J. Math. Soc. Japan
Vol. 35, No. 3, 1983

Infinite families of nilpotent Lie algebras

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(Received May 14, 1982)

0. Introduction.

As early as 1891 K. A. Umlauf [**12**], a student of Engel, classified all the complex nilpotent Lie algebras up to dimension 6: they are finite in number. At the dimensions 7, 8 and 9 he found several infinite families of nilpotent Lie algebras. Many authors studied this phenomenon. Among them are: N. Bourbaki ([**1**], p. 122 ex. n° 18), Chong-Yun Chao [**3**], P. De La Harpe [**4**], G. Favre [**5**], M. A. Gauger [**6**], T. Skjelbred and T. Sund [**10**], [**11**].

We prove a criterion (involving a cohomology space and some orbits of an open set in a Grassmanian under the action of an algebraic group) for the existence, at a given dimension, of infinite families of non isomorphic nilpotent Lie algebras without direct abelian factor. By applying this criterion to a special case we recover a theorem of Gauger [**6**] with supplementary information concerning nilpotent Lie algebras of dimension $\geqq 9$.

This work is chapter III of my thesis [**8**] written under the guidance of Professor M. P. Malliavin to whom I am grateful.

I would like to thank my friends and colleagues J. Alev and T. Levasseur who helped me a lot.

1. Preliminaries.

1.1. All through Sections 1 and 2, F denotes a commutative field and all Lie algebras are of finite dimension over F; from Section 2 onwards, F will be assumed algebraically closed. If $\mathfrak{g}$ is a Lie algebra we denote its center by $Z(\mathfrak{g})$ and its automorphism group by $\mathrm{Aut}\,\mathfrak{g}$. The abelian Lie algebra of dimension n will be denoted by $\mathfrak{g}_1^n$. If E is a vector space over F, $\overset{2}{\wedge}E$ is the homogeneous component of degree 2 of the Grassmann algebra $\wedge E$.

1.2. For $p\in \boldsymbol{N}^*$, Let $C^2(\mathfrak{g}, F^p)$ be the vector space of 2-cochains (i.e. bilinear alternate map from $\mathfrak{g}\times\mathfrak{g}$ into F^p), $Z^2(\mathfrak{g}, F^p)$ the space of 2-cocycles, (i.e. the

The author gratefully acknowledges the support of a grant from the French Government's "Délegation Générale à la Recherche Scientifique et Technique-Paris" (Contract n° 77167).

elements B of $C^2(\mathfrak{g}, F^p)$ such that $B([X, Y], Z)+B([Y, Z], X)+B([Z, X], Y)=0\ \forall X, Y, Z\in\mathfrak{g})$, $B^2(\mathfrak{g}, F^p)$ the space of 2-coboundaries, (i.e. the elements B of $C^2(\mathfrak{g}, F^p)$ for which there exists $f\in\mathrm{Hom}(\mathfrak{g}, F^p)$ such that $B(X, Y)=f([X, Y])\ \forall X, Y\in\mathfrak{g})$. Note that $C^2(\mathfrak{g}, F^p)=C^2(\mathfrak{g}, F)^p$, $Z^2(\mathfrak{g}, F^p)=Z^2(\mathfrak{g}, F)^p$, $B^2(\mathfrak{g}, F^p)=B^2(\mathfrak{g}, F)^p$. Denote the cohomology space of degree 2 of $\mathfrak{g}$ with values in F^p by $H^2(\mathfrak{g}, F^p)$ (this is the quotient $Z^2(\mathfrak{g}, F^p)/B^2(\mathfrak{g}, F^p)$) and indicate the canonical morphism $Z^2(\mathfrak{g}, F^p)\rightarrow H^2(\mathfrak{g}, F^p)$ by $B\rightarrow[B]$ [2].

