Let $\hat{x} = f\tilde{x}$, $f \geqq 0$, and $[\![f = 0]\!] = [\![\tilde{x} = 0]\!]$. If $\tilde{x} \neq 0$ and there exists $\hat{y}$ with $\hat{y} \neq 0 \wedge \hat{\tau}(\hat{y}) < \infty \wedge \hat{y} \leqq \tilde{x}$, then $f\hat{y} \neq 0 \wedge \hat{\tau}(f\hat{y}) < \infty \wedge f\hat{y} \leqq f\tilde{x}$ since $\tilde{x} \neq 0 \to f \neq 0$. Conversely if $\hat{x} \neq 0$ and there exists $\hat{y}$ with $\hat{y} \neq 0 \wedge \hat{\tau}(\hat{y}) < \infty \wedge \hat{y} \leqq \hat{x}$, then $\frac{1}{f}\hat{y} \neq 0 \wedge \hat{\tau}\left(\frac{1}{f}\hat{y}\right) < \infty \wedge \frac{1}{f}\hat{y} \leqq \tilde{x}$ since $\tilde{x} \neq 0 \to f \neq 0$. Therefore, '$\hat{\tau}$ is semi-finite' is equivalent to the statement that for every $x \in \mathcal{M}_+$,

$$[\![\tilde{x} \neq 0 \to \exists \hat{y} \in \tilde{\mathcal{M}}_+(\hat{y} \neq 0 \wedge \hat{\tau}(\hat{y}) < \infty \wedge \hat{y} \leqq \tilde{x})]\!] = 1$$

i. e.,

$$z(x) \leqq [\![\exists \hat{y} \in \tilde{\mathcal{M}}_+(\hat{y} \neq 0 \wedge \hat{\tau}(\hat{y}) < \infty \wedge \hat{y} \leqq \tilde{x})]\!].$$

Since $[\![\hat{y} \leqq \tilde{x}]\!] \leqq z(x)$, $\leqq$ can be replaced by $=$. Since for every $\hat{y} \in \tilde{\mathcal{M}}_+$, $0 \leqq \hat{y} \leqq \tilde{x}$ implies that $\hat{y}$ is of the form $\tilde{y}$, the semi-finiteness of $\hat{\tau}$ can be expressed by the following form. For every $x \in \mathcal{M}_+$,

$$\begin{aligned} z(x) &= \bigvee_{y \in \mathcal{M}_+} [\![\tilde{y} \neq 0 \wedge \hat{\tau}(\tilde{y}) < \infty \wedge \tilde{y} \leqq \tilde{x}]\!] \\ &= \bigvee_{y \in \mathcal{M}_+} \bigvee_{n < \omega} [\![\tilde{y} \neq 0 \wedge \hat{\tau}(\tilde{y}) < n \wedge \tilde{y} \leqq \tilde{x}]\!]. \end{aligned}$$

Now if $y \in S$, then $\bigvee_{n<\omega} [\![\hat{\tau}(\tilde{y}) < n]\!] = 1$. On the other hand, if $[\![\tilde{y} \neq 0 \wedge \hat{\tau}(\tilde{y}) < n \wedge \tilde{y} \leqq \tilde{x}]\!] = b$, then $[\![(by)^{\sim} \neq 0]\!] = b$ and $[\![\hat{\tau}((by)^{\sim}) < n \wedge (by)^{\sim} \leqq \tilde{x}]\!] = 1$. Therefore, the semi-finiteness of $\hat{\tau}$ is equivalent to the statement that for every $x \in \mathcal{M}_+$

$$z(x)=\bigvee_{y\in S}[\![\tilde{y}\neq 0\wedge\tilde{y}\leqq\tilde{x}]\!].$$

Since $[\![\tilde{y}\neq 0\wedge\tilde{y}\leqq\tilde{x}]\!]=b$ iff $[\![(by)^{\sim}\neq 0]\!]=b$ and $[\![(by)^{\sim}\leqq\tilde{x}]\!]=1$,

$$z(x)=\bigvee\{[\![\tilde{y}\neq 0]\!] \mid y\in S \text{ and } y\leqq x\}$$

$$=\sup\{z(y) \mid y\in S \text{ and } y\leqq x\}.$$

This proves the equivalence between the semi-finiteness of $\hat{\tau}$ and the semi-finiteness of T.

A generalized $\mathcal{Z}$-valued trace T on $\mathcal{M}$ is said to be finite if T is a mapping from $\mathcal{M}_+$ into $\bar{\mathcal{Z}}_{0+}$. Then '$\hat{\tau}$ is finite' is equivalent to the finiteness of T. If $\hat{\tau}$ or T is finite, then $\hat{\tau}$ or T can be extended uniquely to a positive linear functional on $\tilde{\mathcal{M}}$ or $\mathcal{M}$ respectively. If $\hat{\tau}$ is finite and faithful, then $\hat{\tau}(1)>0$. Let $f=\hat{\tau}(1)$. Now by rewriting $\frac{1}{f}\hat{\tau}$ by $\hat{\tau}$, if necessary, we may assume that $\hat{\tau}(1)=1$ i.e., $T(1)=1$. $\hat{\tau}$ (or T) satisfying $\hat{\tau}(1)=1$ (or $T(1)=1$) is said to be canonical. For canonical trace, $|\hat{\tau}(\tilde{x})|\leqq(\|x\|)^{\vee}$ for every $x\in\mathcal{M}$. Therefore, $T(x)$ belongs to $\mathcal{Z}$ i.e., $T:\mathcal{M}\to\mathcal{Z}$.

