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The mapping cone method and the Hattori-Villamayor-Zelinsky sequences

By Mitsuhiro TAKEUCHI

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The mapping cone method, which is originally due to MacLane [8], is fully developed in Hattori [4]. Let U be the multiplicative group and let Pic be the Picard group functor. Assume we have an exact sequence of abelian group functors on commutative rings:

$$0 \longrightarrow U \longrightarrow A \xrightarrow{f} B \longrightarrow \mathrm{Pic} \longrightarrow 0. \tag{1}$$

(Amitsur case). Let S/R be an extension of commutative rings, and let $S^n = S\otimes_R \cdots \otimes_R S$ (n terms) for $n=1, 2, \cdots$. Applying (1) to the Amitsur semi-simplicial complex

$$S \rightrightarrows S^2 \overset{\textstyle\rightarrow}{\rightrightarrows} S^3 \overset{\textstyle\rightrightarrows}{\rightrightarrows} \cdots$$

we get an exact sequence of complexes

$$0 \longrightarrow U(S^{\cdot}) \longrightarrow A(S^{\cdot}) \xrightarrow{f} B(S^{\cdot}) \longrightarrow \mathrm{Pic}(S^{\cdot}) \longrightarrow 0 \tag{2}$$

which yields, in view of [4, Theorem 1.3], a long exact sequence

$$\cdots \longrightarrow H^n(S/R, U) \longrightarrow H^n(M(f)) \longrightarrow H^{n-1}(S/R, \mathrm{Pic}) \longrightarrow H^{n+1}(S/R, U) \longrightarrow \cdots \tag{3}$$

where $M(f)$ is the mapping cone of (2) (with degree lowered by one) and $H^{\cdot}(S/R, -)$ means the Amitsur cohomology.

(Galois case). Let G be a group acting as automorphisms of a commutative ring R. (1) gives an exact sequence of G-modules

$$0 \longrightarrow U(R) \longrightarrow A(R) \xrightarrow{f} B(R) \longrightarrow \mathrm{Pic}(R) \longrightarrow 0. \tag{4}$$

Applying [4, Proposition 2.1] to (4), we get a long exact sequence

$$\cdots \longrightarrow H^n(G, U(R)) \longrightarrow H^n(G, f) \longrightarrow H^{n-1}(G, \mathrm{Pic}(R)) \longrightarrow H^{n+1}(G, U(R)) \longrightarrow \cdots \tag{5}$$

where $H^{\cdot}(G, U(R))$ and $H^{\cdot}(G, \mathrm{Pic}(R))$ are the Galois cohomology groups.

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($H^n(G, f)$ here means $H^{n-1}(G, f)$ of [**4**].)

On the other hand, we have the Amitsur Pic-U sequence [**12**], [**3**]

$$(6)\quad \cdots \longrightarrow H^n(S/R, U) \longrightarrow H^n(J) \longrightarrow H^{n-1}(S/R, \mathrm{Pic}) \longrightarrow H^{n+1}(S/R, U) \longrightarrow \cdots$$

in the Amitsur case, and the Galois Pic-U sequence [**2**]

$$(7)\quad \cdots \longrightarrow H^n(G, U(R)) \longrightarrow \boldsymbol{H}^n(R, G) \longrightarrow H^{n-1}(G, \mathrm{Pic}(R)) \longrightarrow H^{n+1}(G, U(R)) \longrightarrow \cdots$$

in the Galois case. The above sequences are generalizations of the Chase-Rosenberg seven term exact sequences.

The purpose of this paper is to show that there is an exact sequence (1) such that there are isomorphisms of sequences

$$\begin{array}{llccccccccc}
(3) & \cdots \longrightarrow & H^n(S/R, U) & \longrightarrow & H^n(M(f)) & \longrightarrow & H^{n-1}(S/R, \mathrm{Pic}) & \longrightarrow & H^{n+1}(S/R, U) & \longrightarrow & \cdots \\
 & & \| & & \wr\| & & \| & & \| & & \\
(6) & \cdots \longrightarrow & H^n(S/R, U) & \longrightarrow & H^n(J) & \longrightarrow & H^{n-1}(S/R, \mathrm{Pic}) & \longrightarrow & H^{n+1}(S/R, U) & \longrightarrow & \cdots
\end{array}$$

for any ring extension S/R, and

$$\begin{array}{llccccccccc}
(5) & \cdots \longrightarrow & H^n(G, U(R)) & \longrightarrow & H^n(G, f) & \longrightarrow & H^{n-1}(G, \mathrm{Pic}(R)) & \longrightarrow & H^{n+1}(G, U(R)) & \longrightarrow & \cdots \\
 & & \| & & \wr\| & & \| & & \| & & \\
(7) & \cdots \longrightarrow & H^n(G, U(R)) & \longrightarrow & \boldsymbol{H}^n(R, G) & \longrightarrow & H^{n-1}(G, \mathrm{Pic}(R)) & \longrightarrow & H^{n+1}(G, U(R)) & \longrightarrow & \cdots
\end{array}$$

for any pair (G, R) with group G acting on ring R.

