J. Math. Soc. Japan
Vol. 35, No. 2, 1983

Equational theories and universal theories of fields

By Hiroakira ONO

(Received Jan. 11, 1982)

1. Introduction.

In this paper, we will investigate equational theories and universal theories of fields. To begin with, we will explain how to deal with problems connected with the multiplicative inverse in field theory. The multiplicative inverse x^{-1} on a given field is defined for all x but zero element 0, so the function $^{-1}$ will be regarded as a partial function. On the other hand, in the customary treatment in logic, every function symbol is interpreted as a total function on a given structure. Thus, the language $\mathcal{L}$ for ring theory which consists of $\{+, -, \cdot, 0, 1\}$ will be usually employed when we formulate the theory of fields. In this language, the existence of the inverse will be represented as

$$(1) \qquad \forall x \exists z(\neg(x=0) \rightarrow xz=1).$$

But, if we will restrict our attention only to universal theories of fields, then we will have to deal with not only the class of fields but also some broader class of algebraic structures, since (1) can not be expressed by a universal formula, i.e., a formula of the form $\forall x_1 \cdots \forall x_n\, \varphi(x_1, \cdots, x_n)$ for some open formula $\varphi(x_1, \cdots, x_n)$. In the above case, we can take the class of integral domains for this, because the axioms of integral domains, which we denote by Θ in the following, can be expressed by universal formulas and moreover it can be shown that the set of universal formulas of $\mathcal{L}$ valid in all fields (of characteristic p) is equal to the set of universal formulas valid in all integral domains (of characteristic p).

On the other hand, it will be possible to treat equational theories of fields, if we determine the value of 0^{-1} in any way. In this way, Komori introduced in [**9**] the notion of *pseudo-fields* and proved that for any equation if it holds in every skew field then it holds also in every pseudo-field and vice versa. In particular, he introduced the notion of *desirable fields*, the skew fields in which $0^{-1}=0$ holds. Following his idea, we will introduce an axiom system Σ for the equational theory of commutative regular rings in the language $\mathcal{L}'=\{+, -, \cdot, 0, 1, ^{-1}\}$. Thus, we will study equational theories of (commutative)

This research was partially supported by Grant-in-Aid for Scientific Research (No. 56340009), Ministry of Education.

fields over Σ and universal theories of (commutative) fields over Θ in this paper. It turns out in § 3 that if restricting only to fields these equational theories and universal theories are essentially equivalent. More precisely, there exists a one-to-one correspondence between equational theories and universal theories of fields, by which both the decidability and the recursive axiomatizability are preserved. In § 4, the axiomatization of equational and universal theories of some special fields will be given explicitly. Then, some decidability results will be obtained. In § 5, we will give a classification of the whole class of equational and universal theories of fields, by means of absolute number fields, following the results due to Ax [**2**] and Wheeler [**21**]. Then we will show that there always exists the smallest equational (or universal) theory among equational (or universal) theories determined by fields having a fixed absolute number field. Furthermore, it will be shown that any two equational (or universal) theories of fields having an isomorphic infinite absolute number field of characteristic $p>0$ are equal, by using Riemann hypothesis for curves proved by Weil.

2. Preliminaries.

We will introduce the axiom system Σ of commutative regular rings in the following. Recall that a commutative ring R is *regular* if x^2 divides x for each $x \in R$. Let $\mathcal{L}'$ be the language obtained from the language $\mathcal{L}$ for ring theory by adding a unary function symbol $^{-1}$ for multiplicative inverse. We define Σ to be the axiom system consisting of

1) usual axioms of commutative rings,

2a) $x^2x^{-1}=x$,

2b) $(x^{-1})^2x=x^{-1}$.

It can be easily verified that Σ is logically equivalent to the axiom system of *desirable pseudo-fields* by Komori [**9**], if we add the axiom of commutativity to the latter (see also [**10**]). It is easy to see the following.

LEMMA 2.1. 1) *Any integral domain which is a model of Σ is a field.* (*In general, any integral domain which is also a commutative regular ring is a field.*)

2) *A field F is a model of Σ if and only if* $0^{-1}=0$ *holds in F, i.e., F is a desirable commutative field in* [**9**].

Our claim $0^{-1}=0$ is not so special, because the theory of fields with the axiom $0^{-1}=0$ in the language $\mathcal{L}'$ is a conservative extension of the usual theory of fields in $\mathcal{L}$, since the former can be considered to be an extension by definitions of the latter in the sense of [**18**] § 4.6.

An *equation* of $\mathcal{L}'$ is a formula of the form $t=s$, where t and s are terms of $\mathcal{L}'$. An equation ε is said to hold in a model R of Σ if ε holds for every assignment of values in R to variables appearing in ε. Define $\mathrm{Eq}(R)$ to be the set of all equations which hold in R. Let U be any class of models of Σ. By abuse of symbol, we write $\mathrm{Eq}(U)$ for the set of all equations which hold in every ring in U. Thus, $\mathrm{Eq}(U)=\bigcap_{R\in U}\mathrm{Eq}(R)$ holds. We call $\mathrm{Eq}(R)$ (or $\mathrm{Eq}(U)$) the *equational theory determined by* a ring R (or a class U of rings, respectively). As for the general aspect of equational theories, see Grätzer [6] Appendix 4. For any set Γ of equations including Σ, we define $\mathrm{mod}\,\Gamma$ to be the class of all models of Γ and $\mathrm{mod}^*\Gamma$ to be the class of all fields in $\mathrm{mod}\,\Gamma$. We call a ring R a *subdirect product* of a class of rings $\{S_i\}_i$ if there exists a monomorphism $h: R\rightarrow S=\prod_i S_i$ such that $\pi_i\circ h$ is an epimorphism for each i, where π_i is a canonical mapping from S to S_i. The first part of the following proposition is a version of Birkhoff's theorem (see e.g. [12] Corollary 1 in § 2.1 and Proposition 4 in § 2.2).

PROPOSITION 2.2. *Let Γ be any set of equations such that $\Sigma\subset\Gamma$. Then every commutative regular ring R in* $\mathrm{mod}\,\Gamma$ *is a subdirect product of some fields $\{F_i\}_i$ in* $\mathrm{mod}\,\Gamma$. *Moreover,* $\mathrm{Eq}(R)=\bigcap_i\mathrm{Eq}(F_i)$ *holds.*

We can take $\{R/p\,;\,p\in I(R)\}$ for $\{F_i\}_i$ in the above proposition, where $I(R)$ denotes the set of all prime ideals of R. In this case, the monomorphism h: $R\rightarrow\prod_{p\in I(R)}R/p$ is called the *canonical representation* of R. By Proposition 2.2, the following result by Komori [9] (Theorem 3.1 and Corollary 3.3) can be derived.

COROLLARY 2.3. *Let Γ be any set of equations such that $\Sigma\subset\Gamma$. Then* $\mathrm{Eq}(\mathrm{mod}\,\Gamma)=\mathrm{Eq}(\mathrm{mod}^*\Gamma)$ *holds. Hence, the class* $\mathrm{mod}\,\Gamma$ *is the minimum equational class including* $\mathrm{mod}^*\Gamma$.

Clearly, any direct product of rings in $\mathrm{mod}\,\Gamma$ is also in $\mathrm{mod}\,\Gamma$. So, we have the following.

LEMMA 2.4. *For each set of rings $\{R_i\}_i$ in* $\mathrm{mod}\,\Gamma$ *there exists a ring R in* $\mathrm{mod}\,\Gamma$ *such that*

$$\bigcap_i\mathrm{Eq}(R_i)=\mathrm{Eq}(R),$$

where Γ is a set of equations such that $\Sigma\subset\Gamma$.

For each integral domain G, define $\mathrm{Th}(G)$ (or $\mathrm{T}_\forall(G)$) to be the set of all formulas (or, of all universal formulas, respectively) in the language $\mathcal{L}$ which hold in G. We say $\mathrm{Th}(G)$ (or $\mathrm{T}_\forall(G)$) the *first-order theory* (or the *universal theory*) *determined by* G. Similarly as the case for equational theories, $\mathrm{Th}(U)$ and $\mathrm{T}_\forall(U)$ can be defined also for a class U of integral domains. The usual axiom system of integral domains, consisting of finitely many universal formulas, will be denoted by Θ. We can not show the corresponding result to the second part of Proposition 2.2 for universal theories. But, we can show easily the following lemma, which plays an alternative role of Corollary 2.3.