1.3. If $\phi\in\mathrm{Aut}\,\mathfrak{g}$ and $B\in C^2(\mathfrak{g}, F^p)$ one defines $B^\phi\in C^2(\mathfrak{g}, F^p)$ by $B^\phi(X, Y)=B(\phi X, \phi Y)\ \forall X, Y\in\mathfrak{g}$. Thus $\mathrm{Aut}\,\mathfrak{g}$ operates on $C^2(\mathfrak{g}, F^p)$. It is easy to see that $Z^2(\mathfrak{g}, F^p)$ and $B^2(\mathfrak{g}, F^p)$ are stabilized by this action, so that there is an induced action of $\mathrm{Aut}\,\mathfrak{g}$ on $H^2(\mathfrak{g}, F^p)$: $(\phi, [B])\mapsto[B^\phi]$.

1.4. Let $G_p(H^2(\mathfrak{g}, F))$ be the Grassmanian of subspaces of dimension p in $H^2(\mathfrak{g}, F)$. One makes $\mathrm{Aut}\,\mathfrak{g}$ operate on $G_p(H^2(\mathfrak{g}, F))$ in the following way:

$$(\phi, V=[B_1]F\oplus\cdots\oplus[B_p]F)\mapsto V^\phi=[B_1^\phi]F\oplus\cdots\oplus[B_p^\phi]F.$$

This definition is legitimate because $([B_1^\phi], \cdots, [B_p^\phi])$ is free if $([B_1], \cdots, [B_p])$ is free. Denote the orbit of $V\in G_p(H^2(\mathfrak{g}, F))$ under the action of $\mathrm{Aut}\,\mathfrak{g}$ by $\mathrm{Orb}(V)$.

1.5. LEMMA. *For* $B\in C^2(\mathfrak{g}, F^p)$, *let* $\mathfrak{g}_B^\perp=\{X\in\mathfrak{g};\ B(X, Y)=0\ \forall Y\in\mathfrak{g}\}$ (*note that* $\mathfrak{g}^\perp_{(B_1,\cdots,B_p)}=\mathfrak{g}^\perp_{B_1}\cap\cdots\cap\mathfrak{g}^\perp_{B_p}$ *if* $B_i\in C^2(\mathfrak{g}, F)$). *If* $(B_1, \cdots, B_p), (B'_1, \cdots, B'_p)\in Z^2(\mathfrak{g}, F^p)$ *are such that*

$$[B_1]F\oplus\cdots\oplus[B_p]F=[B'_1]F\oplus\cdots\oplus[B'_p]F$$

then

$$\mathfrak{g}^\perp_{(B_1,\cdots,B_p)}\cap Z(\mathfrak{g})=\mathfrak{g}^\perp_{(B'_1,\cdots,B'_p)}\cap Z(\mathfrak{g}).$$

PROOF. There exists an invertible matrix (ϕ_i^j) such that $[B'_i]=\sum_{j=1}^p\phi_i^j[B_j]$ therefore there exists $f_i\in\mathrm{Hom}(\mathfrak{g}, F)$ such that $B'_i(X, Y)=\sum\phi_i^jB_j(X, Y)+f_i([X, Y])\ \forall X, Y\in\mathfrak{g}\ \forall i=1\cdots p$. Hence "$B_i(X, Y)=0$ and $[X, Y]=0$" is equivalent to "$B'_i(X, Y)=0$ and $[X, Y]=0$" which is the conclusion. This result allows us to define

$$U_p(\mathfrak{g})=\{[B_1]F\oplus\cdots\oplus[B_p]F\in G_p(H^2(\mathfrak{g}, F));\ \mathfrak{g}^\perp_{(B_1,\cdots,B_p)}\cap Z(\mathfrak{g})=(0)\}.$$

1.6. LEMMA. *The set* $U_p(\mathfrak{g})$ *is stable under the action of* $\mathrm{Aut}\,\mathfrak{g}$.

PROOF. Let $\phi\in\mathrm{Aut}\,\mathfrak{g}$, $V=[B_1]F\oplus\cdots\oplus[B_p]F\in U_p(\mathfrak{g})$ and $B=(B_1\cdots B_p)$; obviously, one has $\mathfrak{g}^\perp_{B^\phi}=\phi^{-1}\mathfrak{g}^\perp_B$ and $\phi^{-1}(Z(\mathfrak{g}))=Z(\mathfrak{g})$. Therefore $\mathfrak{g}^\perp_{B^\phi}\cap Z(\mathfrak{g})=\phi^{-1}(\mathfrak{g}^\perp_B\cap Z(\mathfrak{g}))$ which proves that $V\in U_p(\mathfrak{g})$ if and only if $V^\phi\in U_p(\mathfrak{g})$.