Similar results are proved by Hattori [4, **5**] in some arithmetic cases, and used to give many applications in algebraic number theory. Our method is based on the coherence theorem in categories with abelian group structure due to Ulbrich [**11**]. The article was prepared while K.-H. Ulbrich visited Princeton in March, 1981. I am grateful to him for many useful comments.

§ 1. Construction.

Fix an infinite set Ω. For a commutative ring R, let $R\Omega$ be the free R-module with basis Ω. Let I_R be the set of all direct summand R-submodules $M \subset R\Omega$ which are invertible, i.e., projective of rank one. Let $\mathcal{P}ic(R)$ be the category of all invertible R-modules and isomorphisms.

1.1. DEFINITION. A *group-like* set is a quadruple $(G, +, -, 0)$ where G is a set, $0 \in G$, and

$$+ : G \times G \longrightarrow G, \qquad - : G \longrightarrow G$$

are maps.

Homomorphisms of group-like sets are defined in an obvious manner. For each set I, there is a group-like set $F(I)$ containing I such that for any group-like set G, any map $I \to G$ extends uniquely to a homomorphism $F(I) \to G$. $F(I)$ is called the *free* group-like set on I.

For a commutative ring R, $\mathcal{P}ic(R)$ has an abelian group structure [**10**]. We denote the structure functors by

$$+: \mathcal{P}ic(R) \times \mathcal{P}ic(R) \longrightarrow \mathcal{P}ic(R),$$

$$-: \mathcal{P}ic(R) \longrightarrow \mathcal{P}ic(R)$$

where $M+N=M\otimes_R N$ and $-M=\mathrm{Hom}_R(M, R)$. Thus $\mathrm{Ob}(\mathcal{P}ic(R))$ is a group-like class with R as 0. Let

$$\varepsilon: F(I_R) \longrightarrow \mathrm{Ob}(\mathcal{P}ic(R))$$

be the homomorphism where $\varepsilon | I_R$ is the inclusion. We will use map ε to define a new category $\overline{\mathcal{P}ic}(R)$.

Take $F(I_R)$ as the set of objects in $\overline{\mathcal{P}ic}(R)$. For u, v in $F(I_R)$, let

$$\overline{\mathcal{P}ic}(R)(u, v) = \mathcal{P}ic(R)(\varepsilon(u), \varepsilon(v)).$$

With composite obviously defined, we have a small category $\overline{\mathcal{P}ic}(R)$ together with an equivalence functor

$$\varepsilon: \overline{\mathcal{P}ic}(R) \longrightarrow \mathcal{P}ic(R)$$

where $\varepsilon(f)=f$ for any morphism f in $\overline{\mathcal{P}ic}(R)$.

$\overline{\mathcal{P}ic}(R)$ inherits an abelian group structure from $\mathcal{P}ic(R)$ as follows: If $f: u \to v$ and $g: u' \to v'$ are maps in $\overline{\mathcal{P}ic}(R)$, define $f+g: u+u' \to v+v'$ and $-f: -u \to -v$ by the rule $\varepsilon(f+g)=\varepsilon(f)+\varepsilon(g)$ and $\varepsilon(-f)=-\varepsilon(f)$. This gives rise to functors $+: \overline{\mathcal{P}ic}(R) \times \overline{\mathcal{P}ic}(R) \to \overline{\mathcal{P}ic}(R)$ and $-: \overline{\mathcal{P}ic}(R) \to \overline{\mathcal{P}ic}(R)$. For $u, v, w \in F(I_R)$, the natural isomorphisms

$$a_{u,v,w}: (u+v)+w \longrightarrow u+(v+w),$$

$$c_{u,v}: u+v \longrightarrow v+u,$$

$$e_u: u+0 \longrightarrow u,$$

$$i_u: u+(-u) \longrightarrow 0$$

are defined by $\varepsilon(a_{u,v,w})=a_{\varepsilon(u),\varepsilon(v),\varepsilon(w)}$, $\varepsilon(c_{u,v})=c_{\varepsilon(u),\varepsilon(v)}$, etc., by using the corresponding natural isomorphisms $a_{P,Q,N}$, $c_{P,Q}$, etc. in $\mathcal{P}ic(R)$. This gives $\overline{\mathcal{P}ic}(R)$ an abelian group structure, and $\varepsilon: \overline{\mathcal{P}ic}(R) \to \mathcal{P}ic(R)$ becomes a homomorphism [**10**] whose structure natural transformations are identities. Such a homomorphism is called *strict*.