LEMMA 2.5. *Let Ψ be a set of formulas including Θ. If for any* $G\in\mathrm{mod}\,\Psi$

there exists a field $F \in \mathrm{mod}\,\Psi$ *such that* $G \subset F$, *then* $\mathrm{T}_{\forall}(\mathrm{mod}\,\Psi)=\mathrm{T}_{\forall}(\mathrm{mod}^*\Psi)$ *holds.*

3. Relations between equational theories and universal theories.

Hereafter, we sometimes say a member of $\mathrm{mod}\,\Sigma$ (or of $\mathrm{mod}^*\Sigma$) merely a commutative regular ring (or a field, respectively) when no confusions will occur. Conversely, we suppose that each field is a member of $\mathrm{mod}^*\Sigma$ by claiming $0^{-1}=0$. In this section, we will show that universal theories of fields in the language $\mathcal{L}$ are essentially equivalent to equational theories of fields in $\mathcal{L}'$.

LEMMA 3.1. *Let $t=t(x_1, \cdots, x_n)$ be any term in $\mathcal{L}'$ and y be a variable not appearing in t. Then there exists an effective way of getting an open formula $\varphi_t(x_1, \cdots, x_n, y)$ such that*

1) *every atomic formula of $\varphi_t(x_1, \cdots, x_n, y)$ is of the form either $s=0$ or $y=s$ or $y=s_1 s_2^{-1}$, where s, s_1 and s_2 are terms having neither any occurrence of $^{-1}$ nor of y,*

2) *in any field of* $\mathrm{mod}^*\Sigma$, *it holds that*

$$y=t \longleftrightarrow \varphi_t(x_1, \cdots, x_n, y).$$

PROOF. Our lemma can be shown by using induction. Since it will be tedious to give a complete proof, we will show this only for the case where $t=x_1 x_2^{-1} x_3^{-1}+(x_3^{-1})^{-1} x_1^{-1}$, as an example. Since $(xy)^{-1}=x^{-1}y^{-1}$ and $(x^{-1})^{-1}=x$ hold in every field in $\mathrm{mod}^*\Sigma$ (see [9] p. 14), t is equal to $x_1(x_2x_3)^{-1}+x_3x_1^{-1}$. So, by using $0^{-1}=0$, $y=t$ holds if and only if the following holds;

$$\begin{aligned}&(x_2x_3=0\wedge x_1=0\wedge y=0)\vee(x_2x_3=0\wedge\neg(x_1=0)\wedge y=x_3x_1^{-1})\\ &\vee(\neg(x_2x_3=0)\wedge x_1=0\wedge y=x_1(x_2x_3)^{-1})\\ &\vee(\neg(x_2x_3=0)\wedge\neg(x_1=0)\wedge y=(x_1^2+x_2x_3^2)(x_1x_2x_3)^{-1}).\end{aligned}$$

Now let $\varphi_t(x_1, \cdots, x_n, y)$ be the right side of the above equivalence. Then we have our lemma.

COROLLARY 3.2. *For any equation $t_1=t_2$ in $\mathcal{L}'$, there exists an effective way of getting an open formula $\psi_{t_1=t_2}$ in $\mathcal{L}$ such that $t_1=t_2 \leftrightarrow \psi_{t_1=t_2}$ holds in any field in* $\mathrm{mod}^*\Sigma$.

PROOF. By Lemma 3.1, there exists an effective way of getting an open formula $\varphi_{t_1-t_2}(\boldsymbol{x}, y)$ such that

$$y=t_1-t_2 \longleftrightarrow \varphi_{t_1-t_2}(\boldsymbol{x}, y)$$

holds. Let $\psi_{t_1=t_2}$ be the formula obtained from $\varphi_{t_1-t_2}(\boldsymbol{x}, y)$ by replacing each atomic formula of the form $y=s$ by $s=0$ and $y=s_1s_2^{-1}$ by $s_1=0\vee s_2=0$. Then it is obvious that this formula $\psi_{t_1=t_2}$ satisfies the condition in our corollary.

From the above corollary it follows that for any equation $t=s$, $t=s \in \mathrm{Eq}(F)$

if and only if $\forall \boldsymbol{x}\phi_{t=s}(\boldsymbol{x})\in \mathrm{T}_{\forall}(F)$ for every field F. Conversely, let φ be any universal formula in $\mathcal{L}$. By Wheeler [21] Lemma 6.1, φ is equivalent to a conjunction of formulas of the form

$$\forall \boldsymbol{y}\neg(p_1(\boldsymbol{y})=0\wedge \cdots \wedge p_n(\boldsymbol{y})=0)\,, \tag{1}$$

where each $p_i(\boldsymbol{y})$ is a polynomial in $\boldsymbol{Z}[\boldsymbol{y}]$. Moreover, there exists an effective way of getting these formulas of the form (1) from a given φ. But each formula of the form (1) is also equivalent to an equation

$$(p_1(\boldsymbol{y})p_1(\boldsymbol{y})^{-1}-1)\cdot \cdots \cdot(p_n(\boldsymbol{y})p_n(\boldsymbol{y})^{-1}-1)=0\,, \tag{2}$$

in any field other than zero field, i.e., the field satisfying $0=1$. For, $xx^{-1}=1$ means $\neg(x=0)$ in any field other than zero field. Thus, we have the following.

LEMMA 3.3. *For each universal formula φ in $\mathcal{L}$, there exists an effective way of getting a finite set of equations $\{t_i=s_i\,;\, i=1, \cdots, k\}$ such that*

$\varphi\in \mathrm{T}_{\forall}(F)$ if and only if $t_i=s_i\in \mathrm{Eq}(F)$ for each $i=1, \cdots, k$,

for any field F other than zero field.

By Corollary 3.2 and Lemma 3.3, we can show the following theorem immediately.

THEOREM 3.4. *For all fields F and F' other than zero field,*

$$\mathrm{Eq}(F)\subset \mathrm{Eq}(F') \qquad \textit{if and only if } \mathrm{T}_{\forall}(F)\subset \mathrm{T}_{\forall}(F')\,.$$

Hence, $\mathrm{Eq}(F)=\mathrm{Eq}(F')$ if and only if $\mathrm{T}_{\forall}(F)=\mathrm{T}_{\forall}(F')$.

Let Γ_0 and Γ be two sets of equations (or universal formulas). If $\Gamma=\mathrm{Eq}(\mathrm{mod}\,\Gamma_0)$ (or, $\Gamma=\mathrm{T}_{\forall}(\mathrm{mod}\,\Gamma_0)$, respectively) holds, we say Γ_0 is a *set of axioms for Γ*. If there exists a finite (or recursive) set Γ_0 of axioms for an equational or a universal theory Γ, we say that Γ is *finitely* (or *recursively*) *axiomatized*.