The set of orbits will be denoted by $U_p(\mathfrak{g})/\mathrm{Aut}\,\mathfrak{g}$.

1.7. THEOREM (T. Skejelbred and T. Sund [11]). *There exists a canonical one-to-one map from* $U_p(\mathfrak{g})/\mathrm{Aut}\,\mathfrak{g}$ *onto the set of isomorphism classes of Lie algebras without direct abelian factor which are central extensions of* $\mathfrak{g}$ *by* F^p *and have*

p-dimensional center.

2. The criterion.

2.1. All through Section 2 we will assume F algebraically closed. All the topological notions are relative to the Zariski topology. The set $G_p(H^2(\mathfrak{g}, F))$ is an algebraic variety and $\mathrm{Aut}\,\mathfrak{g}$ is an algebraic group acting morphically in all spaces considered in Section 1 [**7**].

2.2. LEMMA. *The set $U_p(\mathfrak{g})$ is open in the Grassmanian variety $G_p(H^2(\mathfrak{g}, F))$.*

PROOF. Let $\tilde{B}: \mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}, F^p)$ be defined by $\tilde{B}(X)(Y)=B(X, Y)$. One has $\mathrm{Ker}\,\tilde{B}=\mathfrak{g}_B^{\perp}$. Let $\bar{B}=\tilde{B}/Z(\mathfrak{g})$. Then

$$U_p(\mathfrak{g})=\{[B_1]F\oplus \cdots \oplus[B_p]F\in G_p(H^2(\mathfrak{g}, F))\,;\ \mathrm{Ker}\,\bar{B}=(0)\}.$$

Since $\bar{B}$ is linear from $Z(\mathfrak{g})$ into $\mathrm{Hom}(\mathfrak{g}, F^p)$, the matrix of $\bar{B}$ is $k\times np$ where $k=\dim Z(\mathfrak{g})$ and $n=\dim \mathfrak{g}$. Let $D_1(B_1, \cdots, B_p), \cdots, D_q(B_1, \cdots, B_p)$ be the determinants of the $k\times k$ matrices extracted from the matrix of $\bar{B}$. One has

$$G_p(H^2(\mathfrak{g}, F))\setminus U_p(\mathfrak{g})=\{[B_1]F\oplus \cdots \oplus[B_p]F\in G_p(H^2(\mathfrak{g}, F))\,;$$
$$D_1(B_1, \cdots, B_p)= \cdots =D_q(B_1, \cdots, B_p)=0\}\,;$$

hence the complement of $U_p(\mathfrak{g})$ is closed.

2.3. THEOREM. *Let $\mathfrak{g}$ be a Lie algebra of finite dimension over a commutative and algebraically closed field F. Let $p\in \boldsymbol{N}^*$ and assume that $\mathfrak{g}$ has at least one central extension by F^p without direct abelian factor and with p-dimensional center (i.e. $U_p(\mathfrak{g})\neq\emptyset$ by 1.7). Let s be the maximum of the dimensions of* $\mathrm{Aut}\,\mathfrak{g}$-*orbits of $U_p(\mathfrak{g})$. If $s<p(\dim H^2(\mathfrak{g}, F)-p)$ then there exist infinite families of Lie algebras which are*

— *two by two non isomorphic*
— *without direct abelian factor*
— *central extensions of $\mathfrak{g}$ by F^p*
— *with p-dimensional center.*