Now let us assume that $\mathcal{Z}_0$ is the center of $\mathcal{M}$ i.e., $\mathcal{Z}_0=\mathcal{Z}$. In this case, a finite canonical faithful trace $\hat{\tau}$ on $\tilde{\mathcal{M}}$ corresponds to a canonical center valued trace T on $\mathcal{M}$. The uniqueness and normality of a canonical center valued trace follows from the uniqueness and normality of a finite canonical faithful trace on a factor. The existence of the canonical center valued trace on $\mathcal{M}$ is equivalent to the finiteness of $\mathcal{M}$, since for every factor, the existence of finite faithful trace on $\mathcal{M}$ is equivalent to the finiteness of $\mathcal{M}$.

If $\mathcal{Z}_0$ is the center of $\mathcal{M}$, then a generalized $\mathcal{Z}_0$-valued trace on $\mathcal{M}$ is called a generalized center-valued trace on $\mathcal{M}$. Since the semi-finiteness of $\mathcal{M}$ is equivalent to the existence of a faithful semi-finite normal trace on $\mathcal{M}$, the semi-finiteness is equivalent to the existence of a faithful semi-finite normal generalized $\mathcal{Z}_0$-valued trace.

Since any semi-finite normal faithful traces on a semi-finite factor are proportional, a faithful semi-finite normal generalized center valued trace is unique up to a multiplication by $\bar{\mathcal{Z}}^{>0}$ in the sense that if T_1 and T_2 are two such generalized center valued traces then there exists $f\in\bar{\mathcal{Z}}^{>0}$ such that

$$T_1(x)=f\cdot T_2(x) \qquad \text{for} \quad x\in\mathcal{M}_+.$$

DEFINITION. A map $H:\mathcal{M}_+\to\hat{\mathcal{Z}}_{0+}$ is said to be a generalized $\mathcal{Z}_0$-valued weight if it satisfies the following conditions.

1) $H(x+y)=H(x)+H(y)$ for $x, y\in\mathcal{M}_+$

2) $H(ax)=aH(x)$ for $a\in\mathcal{Z}_{0+}$, $x\in\mathcal{M}_+$.

H is said to be faithful, if for every $x\in\mathcal{M}_+$, $H(x)=0$ implies $x=0$. H is said

to be normal if $H(\sup_i x_i)=\sup_i H(x_i)$ for any bounded increasing net $\{x_i\}$ in $\mathcal{M}_+$. H is said to be semi-finite if $S''=\mathcal{M}$ where $S=\{y\in\mathcal{M}_+ \mid H(y)\in\mathcal{Z}_{0+}\}$. In the same way as in the case of traces, a weight $\hat{\varphi}$ on $\tilde{\mathcal{M}}_+$ in $V^{(\mathcal{B})}$ corresponds to a generalized $\mathcal{Z}_0$-valued weight H on $\mathcal{M}_+$. The faithfulness, normality, and semi-finiteness of $\hat{\varphi}$ is also equivalent to the faithfulness, normality, and semi-finiteness of H respectively. Since there exists a normal semi-finite faithful weight on every von Neumann algebra, there exists a normal semi-finite faithful generalized $\mathcal{Z}_0$-valued weight on every von Neumann algebra whose center includes $\mathcal{Z}_0$. If $\mathcal{Z}_0$ is the center, then a generalized $\mathcal{Z}_0$-valued weight is called as a generalized center valued weight.

§ 5. Two examples of direct translations.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two von Neumann algebras whose centers include $\mathcal{Z}_0$. Let $\mathcal{B}$ be the complete Boolean algebra of all projections in $\mathcal{Z}_0$. Then we can build $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$ in $V^{(\mathcal{B})}$ corresponding to $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Then $\tilde{\mathcal{M}}_1\overline{\otimes}\tilde{\mathcal{M}}_2$ is a von Neumann algebra in $V^{(\mathcal{B})}$. Therefore, we have a von Neumann algebra $\mathcal{M}$ such that $\tilde{\mathcal{M}}$ in $V^{(\mathcal{B})}$ is isomorphic to $\tilde{\mathcal{M}}_1\overline{\otimes}\tilde{\mathcal{M}}_2$. Obviously the center of $\mathcal{M}$ include $\mathcal{Z}_0$. Let $x_1\in\mathcal{M}_1$ and $x_2\in\mathcal{M}_2$. Then $\tilde{x}_1\otimes\tilde{x}_2\in\tilde{\mathcal{M}}_1\overline{\otimes}\tilde{\mathcal{M}}_2$ and $\|\tilde{x}_1\otimes\tilde{x}_2\|_{\mathcal{B}}\leqq(\|x_1\|\cdot\|x_2\|)^{\vee}$. Therefore, there exists a unique $x\in\mathcal{M}$ such that $\tilde{x}=\tilde{x}_1\otimes\tilde{x}_2$. We denote $\mathcal{M}$ by $\mathcal{M}_1\overline{\otimes}_{\mathcal{Z}_0}\mathcal{M}_2$ and the x constructed in this way by $x_1\otimes_{\mathcal{Z}_0}x_2$. Then we have the following properties.