THEOREM 3.5. *Let F be any field.*

1) *The equational theory $\mathrm{Eq}(F)$ is decidable if and only if the universal theory $\mathrm{T}_{\forall}(F)$ is decidable.*

2) *If $\mathrm{T}_{\forall}(F)$ is finitely (or recursively) axiomatized then $\mathrm{Eq}(F)$ is also finitely (or recursively) axiomatized. Conversely, if $\mathrm{Eq}(F)$ is recursively axiomatized then so is $\mathrm{T}_{\forall}(F)$.*

PROOF. It is easy to see that 1) follows from Corollary 3.2 and Lemma 3.3, since every construction has been done effectively. So, we will show 2). We have only to show this for the case where F is a field other than zero field. Suppose that $\mathrm{T}_{\forall}(F)$ is axiomatized by a set Γ of universal formulas. Of course, we can assume that $\Theta\subset\Gamma$. For each $\varphi\in\Gamma-\Theta$, there exists a finite set $\Delta(\varphi)$ of equations, for which Lemma 3.3 holds. Now define a set Δ of equations by $\Delta=\Sigma\cup\bigcup_{\varphi}\Delta(\varphi)$, where φ ranges over formulas in $\Gamma-\Theta$ and Σ denotes the axiom system of commutative regular rings. We will show that Δ is a set of axioms for $\mathrm{Eq}(F)$. Let $\tilde{\Delta}=\mathrm{Eq}(\mathrm{mod}\,\Delta)$. Clearly, $\tilde{\Delta}\subset\mathrm{Eq}(F)$ since F is a model of Δ. By Lemma 2.4, there exists a model R of Δ such that $\tilde{\Delta}=\mathrm{Eq}(R)$. By Proposition 2.2, there exists a set of fields $\{F_i\}_i$ in $\mathrm{mod}^*\Delta$ such that $\tilde{\Delta}=\mathrm{Eq}(R)=\bigcap_i \mathrm{Eq}(F_i)$. Here we can suppose that each F_i is not a zero field. Since each equation in $\Delta(\varphi)$ is valid in F_i, φ is also valid in F_i, for each $\varphi\in\Gamma-\Theta$. Thus, F_i is a model of Γ, i.e., $\mathrm{T}_\forall(F)\subset\mathrm{T}_\forall(F_i)$. By Theorem 3.4, $\mathrm{Eq}(F)\subset\mathrm{Eq}(F_i)$ for each i. Hence, $\mathrm{Eq}(F)\subset\bigcap_i \mathrm{Eq}(F_i)=\tilde{\Delta}$. Thus, $\tilde{\Delta}=\mathrm{Eq}(F)$. So, Δ is a set of axioms for $\mathrm{Eq}(F)$. When Γ is finite (or recursive), Δ is also finite (or recursive). Conversely, suppose that $\mathrm{Eq}(F)$ has a recursive set Δ of axioms. Let Γ be the axiom system obtained from Θ by adding axioms

1) $\forall x\exists z(\neg(x=0)\longrightarrow xz=1)$,

2) ψ_ε for each equation ε in Δ,

where ψ_ε denotes the open formula corresponding to ε, which is defined in Corollary 3.2. Let $\Phi=\mathrm{Th}(\mathrm{mod}\,\Gamma)$. Since Γ is recursive, Φ is recursively enumerable and hence $\Phi_\forall$, the set of all universal formulas in Φ, is also recursively enumerable. So, $\Phi_\forall$ is recursively axiomatized (cf. e.g. [18] p. 138). Now we will show that $\mathrm{T}_\forall(F)=\Phi_\forall$. Since the field F is a model of Γ, $\Phi\subset\mathrm{Th}(F)$ and therefore $\Phi_\forall\subset\mathrm{T}_\forall(F)$. On the other hand, we can suppose that $\Phi_\forall$ is of the form $\bigcap_j \mathrm{T}_\forall(F_j)$, where $\{F_j\}_j$ is a set of models of Γ, since $\Phi=\mathrm{Th}(\mathrm{mod}\,\Gamma)$ holds. So, every F_j is a field satisfying both ψ_ε and ε. Hence, $\mathrm{Eq}(F)=\tilde{\Delta}\subset\mathrm{Eq}(F_j)$ for each j. By Theorem 3.4, $\mathrm{T}_\forall(F)\subset\mathrm{T}_\forall(F_j)$ holds. Thus $\mathrm{T}_\forall(F)\subset\bigcap_j\mathrm{T}_\forall(F_j)=\Phi_\forall$. Hence, $\mathrm{T}_\forall(F)$ $(=\Phi_\forall)$ is recursively axiomatized.

Let Γ_0 be the axiom system obtained from Γ in the above proof by deleting the axiom 1). In general, Γ_0 can not be a set of axioms for $\mathrm{T}_\forall(F)$. We will show this when F is the field $\boldsymbol{R}$ of real numbers, as an example. By Tarski's result on the completeness of the first-order theory of real closed fields, $\mathrm{Eq}(\boldsymbol{R})=\mathrm{Eq}(F)$ and $\mathrm{T}_\forall(\boldsymbol{R})=\mathrm{T}_\forall(F)$ hold for each real closed field F. Due to Artin-Schreier, formally real fields are characterized as fields in which

$$\neg(1+x_1^2+\cdots+x_n^2=0)$$

holds for each n. By using the fact that any formally real field can be embedded in a real closed field, we have the following theorem.

THEOREM 3.6. *The equational theory* $\mathrm{Eq}(\boldsymbol{R})$ *is axiomatized by* Σ *with axioms*

$$(1+\sum_{i=1}^n x_i^2)(1+\sum_{i=1}^n x_i^2)^{-1}=1$$

for any natural number n.

On the other hand, the axiom system Θ with axioms

$$\neg(1+\sum_{i=1}^{n} x_i^2=0) \qquad \text{for each natural number } n \tag{1}$$

can not characterize $T_{\forall}(\boldsymbol{R})$, since an integral domain satisfying the condition (1) need not be ordered. But, we can give explicitly an axiom system for $T_{\forall}(\boldsymbol{R})$ by using the following two propositions.

PROPOSITION 3.7 (see [5] Chapter VII §2 Corollary 8). *A ring R with 1 and without zero divisors can be ordered if and only if no sum of even products vanishes, more precisely, for every finite set of elements $a_1, \cdots, a_n$ of R ($a_i \neq 0$), no sum of products containing each a_i an even number of times is equal to* 0.

PROPOSITION 3.8 (see [5] Chapter VI §3 Corollary 5). *Every total ordering of an integral domain can be uniquely extended to a total ordering of its field of quotients.*

THEOREM 3.9. *The universal theory $T_{\forall}(\boldsymbol{R})$ is axiomatized by Θ with an infinite set Δ of universal formulas which says that no sum of even products vanishes.*

PROOF. It is obvious that $\boldsymbol{R}$ satisfies every axiom in $\Theta\cup\Delta$. For the completeness, we will show that for each universal formula φ in $T_{\forall}(\boldsymbol{R})$, φ is valid in every model of $\Theta\cup\Delta$. Let G be an arbitrary integral domain in which every formula in Δ is valid. Then by Proposition 3.7, G can be ordered. Thus G can be embedded in a formally real field F by Proposition 3.8. Then F can be in turn embedded in a real closed field $\tilde{F}$. Since both $\boldsymbol{R}$ and $\tilde{F}$ are real closed, φ is also valid in $\tilde{F}$ by the completeness of the first-order theory of real closed fields. But φ is a universal formula and G is a substructure of $\tilde{F}$. Hence φ is also valid in G.

We don't know whether the finite axiomatizability of $\mathrm{Eq}(F)$ implies that of $T_{\forall}(F)$ for any field F.

4. Axiom systems for some equational theories.

To begin with, we will give a finite set of axioms for each equational theory determined by a finite commutative regular ring. We remark that Kruse proved in [11] that every equational theory (in $\mathcal{L}$) determined by a finite ring can be finitely axiomatized.

It is obvious that the equational theory of the zero ring can be characterized by Σ with a single axiom $0=1$. So in the following we will consider only nonzero rings, i.e. rings in which $\neg(0=1)$ holds. Suppose that a finite commutative regular ring R in mod Σ is given. Let $h: R\rightarrow \prod_{i\in I} F_i$ be the canonical representation of R. Since R is finite, all fields F_i and the index set I are both finite. Define

$$\mathrm{Ch}(R)=\{\mathrm{char}(F_i)\,;\ i\in I\},$$

where char(F) means the characteristic of a field F. Clearly, each element p in Ch(R) is positive. We assume that

$$\mathrm{Ch}(R)=\{p_1,\cdots,p_m\},$$

where $p_i\neq p_j$ if $i\neq j$. We define $p_R=\prod_{j=1}^{m}p_j$. For each $p_j\in\mathrm{Ch}(R)$, let $F_{j1},\cdots,F_{jk}$ be an enumeration of all fields in $\{F_i\,;\,i\in I\}$ whose characteristic are equal to p_j. Moreover, we suppose that each F_{ji} is the finite field $\boldsymbol{F}_{p_j^{n_i}}$ with $p_j^{n_i}$ elements. Let $f_d^{(j)}(x)$ be the product of all irreducible monics of degree d in the polynomial ring $\boldsymbol{F}_{p_j}[x]$. We define

$$g_j(x)=\prod_d f_d^{(j)}(x),$$

where d ranges over every divisor of n_i for $i=1,\cdots,k$. By using Chinese remainder theorem, we can construct a polynomial $g_R(x)\in\boldsymbol{Z}[x]$ such that

$$g_R(x)\equiv g_j(x)\pmod{p_j}\qquad\text{for each } j=1,\cdots,m.$$

Now the axiom system $\Sigma_0(R)$ is defined to be the axiom system Σ with the following two axioms;

1) $p_R=0$,
2) $g_R(x)=0$.