PROOF. Since $U_p(\mathfrak{g})$ is open (2.2) and not empty in $G_p(H^2(\mathfrak{g}, F))$ one has $\dim U_p(\mathfrak{g})=\dim G_p(H^2(\mathfrak{g}, F))=p(\dim H^2(\mathfrak{g}, F)-p)$ [**7**]. If $s<p(\dim H^2(\mathfrak{g}, F)-p)$ then $\dim \mathrm{orb}\,V<\dim U_p(\mathfrak{g})$ for all $V\in U_p(\mathfrak{g})$; but $U_p(\mathfrak{g})$ is the union of all $\mathrm{orb}\,V$ for $V\in U_p(\mathfrak{g})$ and a variety cannot be the union of a finite number of sub-varieties if the dimensions of the subvarieties are strictly less than the dimension of the variety. Therefore there are infinitely many orbits in $U_p(\mathfrak{g})$. The conclusion follows then from 1.7.

2.4. REMARK. The hypothesis $U_p(\mathfrak{g})\neq\emptyset$ is essential as one can see from the case of Heisenberg Lie algebras which do not have central extensions by F^p without direct abelian factor and with p-dimensional center ([**8**], [**9**]).

2.5. LEMMA. *If $k\in \boldsymbol{N}^*$ and $p\in \boldsymbol{N}^*$ are such that $p\leqq\binom{k}{2}$ then $U_p(\mathfrak{g}_1^k)\neq\emptyset$.*

PROOF. Let $\mathfrak{g}=\mathfrak{g}_1^k$ and denote $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. One has $B^2(\mathfrak{g}, F)=(0)$, $Z^2(\mathfrak{g}, F)=\overset{2}{\wedge}\mathfrak{g}^*$, $H^2(\mathfrak{g}, F)=\overset{2}{\wedge}\mathfrak{g}^*$, $[B]=B$ (by identification). If $\{e_1, \cdots, e_k\}$ is a basis of $\mathfrak{g}$ and $\{e^{*1}, \cdots, e^{*k}\}$ its dual, then $\{e^{*i}\wedge e^{*j};\ 1\leqq i<j\leqq k\}$ is a basis of $\overset{2}{\wedge}\mathfrak{g}^*$. Let $B\in Z^2(\mathfrak{g}, F^p)$ then $\mathfrak{g}_B^\perp\cap Z(\mathfrak{g})=\mathfrak{g}_B^\perp$ therefore $U_p(\mathfrak{g})=\{B_1F\oplus\cdots\oplus B_pF\in G_p(\overset{2}{\wedge}\mathfrak{g}^*);\ \mathfrak{g}^\perp_{(B_1,\cdots,B_p)}=(0)\}$.

First case: k even. Let $B_1=e^{*1}\wedge e^{*2}+\cdots+e^{*k-1}\wedge e^{*k}$ and $X=\sum_{i=1}^{k}X^ie_i\in\mathfrak{g}$. Then

$$B_1(X, e_j)=-X^{j+1} \quad \text{if } j \text{ even and } 2\leqq j\leqq k$$
$$=X^{j-1} \quad \text{if } j \text{ odd and } 1\leqq j\leqq k-1\,.$$

Thus $\mathfrak{g}_{B_1}^\perp=(0)$. Let $\{B_2, \cdots, B_p\}\subset\overset{2}{\wedge}\mathfrak{g}^*$ such that $\{B_1, \cdots, B_p\}$ is free then $\mathfrak{g}^\perp_{(B_1,\cdots,B_p)}=\mathfrak{g}^\perp_{B_1}\cap\cdots\cap\mathfrak{g}^\perp_{B_p}=(0)$ thus $B_1F\oplus\cdots\oplus B_pF\in U_p(\mathfrak{g})$ therefore $U_p(\mathfrak{g})\neq\emptyset$.

Second case: k odd. Let $B_1=e^{*1}\wedge e^{*2}+\cdots+e^{*k-2}\wedge e^{*k-1}$, $B_2=e^{*1}\wedge e^{*k}$. As in the first case one proves that $\mathfrak{g}^\perp_{B_1}\cap\mathfrak{g}^\perp_{B_2}=(0)$. Let $\{B_3, \cdots, B_p\}\subset\wedge\mathfrak{g}^*$ such that $\{B_1, \cdots, B_p\}$ is free then $B_1F\oplus\cdots\oplus B_pF\in U_p(\mathfrak{g})$ thus $U_p(\mathfrak{g})\neq\emptyset$.