1) $(ax_1+by_1)\otimes_{\mathcal{Z}_0}x_2=a(x_1\otimes_{\mathcal{Z}_0}x_2)+b(y_1\otimes_{\mathcal{Z}_0}x_2)$
2) $x_1\otimes_{\mathcal{Z}_0}(ax_2+by_2)=ax_1\otimes_{\mathcal{Z}_0}x_2+bx_1\otimes_{\mathcal{Z}_0}y_2$
3) $(x_1\otimes_{\mathcal{Z}_0}x_2)(y_1\otimes_{\mathcal{Z}_0}y_2)=x_1y_1\otimes_{\mathcal{Z}_0}x_2y_2$
4) $(x_1\otimes_{\mathcal{Z}_0}x_2)^*=x_1^*\otimes_{\mathcal{Z}_0}x_2^*$
5) $\|x_1\otimes_{\mathcal{Z}_0}x_2\|\leqq\|x_1\|\cdot\|x_2\|$
6) $x_1\otimes_{\mathcal{Z}_0}x_2=0$ iff $\exists b_1, b_2\in\mathcal{B}(b_1\cdot b_2=0,\ b_1x_1=x_1,$ and $b_2x_2=x_2)$

where $a, b\in\mathcal{Z}_0$, $x_1, y_1\in\mathcal{M}_1$, and $x_2, y_2\in\mathcal{M}_2$. Obviously the set $\{x_1\otimes_{\mathcal{Z}_0}x_2 \mid x_1\in\mathcal{M}_1$ and $x_2\in\mathcal{M}_2\}$ generates $\mathcal{M}_1\overline{\otimes}_{\mathcal{Z}_0}\mathcal{M}_2$.

Since every factor of type II_∞ is obtained as $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$, where $\mathcal{M}_1$ is a factor of type II_1 and $\mathcal{M}_2$ is a factor of type I_∞, we immediately get the following theorem.

THEOREM. *Every von Neumann algebra of type II_∞ with the center $\mathcal{Z}$ is obtained as $\mathcal{M}_1\overline{\otimes}_{\mathcal{Z}}\mathcal{M}_2$, where $\mathcal{M}_1$ is a von Neumann algebra of type II_1 with the center $\mathcal{Z}$ and $\mathcal{M}_2$ is of the form of $\mathcal{Z}'$ in some $\mathcal{L}(\mathcal{H})$, $\mathcal{Z}\subseteqq\mathcal{L}(\mathcal{H})$, and $\mathcal{Z}'$ is properly infinite.*

Now let $\mathfrak{N}$ be a von Neumann algebra, $\mathcal{Z}$ an abelian von Neumann algebra, and U a function on $\bar{\mathcal{Z}}^{>0}$ whose value is always a unitary operator. Then $\mathfrak{N}\times_U\mathcal{Z}$ denotes the von Neumann algebra generated by $\mathfrak{N}\cup\{U(f) \mid f\in\bar{\mathcal{Z}}^{>0}\}$.

Then our machinery provides us with the following theorem from a special case of Takesaki's theorem in [4].

THEOREM. *Every von Neumann algebra of type III with the center $\mathcal{Z}$ is of the form $\mathfrak{N}\times_U\mathcal{Z}$ satisfying the following conditions:*

1) *$\mathfrak{N}$ is a von Neumann algebra of type II_∞ whose center includes $\mathcal{Z}$.*

2) $f\in\bar{\mathcal{Z}}^{>0}\Rightarrow U(f)\mathfrak{N}U^*(f)=\mathfrak{N}$.

3) $b\in\mathcal{B}$, $f, g\in\bar{\mathcal{Z}}^{>0}$, $bf=bg\Rightarrow bU(f)=bU(g)$.

4) $f_1, f_2, \cdots, f\in\bar{\mathcal{Z}}^{>0}$, $\lim_{i\to\infty} f_i=f$ *a.e.*

$$\Rightarrow \lim_{i\to\infty} U(f_i)=U(f) \text{ in the strong topology.}$$

5) *There exists a $\hat{\mathcal{Z}}_+$-valued faithful normal trace T on $\mathfrak{N}$ such that* $T(U(f)xU^*(f))=fT(x)$.

6) $U(f_1, f_2)=U(f_1)U(f_2)$.

Furthermore, the composition is uniquely determined.

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Gaisi TAKEUTI

Department of Mathematics
University of Illinois
Urbana, Illinois 61801
U.S.A.