THEOREM 4.1. *For any finite commutative regular ring R in* mod Σ, $\Sigma_0(R)$ *is a set of axioms for* Eq(R).

PROOF. Let $h:R\to\prod_{i\in I}F_i$ be the canonical representation of R. By Proposition 2.2, $\mathrm{Eq}(R)=\bigcap_{i\in I}\mathrm{Eq}(F_i)$. Using the same notations as those in the above definition, we suppose that a given field F_i is of the form $\boldsymbol{F}_{p_j^n}$ for some $j=1,\cdots,m$. Since $p_j=\mathrm{char}(F_i)\in\mathrm{Ch}(R)$ and $p_j\,|\,p_R$, $p_R=0$ holds in F_i. By the definition of $g_R(x)$, $g_R(x)=0\in\mathrm{Eq}(F_i)$ if and only if $g_j(x)=0\in\mathrm{Eq}(F_i)$. On the other hand, since

$$x^{p_j^n}-x=\prod_{d\mid n}f_d^{(j)}(x)$$

holds in F_i (see e.g. [7] p. 61), it holds that

$$x^{p_j^n}-x\,|\,g_j(x)$$

by the definition of $g_j(x)$. Moreover, since $x^{p_j^n}-x=0\in\mathrm{Eq}(F_i)$, $g_j(x)=0\in\mathrm{Eq}(F_i)$ and hence $g_R(x)=0\in\mathrm{Eq}(F_i)$. Thus, $\Sigma_0(R)\subset\mathrm{Eq}(F_i)$ for each $i\in I$. Therefore $\Sigma_0(R)\subset\mathrm{Eq}(R)$, or equivalently,

$$\mathrm{Eq}(\mathrm{mod}\,\Sigma_0(R))\subset\mathrm{Eq}(R). \tag{1}$$

Next, by Lemma 2.4, there exists a commutative regular ring R' such that $\mathrm{Eq}(\mathrm{mod}\,\Sigma_0(R))=\mathrm{Eq}(R')$. Then there exists a set of fields $\{K_h\}_{h\in H}$ such that $\mathrm{Eq}(R')=\bigcap_{h\in H}\mathrm{Eq}(K_h)$. Now suppose that ε is an equation valid in R. Since $g_R(x)=0\in\Sigma_0(R)\subset\mathrm{Eq}(K_h)$, every element of K_h must be a root of $g_R(x)$, so K_h is finite for each $h\in H$. On the other hand, since $p_R=0\in\mathrm{Eq}(K_h)$, $\mathrm{char}(K_h)\in\mathrm{Ch}(R)$. Let $p_j=\mathrm{char}(K_h)$. Then, $g_R(x)=0\in\mathrm{Eq}(K_h)$ implies $g_j(x)=0\in\mathrm{Eq}(K_h)$. Now let us suppose that $K_h=\boldsymbol{F}_{p_j n}$ for some $n>0$. Since in K_h, $x^{p_j{}^n}-x=\prod_{d|n}f_d{}^{(j)}(x)$ holds, every element of K_h is a root of some $f_d{}^{(j)}(x)$ for $d|n$, and vice versa. Take an arbitrary number d such that $d|n$ and take any irreducible monic $p(x)$ of degree d. Then, $p(a)=0$ for some $a\in K_h$. In this case, $p(x)$ is the minimum polynomial of a. Hence, $p(x)|g_j(x)$. If we let $p(x)$ range over all irreducible monics of degree d in $\boldsymbol{F}_{p_j}[x]$, then we have that $f_d{}^{(j)}(x)|g_j(x)$. Let $\boldsymbol{F}_{p_j n_1},\cdots,\boldsymbol{F}_{p_j n_k}$ be an enumeration of all fields in $\{F_i\}_{i\in I}$ whose characteristics are equal to p_j. By the definition of the polynomial $g_j(x)$, d is a divisor of some n_s for $s=1,\cdots,k$. In particular, if we take n for d, then there must be some $s\in\{1,\cdots,k\}$ such that $n|n_s$. This means that $K_h=\boldsymbol{F}_{p_j n}$ is a subfield of some $\boldsymbol{F}_{p_j n_s}$. Since ε is an equation valid in R and $\mathrm{Eq}(R)\subset\mathrm{Eq}(\boldsymbol{F}_{p_j n_s})$, ε is valid in $\boldsymbol{F}_{p_j n_s}$ and hence it is also valid in K_h. This holds for all $h\in H$. Since $\mathrm{Eq}(R')=\bigcap_{h\in H}\mathrm{Eq}(K_h)$, $\varepsilon\in\mathrm{Eq}(R')=\mathrm{Eq}(\mathrm{mod}\,\Sigma_0(R))$. Hence,

$$\mathrm{Eq}(R)\subset\mathrm{Eq}(\mathrm{mod}\,\Sigma_0(R))\,. \tag{2}$$

By (1) and (2), $\Sigma_0(R)$ is a set of axioms for $\mathrm{Eq}(R)$.

As for the axiomatization of universal theories of finite integral domains, we remark that every finite integral domain is a field. So, we can get immediately an axiom system of a universal theory of each finite integral domain, by using Theorem 4.1.

REMARK. A set of equations Γ is called to be *strictly consistent* if there exists an algebra $A\in\mathrm{mod}\,\Gamma$ such that $|A|>1$. For a strictly consistent set of equations Γ, we say Γ is *equationally complete*, if whenever $\Gamma\subset\Gamma'$, where Γ' is strictly consistent, then $\Gamma=\Gamma'$ (see [**6**] § 27). An algebra A is equationally complete, if the equational theory $\mathrm{Eq}(A)$ is equationally complete. Given a positive integer p, a commutative ring R is called a *p-ring* if $p\cdot x=0$ and $x^p=x$ for every element x, and R is called a *p-zero-ring* if $p\cdot x=0$ and $x\cdot y=0$ hold for all elements x and y. In [**20**], Tarski proved that a ring R with more than one element is equationally complete if and only if R is either a p-ring or a p-zero-ring for some prime number p. We can show easily the corresponding result as follows:

A commutative regular ring with more than one element is equationally complete if and only if it is a prime field.

Let S_1 and S_2 be classes of structures. We say that S_1 and S_2 are *mutually model consistent*, if every structure in S_1 is embeddable in a structure in S_2 and vice versa. We can show easily the following lemma.

LEMMA 4.2. *If two classes of structures (defined in the same language) S_1 and S_2 are mutually model consistent, then* $\mathrm{Eq}(S_1)=\mathrm{Eq}(S_2)$ *and* $\mathrm{T}_\forall(S_1)=\mathrm{T}_\forall(S_2)$ *hold. In such a case, moreover if* $\mathrm{Eq}(S_2)$ *(or* $\mathrm{T}_\forall(S_2)$*) is decidable, then so is* $\mathrm{Eq}(S_1)$ *(or* $\mathrm{T}_\forall(S_1)$*, respectively).*

By using this lemma, we can derive some decidability results on equational and universal theories.