2.6. THEOREM. *Let $\mathfrak{g}_1^{n-p}$ be the abelian Lie algebra of dimension $n-p$ over an algebraically closed field F.*

(i) *For all $(n, p)\in\boldsymbol{N}^*\times\boldsymbol{N}^*$ verifying $(n-p)^2\leqq p\left(\binom{n-p}{2}-p\right)$ there exist infinite families of nilpotent Lie algebras of dimension n which are:*

— *two by two non isomorphic*

— *central extensions of $\mathfrak{g}_1^{n-p}$ by F^p*

— *without abelian direct factor*

— *with p-dimensional center.*

(ii) *For all $n\geqq 9$ there exist $p\geqq 3$ such that $(n-p)^2\leqq p\left(\binom{n-p}{2}-p\right)$, therefore, from dimension 9 onwards, one gets infinite families as described above.*

PROOF. (i) Let $(n, p)\in\boldsymbol{N}^*\times\boldsymbol{N}^*$ verifying $(n-p)^2\leqq p\left(\binom{n-p}{2}-p\right)$; this implies in particular $n-p\geqq 2$ and $p\leqq\binom{n-p}{2}$; by 2.5 one has $U_p(\mathfrak{g}_1^{n-p})\neq\emptyset$. The center of $\mathrm{Aut}\,\mathfrak{g}_1^{n-p}$ is isomorphic with $F\setminus\{0\}$ (thus of dimension 1) and operates trivially on $U_p(\mathfrak{g}_1^{n-p})$. Thus $\dim\mathrm{orb}\,V\leqq\dim\mathrm{Aut}\,\mathfrak{g}_1^{n-p}-1$ for all $V\in U_p(\mathfrak{g}_1^{n-p})$. But $\dim\mathrm{Aut}\,\mathfrak{g}_1^{n-p}=(n-p)^2$, hence $s\leqq(n-p)^2-1$ where s is the dimension of an orbit of maximal dimension. Thus $s<p\left(\binom{n-p}{2}-p\right)$ by the hypothesis on (n, p). On the other hand $B^2(\mathfrak{g}_1^{n-p}, F)=(0)$, $Z^2(\mathfrak{g}_1^{n-p}, F)=\overset{2}{\wedge}(\mathfrak{g}_1^{n-p})^*$, $H^2(\mathfrak{g}_1^{n-p}, F)=\overset{2}{\wedge}(\mathfrak{g}_1^{n-p})^*$ thus $\dim H^2(\mathfrak{g}_1^{n-p}, F)=\binom{n-p}{2}$, which implies $s<p(\dim H^2(\mathfrak{g}_1^{n-p}, F)-p)$. Since a central extension of the abelian Lie algebra $\mathfrak{g}_1^{n-p}$ by F^p is a nilpotent Lie algebra of dimension n one gets the conclusion by 2.3.

(ii) Let $f(n, p)=p\left(\binom{n-p}{2}-p\right)-(n-p)^2$. Since $f(9, 3)=0$ the assertion is proved for $n=9$ (with $p=3$); assume it is true for a certain n and let $p\geqq 3$

such that $f(n, p)\geqq 0$, then $n-p\geqq 2$. One has $f(n+1, p)=f(n, p)+(n-p)(p-2)-1\geqq 0+2(3-2)-1\geqq 0$ therefore the assertion is proved by induction on n.

2.7. REMARK. By using the argument of algebraic geometry as in 2.3, Gauger [**6**] (Theorem 7.8 and Corollary 7.9, p. 315-316) proves directly the result 2.6 but does not explain if the central extensions are with direct abelian factor or not. This point seems to us very important because if we have infinite families at a certain dimension n_0 then one can obtain infinite families at all dimension $\geqq n_0$ by making a trivial extension. The supplementary information we give comes from the theorem of T. Skjelbred and T. Sund (1.7).

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