THEOREM 4.3. *The equational theory* $\mathrm{Eq}(\mathrm{mod}\,\Sigma)$ *of commutative regular rings and the universal theory* $\mathrm{T}_\forall(\mathrm{mod}\,\Theta)$ *of integral domains are both decidable.*

PROOF. Let U^* be the class of all algebraically closed fields. Then, by using Lemma 4.2,

$$\mathrm{Eq}(\mathrm{mod}\,\Sigma)=\mathrm{Eq}(U^*) \quad \text{and} \quad \mathrm{T}_\forall(\mathrm{mod}\,\Theta)=\mathrm{T}_\forall(U^*)\,.$$

Moreover, the first-order theory $\mathrm{Th}(U^*)$ of algebraically closed fields is decidable. Thus we have our theorem.

By using Corollary 2.3 and Lemma 2.5, it follows from Theorem 4.3 that *both the equational theory and the universal theory of fields are decidable.* Compare these with the undecidability of the first-order theory of integral domains in [**19**] p. 71 or, of fields in [**16**] Theorem 4.3. In Theorem 6.2 of [**9**], Komori proved that the equational theory of (noncommutative) desirable pseudo-fields, i.e., the equational theory of rings satisfying the axioms Σ without the axiom of commutativity, can not be characterized by any class of *finite* desirable pseudo-fields. In contrast with this, it can be shown that the equational theory determined by Σ can be characterized by the set of all finite fields. In the following, P denotes the set of all prime numbers.

LEMMA 4.4. *For any prime number p, let $\tilde{\boldsymbol{F}}_p$ be the algebraic closure of the prime field $\boldsymbol{F}_p$ and $U_p{}^f$ be the set of all finite fields of characteristic p. Then,*

$$\mathrm{Eq}(\tilde{\boldsymbol{F}}_p)=\mathrm{Eq}(U_p{}^f) \quad \textit{and} \quad \mathrm{T}_\forall(\tilde{\boldsymbol{F}}_p)=\mathrm{T}_\forall(U_p{}^f) \quad \textit{hold.}$$

PROOF. Clearly, $\mathrm{Eq}(\tilde{\boldsymbol{F}}_p)\subset\mathrm{Eq}(U_p{}^f)$. Suppose an equation ε $(=\varepsilon(x_1, \cdots, x_n))$ does not belong to $\mathrm{Eq}(\tilde{\boldsymbol{F}}_p)$. Then, there exist $a_1, \cdots, a_n$ in $\tilde{\boldsymbol{F}}_p$ such that $\varepsilon(a_1, \cdots, a_n)$ is false in $\tilde{\boldsymbol{F}}_p$. Of course, $\varepsilon(a_1, \cdots, a_n)$ is also false in $\boldsymbol{F}_p(a_1, \cdots, a_n)$, which is a finite algebraic extension of $\boldsymbol{F}_p$. Thus, ε does not belong to an equational theory of some finite field in $U_p{}^f$. Quite similarly, we can show that $\mathrm{T}_\forall(\tilde{\boldsymbol{F}}_p)=\mathrm{T}_\forall(U_p{}^f)$.

Let U^{num} be the set of all number fields, i.e., finite algebraic extensions of $\boldsymbol{Q}$, and $\tilde{\boldsymbol{Q}}$ be the algebraic closure of $\boldsymbol{Q}$. Then, similarly as the above, we can show that $\mathrm{Eq}(\tilde{\boldsymbol{Q}})=\mathrm{Eq}(U^{\mathrm{num}})$ and $\mathrm{T}_\forall(\tilde{\boldsymbol{Q}})=\mathrm{T}_\forall(U^{\mathrm{num}})$.

By using compactness theorem and the fact that the first-order theory of algebraically closed fields of a given characteristic is complete, we have the following.

LEMMA 4.5. *Let φ be any first-order formula and S be any infinite subset of the set P of prime numbers. If for each prime number $q \in S$ there exists an algebraically closed field of characteristic q in which φ is true, then φ is also true in any algebraically closed field of characteristic* 0.

THEOREM 4.6. *Let U^f be the set of all finite fields. Then,* $\mathrm{Eq}(\mathrm{mod}\,\Sigma)=\mathrm{Eq}(U^f)$ *and* $\mathrm{T}_\forall(\mathrm{mod}\,\Theta)=\mathrm{T}_\forall(U^f)$.

PROOF. As shown in the proof of Theorem 4.3, $\mathrm{Eq}(\mathrm{mod}\,\Sigma)=\mathrm{Eq}(U^*)$. Clearly $\mathrm{Eq}(U^*)=\bigcap_{p\in P}\mathrm{Eq}(\tilde{\boldsymbol{F}}_p)\cap\mathrm{Eq}(\tilde{\boldsymbol{Q}})$. By Lemma 4.5, $\bigcap_{p\in P}\mathrm{Eq}(\tilde{\boldsymbol{F}}_p)\subset\mathrm{Eq}(\tilde{\boldsymbol{Q}})$. Hence, $\mathrm{Eq}(\mathrm{mod}\,\Sigma)=\bigcap_{p\in P}\mathrm{Eq}(\tilde{\boldsymbol{F}}_p)=\bigcap_{p\in P}\mathrm{Eq}(U_p{}^f)=\mathrm{Eq}(U^f)$, by Lemma 4.4. Similarly, we have $\mathrm{T}_\forall(\mathrm{mod}\,\Theta)=\mathrm{T}_\forall(U^f)$.

We define the axiom system Σ_p for the equational theory of fields of characteristic p as follows.

1) If p is a prime number, then Σ_p consists of Σ with an axiom $p=0$.

2) If p is equal to 0, then Σ_p consists of Σ with axioms $q\cdot q^{-1}=1$ for every prime number q.

Similarly, the axiom system Θ_p for the universal theory of fields of characteristic p can be defined by replacing Σ by Θ and $q\cdot q^{-1}=1$ by $\neg(q=0)$, in the above definition. Let U_p (or $U_p{}^*$) be the class of all fields (or, all algebraically closed fields) of characteristic p. Similarly as the above, we can show the following.

THEOREM 4.7. 1) *Let F be any algebraically closed field of characteristic p* $(\geqq 0)$. *Then,*

$$\mathrm{Eq}(\mathrm{mod}\,\Sigma_p)=\mathrm{Eq}(U_p)=\mathrm{Eq}(U_p{}^*)=\mathrm{Eq}(F) \quad \textit{and}$$

$$\mathrm{T}_\forall(\mathrm{mod}\,\Theta_p)=\mathrm{T}_\forall(U_p)=\mathrm{T}_\forall(U_p{}^*)=\mathrm{T}_\forall(F) \quad \textit{hold.}$$

Moreover, if $p\neq 0$ then $\mathrm{Eq}(\mathrm{mod}\,\Sigma_p)=\mathrm{Eq}(U_p{}^f)$ *and* $\mathrm{T}_\forall(\mathrm{mod}\,\Theta_p)=\mathrm{T}_\forall(U_p{}^f)$ *hold.*

2) *Both* $\mathrm{Eq}(\mathrm{mod}\,\Sigma_p)$ *and* $\mathrm{T}_\forall(\mathrm{mod}\,\Theta_p)$ *are decidable.*

We can extend Theorem 4.7 to more general cases. Suppose that S is a subset of $P\cup\{0\}$. Then we define an axiom system Σ_S to be the system obtained from Σ by adding Δ_S, where Δ_S is a set of equations defined as follows.

$$\Delta_S=\begin{cases}\{q\cdot q^{-1}=1;\ q\in P-S\} & \text{if either } S \text{ is infinite or } 0\in S,\\[2ex] \{\prod_{i=1}^{k} p_i=0\} & \text{if } S \text{ is a finite subset } \{p_1,\cdots,p_k\} \text{ of } P.\end{cases}$$

Let $U_S{}^*$ be the class of all commutative regular rings which are models of Σ_S and which are integrally closed rings without minimal idempotents. Then the theory $\mathrm{Th}(U_S{}^*)$ is a complete model companion of Σ_S (as a first-order theory in $\mathcal{L}'$) by Lipshitz-Saracino [**15**] (see especially Remark in [**15**]).

THEOREM 4.8. *For any subset* S *of* $P\cup\{0\}$ *and any ring* R *in* $U_S{}^*$, $\mathrm{Eq}(\mathrm{mod}\,\Sigma_S)=\mathrm{Eq}(U_S{}^*)=\mathrm{Eq}(R)$. *Hence, if* S *is recursive then* $\mathrm{Eq}(\mathrm{mod}\,\Sigma_S)$ *is decidable.*

By Corollary 2.3 and Lemma 4.5, we can show the following.

THEOREM 4.9. *Let* S *be a subset of* $P\cup\{0\}$. *Then, there exists a set* U *of finite fields such that* $\mathrm{Eq}(\mathrm{mod}\,\Sigma_S)=\mathrm{Eq}(U)$ *if and only if either* S *is infinite or* $0\notin S$.

In Corollary 4 of [17], Rumely showed that there exists a formula which is true in all function fields and false in all number fields. Here, a function field means a finite algebraic extension of some field of rational functions $F(t)$ over a finite field F. By using the similar technique used in the above, we can show the following.

THEOREM 4.10. *For each universal formula* φ *in* $\mathcal{L}$, *if* φ *is true in all function fields then it is also true in all number fields.*

5. A classification of equational and universal theories of fields.

In this section, we will try to classify the whole class of equational and universal theories of fields by means of absolute number fields. A field is an *absolute number field,* if it is algebraic over its prime field. For each field F, let $\mathrm{Abs}(F)$ denotes the set of all elements of F which are algebraic over the prime field of F. Of course, $\mathrm{Abs}(F)$ is a subfield of F. Clearly, for each absolute number field E and for each field F, $\mathrm{Abs}(F)=E$ if and only if E is algebraically closed in F. Let $[F]$ be the set of all polynomials in $\boldsymbol{Z}[x]$ having a root in F. Hereafter, we will consider only fields satisfying $\neg(0=1)$. By Ax [1] Lemma 5, we have the following.

PROPOSITION 5.1. *Suppose that two fields* F *and* F' *have the same characteristic. Then,* $\mathrm{Abs}(F)$ *is isomorphic to* $\mathrm{Abs}(F')$ *if and only if* $[F]=[F']$.

COROLLARY 5.2. *If* $\mathrm{Eq}(F)=\mathrm{Eq}(F')$, *or equivalently,* $\mathrm{T}_\forall(F)=\mathrm{T}_\forall(F')$ *then* $\mathrm{Abs}(F)$ *is isomorphic to* $\mathrm{Abs}(F')$.

This corollary suggests us that absolute number fields will play an important role in classifying equational or universal theories. We remark here that the converse of Corollary 5.2 does not always hold. Let $\boldsymbol{R}_0$ be the field of real algebraic numbers. Then, there exists a field K such that $\mathrm{Abs}(K)=\boldsymbol{R}_0$ but K is not a formally real field (see [21] p. 225). Indeed, K contains a zero of $1+x^2+y^2$. On the other hand, since $\boldsymbol{R}_0$ is a formally real field, $\neg(1+x^2+y^2=0)$ holds in $\boldsymbol{R}_0$ (cf. Theorem 3.6).

Let E be an absolute number field of characteristic p. Define a set $\varepsilon(E)$ of equations and a set $\omega(E)$ of universal formulas as follows;

$$\varepsilon(E)=\{f(x)f(x)^{-1}=1\,;\ f(x)\in\boldsymbol{Z}[x]-[E]\},$$

$$\omega(E)=\{\forall x \neg(f(x)=0)\,;\ f(x)\in \boldsymbol{Z}[x]-[E]\}.$$

Now let $\Sigma(E)$ be an axiom system $\Sigma_p \cup \varepsilon(E)$, where Σ_p is an axiom system for the equational theory of fields of characteristic p (see Theorem 4.7). We define $\Phi(E)$ to be the first-order theory of fields in $\mathcal{L}$ with the set of axioms $\omega(E)$. Then, $\Theta^*(E)$ denotes the set of all universal consequences of $\Phi(E)$. (The theory $\Theta^*(E)$ is equivalent to the theory S_E in [**21**].)

An extension field K of a field F is called a *totally transcendental extension* of F if F is algebraically closed in K. The class of totally transcendental extensions of F will be denoted by $\mathcal{V}(F)$. For each absolute number field E, define

$$\mathcal{T}(E)=\{\mathrm{Eq}(F)\,;\ F\in\mathcal{V}(E)\},$$

$$\mathcal{T}^*(E)=\{\mathrm{T}_\forall(F)\,;\ F\in\mathcal{V}(E)\}.$$

Clearly, each member of $\mathcal{T}(E)$ (or $\mathcal{T}^*(E)$) contains $\Sigma(E)$ (or $\Theta^*(E)$, respectively). By Theorem 3.4, $\mathcal{T}^*(E)$ is order-isomorphic to $\mathcal{T}(E)$. We will study the structure of $\mathcal{T}(E)$ and $\mathcal{T}^*(E)$ for each absolute number field E, in the following. Clearly, $\mathrm{Eq}(E)$ (or $\mathrm{T}_\forall(E)$) is the largest member (concerning the set inclusion) of $\mathcal{T}(E)$ (or $\mathcal{T}^*(E)$, respectively).

THEOREM 5.3. *Let E be an absolute number field. Then,*

1) *if E is finite then* $\mathrm{Eq}(E(t))$ *is the second largest member of* $\mathcal{T}(E)$,

2) *if E is infinite then* $\mathrm{Eq}(E(t))=\mathrm{Eq}(E)$ *holds,*

where $E(t)$ denotes the field of rational functions over E. The similar result holds also for universal theories.

PROOF. 1) Let E be a field with q elements. Since $E\subset E(t)$, $\mathrm{Eq}(E(t))\subset \mathrm{Eq}(E)$. Moreover, the equation $x^q-x=0$ holds in E but not in $E(t)$. Thus the inclusion is proper. Next, let F be any member of $\mathcal{V}(E)$, other than E. Then F contains E as its absolute number field and F must be a transcendental extension of E. Hence, F contains a subfield which is isomorphic to $E(t)$. Thus, $\mathrm{Eq}(F)\subset\mathrm{Eq}(E(t))$. Therefore, $\mathrm{Eq}(E(t))$ is the second largest among $\mathcal{T}(E)$. The second part of our theorem follows immediately from the following stronger result.

LEMMA 5.4. *Let F be an infinite field and L be an extension of F, which is also a subfield of a pure transcendental extension K of F. Then,* $\mathrm{Eq}(L)=\mathrm{Eq}(F)$ *and* $\mathrm{T}_\forall(L)=\mathrm{T}_\forall(F)$.

PROOF. Clearly, it suffices to show that $\mathrm{Eq}(K)=\mathrm{Eq}(F)$. Suppose that the transcendental degree of K/F is κ. By choosing an appropriate index set I and a nonprincipal ultrafilter D over I, we have an ultrapower $\tilde{F}=\prod_D F$ of F whose cardinality is greater than κ (see e.g. [**3**] Corollary 4.3.8). Since there exists the canonical mapping from F to $\tilde{F}$, F can be regarded as a subfield of a field $\tilde{F}$. Since F is elementarily embedded in $\tilde{F}$, $\mathrm{T}_\forall(\tilde{F})=\mathrm{T}_\forall(F)$. On the other hand, since $\tilde{F}$ contains an algebraically independent set of cardinality κ, $\tilde{F}$ has a subfield isomorphic to K. Hence $\mathrm{T}_\forall(\tilde{F})\subset\mathrm{T}_\forall(K)$. Clearly, $\mathrm{T}_\forall(K)\subset\mathrm{T}_\forall(F)$. Thus, $\mathrm{T}_\forall(K)=\mathrm{T}_\forall(F)$ and $\mathrm{Eq}(K)=\mathrm{Eq}(F)$ hold.

As we have already mentioned, each field F in $\mathcal{V}(E)$ is a model of $\Sigma(E)$ for each absolute number field E. When there exists a field F in $\mathcal{V}(E)$ such that $\mathrm{Eq}(F)=\mathrm{Eq}(\mathrm{mod}\,\Sigma(E))$, $\mathrm{Eq}(F)$ becomes the smallest member in $\mathcal{T}(E)$. In the next theorem, we will show that it always happens. We need some preparations. The following lemma can be proved by using standard algebraic technique (see Lang [13] Chapter III § 1).

LEMMA 5.5. *Suppose that an absolute number field E and a field F are both subfields of a certain field. If* $\mathrm{Abs}(F)\subset E$ *then E is algebraically closed in the composite field FE.*

LEMMA 5.6. *The class $\mathcal{V}(E)$ has the joint embedding property, i.e. for all F_1, $F_2\in\mathcal{V}(E)$, there exists a joint extension F of F_1 and F_2 in $\mathcal{V}(E)$.*

PROOF. Since any absolute number field is perfect, both F_1 and F_2 are regular extensions of E. So, the free composite F of F_1 and F_2 is also a regular extension of E, so in particular, it is a totally transcendental extension of both F_1 and F_2 (see [7] Chapter IV § 11 and [13] Chapter III § 1).

THEOREM 5.7. *For each absolute number field E, there exists a field F in $\mathcal{V}(E)$ such that* $\mathrm{Eq}(F)=\mathrm{Eq}(\mathrm{mod}\,\Sigma(E))$. *Thus, $\mathcal{T}(E)$ has the minimum element* $\mathrm{Eq}(\mathrm{mod}\,\Sigma(E))$. *Similarly, $\mathcal{T}^*(E)$ has the minimum element $\Theta^*(E)$.*

PROOF. By Corollary 2.3, there exists a set of fields $\{F_i\}_{i\in I}$ such that

$$\mathrm{Eq}(\mathrm{mod}\,\Sigma(E))=\bigcap_{i\in I}\mathrm{Eq}(F_i)\,. \tag{1}$$

Since each F_i is a model of $\Sigma(E)$, $\mathrm{Abs}(F_i)\subset E$. We can assume that both F_i and E are subfields of an algebraic closure of F_i. So the composite field F_iE belongs to $\mathcal{V}(E)$ by Lemma 5.5, and hence is a model of $\Sigma(E)$. On the other hand, $\mathrm{Eq}(F_iE)\subset\mathrm{Eq}(F_i)$, since F_i is a subfield of F_iE. Thus,

$$\mathrm{Eq}(\mathrm{mod}\ \Sigma(E))=\bigcap_{i\in I}\mathrm{Eq}(F_iE)\,. \tag{2}$$

Hence, we can suppose from the beginning that each F_i in (1) belongs to $\mathcal{V}(E)$. Moreover, we can suppose also that the index set I is countable, since the set of all equations in $\mathcal{L}'$ is countable. Thus, we can take the set $\boldsymbol{N}$ of natural numbers for I. Now we will define a sequence of fields $\{L_i\}_{i\in N}$ inductively as follows;

1) $L_0=F_0$,

2) L_{i+1} is the free composite of L_i and F_{i+1}.

Then, it holds that

1) each L_i is in $\mathcal{V}(E)$, by Lemma 5.6,

2) each L_{i+1} is an extension of L_i.

Let us define $L=\bigcup_{i\in N} L_i$. By Chang-Łoś-Suszko Theorem, L is also a field whose absolute number field is E, since all axioms of the theory of fields with the absolute number field E can be expressed by $\forall\exists$-sentences (see [3] Theorem 3.2.3). Hence, L belongs to $\mathcal{U}(E)$. On the other hand, L is an extension of each L_i, which in turn is an extension of F_i. Thus, $\mathrm{Eq}(L)\subset\mathrm{Eq}(F_i)$. So, $\mathrm{Eq}(L)\subset\bigcap_{i\in N}\mathrm{Eq}(F_i)$. Hence $\mathrm{Eq}(L)=\mathrm{Eq}(\mathrm{mod}\,\Sigma(E))$. From the similar argument the second part of our theorem follows.

A field F is called to be *absolutely minimal* if $\mathrm{Eq}(F)$ is the minimum element of $\mathcal{T}(\mathrm{Abs}(F))$. A field F is called to be *relatively minimal* if $\mathrm{Eq}(G)=\mathrm{Eq}(F)$ for any totally transcendental extension G of F. Due to [21], we say that a field K is a *maximally totally transcendental extension* of F, if F is algebraically closed in K and F is not algebraically closed in any proper algebraic extension of K.

LEMMA 5.8. 1) *If a field F is absolutely minimal then it is relatively minimal.*

2) *Conversely, if a field F is relatively minimal and is a maximally totally transcendental extension of* $\mathrm{Abs}(F)$, *then it is absolutely minimal. In particular, if an absolute number field is relatively minimal then it is absolutely minimal.*

PROOF. Trivially, 1) holds. Suppose that a field F with the absolute number field E is relatively minimal and is a maximally totally transcendental extension of E. Let K be any absolutely minimal field in $\mathcal{U}(E)$. By Lemma 5.6, there exists a field L in $\mathcal{U}(E)$ which is an extension of both F and K. Clearly, L is also absolutely minimal. Let F' be the relative algebraic closure of F in L. Then F' must be a totally transcendental extension of E, since $L\in\mathcal{U}(E)$. But the extension F'/F is algebraic, so $F'=F$ by the maximality of F. Thus, F is algebraically closed in L, i.e. $L\in\mathcal{U}(F)$. Since F is relatively minimal, $\mathrm{Eq}(F)=\mathrm{Eq}(L)=\mathrm{Eq}(\mathrm{mod}\,\Sigma(E))$. Hence F is also absolutely minimal.

In [21], Wheeler introduced the important notion of *pseudo-algebraically closed fields*. A field F is pseudo-algebraically closed if whenever I is a prime ideal in a polynomial ring $F[x_1,\cdots,x_m]$ and F is algebraically closed in the quotient field of $F[x_1,\cdots,x_m]/I$, then there is a homomorphism from $F[x_1,\cdots,x_m]/I$ into F which is the identity on F. A field F is *weakly pseudo-algebraically closed* if each nonvoid, absolutely irreducible F-variety has an F-rational point. A field is *quasi-perfect* if either it has characteristic 0 or it has at most one purely inseparable extension of degree p where p (>0) is its characteristic. Of course, any perfect field is quasi-perfect. Wheeler proved the following propositions (Theorems 2.2 and 2.3 in [21]).

PROPOSITION 5.9. *The following are equivalent for a field K;*

1) *K is pseudo-algebraically closed,*

2) *K is quasi-perfect and weakly pseudo-algebraically closed,*

3) *K is existentially complete in each totally transcendental extension field.*

PROPOSITION 5.10. *A field K in $\mathcal{V}(F)$ is existentially complete in $\mathcal{V}(F)$ if K is pseudo-algebraically closed and is a maximally totally transcendental extension of F.*

By using these propositions, the following can be proved immediately.

LEMMA 5.11. 1) *Any pseudo-algebraically closed field is relatively minimal.*

2) *Any field F which is existentially complete in $\mathcal{V}(\mathrm{Abs}(F))$ is absolutely minimal.*

Since each field has a totally transcendental extension which is pseudo-algebraically closed, it follows that each field has a totally transcendental extension which is relatively minimal. We can show that neither the converse of 1) nor of 2) holds. For, any totally transcendental extension of a given relatively minimal (or absolutely minimal) field is also relatively minimal (or absolutely minimal, respectively), while there exists a totally transcendental extension of a pseudo-algebraically closed (or an existentially complete) field which is not pseudo-algebraically closed (or existentially complete, respectively).

The following is an important corollary of Weil's Riemann hypothesis for curves.

THEOREM 5.12. *Let E be any infinite absolute number field of characteristic $p>0$. Then $\mathcal{V}(E)$ is a singleton set, i.e. E is absolutely minimal. The similar result holds also for universal theories.*

PROOF. By using Weil's theorem on Riemann hypothesis for curves, Ax proved in [2] that for each infinite algebraic extension E of a finite field, every absolutely irreducible E-variety has an E-rational point, i.e. E is weakly pseudo-algebraically closed. It is obvious that E is also perfect. Thus, E is absolutely minimal by Proposition 5.9 and Lemma 5.11. 2).

The following corollary says that for every field of characteristic $p>0$ having an infinite absolute number field, the converse of Corollary 5.2 holds.

COROLLARY 5.13. *For all fields F and F' having the same characteristic $p>0$, if $\mathrm{Abs}(F)$ is infinite and isomorphic to $\mathrm{Abs}(F')$ then $\mathrm{Eq}(F)=\mathrm{Eq}(F')$ and $\mathrm{T}_\forall(F)=\mathrm{T}_\forall(F')$ hold.*

Compare this with Theorem 4 in Ax [2], which says that $\mathrm{Abs}(F)$ is isomorphic to $\mathrm{Abs}(F')$ if and only if F is elementarily equivalent to F', for all *pseudo-finite* fields F and F'. Here, a pseudo-finite field means a perfect, pseudo-algebraically closed field having precisely one extension of each degree. We cannot expect that the similar result holds for other absolute number fields. But we can see that a lot of absolute number fields of characteristic 0 are absolutely minimal, by using a theorem proved by Jarden [8]. For any field F, we denote by F_s the separable closure of F. For each n-tuple $(\sigma_1, \cdots, \sigma_n)$ in $\mathcal{G}(F_s/F)^n$, where $\mathcal{G}(F_s/F)$ is the Galois group of F_s over F, let $F_s(\sigma_1, \cdots, \sigma_n)$ be the fixed field in

F_s of automorphisms $\sigma_1, \cdots, \sigma_n$.

PROPOSITION 5.14 (Jarden). *If F is a countable Hilbertian field, then for almost all $(\sigma_1, \cdots, \sigma_n) \in \mathcal{G}(F_s/F)^n$ (in the sense of Haar measure μ on $\mathcal{G}(F_s/F)^n$ defined with respect to its Krull topology) the fixed field of $\{\sigma_1, \cdots, \sigma_n\}$, $F_s(\sigma_1, \cdots, \sigma_n)$, is weakly pseudo-algebraically closed.*

Recall that every global field is Hilbertian. (As for Hilbertian fields, see Lang [**14**] Chapter VIII.) In particular, the field $\boldsymbol{Q}$ of rational numbers is Hilbertian. Thus, we have the following.

COROLLARY 5.15. *The absolute number field $\tilde{\boldsymbol{Q}}(\sigma_1, \cdots, \sigma_n)$ is absolutely minimal for almost all $(\sigma_1, \cdots, \sigma_n) \in \mathcal{G}(\tilde{\boldsymbol{Q}}/\boldsymbol{Q})^n$, where $\tilde{\boldsymbol{Q}}$ denotes the algebraic closure of $\boldsymbol{Q}$.*

A field F has *bounded corank* if and only if it has only finitely many separable algebraic extensions of degree n over F for each integer $n \geqq 2$. Remark that any absolute number field of characteristic $p>0$ has bounded corank.

THEOREM 5.16. *Let E be an absolute number field having bounded corank such that $[E]$ is recursive. Then, both* $\mathrm{Eq}\,(\mathrm{mod}\, \Sigma(E))$ *and $\Theta^*(E)$ are decidable.*

PROOF. Since $[E]$ is recursive, the theory T^*_E introduced in [**21**] is recursively axiomatized (see [**21**] §4). Moreover, T^*_E is complete, since E has bounded corank (see Theorem 4.3 in [**21**]). Hence, T^*_E is decidable. Let F be any model of T^*_E. Then $F \in \mathcal{V}(E)$, F is existentially complete in $\mathcal{V}(E)$ and $\mathrm{Th}\,(F) = T^*_E$. So, $\mathrm{Eq}\,(\mathrm{mod}\, \Sigma(E)) = \mathrm{Eq}\,(F)$ and $\Theta^*(E) = \mathrm{T}_\forall(F)$, and hence they are decidable.

By using Theorem 5.12, we can derive the following (cf. Eršov [4] Theorem 2).

COROLLARY 5.17. *Let F be a field of characteristic $p>0$ having an infinite absolute number field. Then, the following three conditions are equivalent;*

1) $\mathrm{T}_\forall(F)$ *is decidable,*
2) $\mathrm{Eq}\,(F)$ *is decidable,*
3) $[F]$ *is recursive.*

REMARK. We don't know whether $\mathrm{Eq}\,(\boldsymbol{Q})$, or equivalently $\mathrm{T}_\forall(\boldsymbol{Q})$, is decidable or not. It should be noticed that the decidability of $\mathrm{T}_\forall(\boldsymbol{Q})$ implies the recursive solvability of the diophantine problem for $\boldsymbol{Q}$ and vice versa. For, in $\boldsymbol{Q}$ and in fact in any formally real field, we can show that every existential formula (in $\mathcal{L}$), i.e. a formula of the form $\exists x_1 \cdots \exists x_n \varphi(x_1, \cdots, x_n)$ with an open formula $\varphi(x_1, \cdots, x_n)$, is logically equivalent to a formula of the form

$$\exists y_1 \cdots \exists y_m (f(y_1, \cdots, y_m) = 0),$$

where $f(y_1, \cdots, y_m)$ is a polynomial with coefficients in $\boldsymbol{Z}$. Now, let F be any formally real field. Then, the field $F(t)$ of rational functions over F is also formally real. Combining the above fact with Theorem 5.3, we can give a new proof of the following result;

for any formally real field F, the diophantine problem for $F(t)$ with co-

efficients in $\boldsymbol{Z}$ is recursively solvable if and only if the diophantine problem for F with coefficients in $\boldsymbol{Z}$ is recursively solvable.

References

[1] J. Ax, Solving diophantine problems modulo every prime, Ann. of Math., **85** (1967), 161-183.
[2] J. Ax, The elementary theory of finite fields, Ann. of Math., **88** (1968), 239-271.
[3] C.C. Chang and H.J. Keisler, Model theory, North-Holland, Amsterdam-New York-Oxford, 1973.
[4] Ju. L. Eršov, Fields with a solvable theory, Soviet Math. Dokl., **8** (1967), 575-576.
[5] L. Fuchs, Partially ordered algebraic systems, Pergamon Press, New York, 1963.
[6] G. Grätzer, Universal algebra, 2nd ed., Springer, New York-Heidelberg-Berlin, 1979.
[7] N. Jacobson, Lectures in abstract algebra III; Theory of fields and Galois theory, D. van Nostrand, Princeton-Toronto-New York-London, 1964.
[8] M. Jarden, Elementary statements over large algebraic fields, Trans. Amer. Math. Soc., **164** (1972), 67-91.
[9] Y. Komori, Free algebras over all fields and pseudo-fields, Rep. Fac. Sci., Shizuoka Univ., **10** (1975), 9-15.
[10] Y. Komori, A relation between strongly regular rings and pseudo-fields, Rep. Fac. Sci., Shizuoka Univ., **11** (1976), 23-24.
[11] R.L. Kruse, Identities satisfied by a finite ring, J. Algebra, **26** (1973), 298-318.
[12] J. Lambek, Lectures on rings and modules, Blaisdell, Waltham-Toronto-London, 1966.
[13] S. Lang, Introduction to algebraic geometry, Interscience, New York, 1958.
[14] S. Lang, Diophantine geometry, Interscience, New York, 1962.
[15] L. Lipshitz and D. Saracino, The model companion of the theory of commutative rings without nilpotent elements, Proc. Amer. Math. Soc., **38** (1973), 381-387.
[16] J. Robinson, Definability and decision problems in arithmetic, J. Symbolic Logic, **14** (1949), 98-114.
[17] R.S. Rumely, Undecidability and definability for the theory of global fields, Trans. Amer. Math. Soc., **262** (1980), 195-217.
[18] J.R. Shoenfield, Mathematical logic, Addison-Wesley, Reading, 1967.
[19] A. Tarski, A. Mostowski and R.M. Robinson, Undecidable theories, North-Holland, Amsterdam, 1953.
[20] A. Tarski, Equationally complete rings and relation algebras, Indag. Math., **18** (1956), 39-46.
[21] W.H. Wheeler, Model-complete theories of pseudo-algebraically closed fields, Ann. Math. Log., **17** (1979), 205-226.

Hiroakira ONO
Faculty of Integral Arts and Sciences
Hiroshima University
Hiroshima 730, Japan