1.2. DEFINITION. Let $\overline{\mathcal{P}ic}(R)^{\mathrm{red}}$ be the smallest subcategory of $\overline{\mathcal{P}ic}(R)$ such that $\mathrm{Ob}(\overline{\mathcal{P}ic}(R)^{\mathrm{red}})=\mathrm{Ob}(\overline{\mathcal{P}ic}(R))$ and $\mathrm{Mor}(\overline{\mathcal{P}ic}(R)^{\mathrm{red}})$ is closed under $+$ and $-$ containing $a_{u,v,w}$, $c_{u,v}$, e_u, i_u together with their inverses for all $u, v, w \in F(I_R)$.

Morphisms in $\overline{\mathcal{P}ic}(R)^{\text{red}}$ are called *reduced.*

The following is a special case of the coherence theorem due to Ulbrich [**11**]. For a simpler proof, see Laplaza [**6**]. Ulbrich also has an improved proof (oral communication). Different approaches to coherence are found in [**1**, pp. 246-247], [**12**, § 3].

1.3. THEOREM. *For any $u, v \in F(I_R)$, there is one reduced morphism $u \to v$ at most.*

We are now ready to define the sequence of abelian groups

$$0 \longrightarrow U(R) \xrightarrow{i_R} A(R) \xrightarrow{f_R} B(R) \xrightarrow{\pi_R} \operatorname{Pic}(R) \longrightarrow 0 \tag{1.4}$$

for any commutative ring R.

Let $B(R) = \mathbf{Z}I_R$ be the free abelian group on I_R and let π_R be the canonic projection. We may view $B(R)$ as the quotient set of $F(I_R)$ by the equivalence relation: $u \sim v$ if there is a reduced morphism $u \to v$. We denote by

$$u \longmapsto [u], \qquad F(I_R) \longrightarrow B(R)$$

the canonical surjection.

Let $A(R)$ be the quotient set of the set $\Lambda(R)$ of all pairs (u, a) with $u \in F(I_R)$ and $a : u \to 0$ in $\overline{\mathcal{P}ic}(R)$ by the equivalence relation: $(u, a) \sim (v, b)$ if there is a reduced morphism $c : u \to v$ such that $a = b \circ c$. Let $[u, a]$ denote the equivalence class of (u, a). We make $A(R)$ into an abelian group. For (u, a), (v, b) in $\Lambda(R)$, let

$$(u, a) + (v, b) = (u + v, \zeta \circ (a + b))$$

where $\zeta : 0 + 0 \to 0$ is the reduced map. If $(u, a) \sim (u', a')$ and $(v, b) \sim (v', b')$, then $(u, a) + (v, b) \sim (u', a') + (v', b')$. Hence addition on $A(R)$

$$[u, a] + [v, b] = \text{class of } (u, a) + (v, b)$$

is well-defined. It follows easily by the definition of $\overline{\mathcal{P}ic}(R)^{\text{red}}$ that $A(R)$ becomes an abelian group. The unit is $[0, \text{id}]$.

We will define homomorphisms f_R and i_R. For $[u, a]$ in $A(R)$, and r in $U(R)$, we put

$$f_R[u, a] = [u], \qquad i_R(r) = [0, r]$$

where we use the usual identification

$$\overline{\mathcal{P}ic}(R)(0, 0) = \mathcal{P}ic(R)(0, 0) = U(R).$$

Maps f_R and i_R are well-defined, and seen to be homomorphisms.

It is easy to show that (1.4) is exact.

Next, we make A and B into group functors on commutative rings so that i_R, f_R, π_R are natural in R.

Let $\phi: R\to S$ be a homomorphism of commutative rings. Extend it to the semilinear map

$$\phi: R\Omega \longrightarrow S\Omega$$

which is the identity on Ω. If $M\in I_R$, then $S\cdot\phi(M)\in I_S$ since $S\otimes_R M\simeq S\cdot\phi(M)$. Put

$$\bar{\phi}: M\longmapsto S\cdot\phi(M), \qquad I_R \longrightarrow I_S$$

and extend it to the homomorphism of group-like sets

$$\bar{\phi}: F(I_R) \longrightarrow F(I_S).$$

We have a homomorphism [10, p. 137]

$$\ddot{\phi}: M\longmapsto S\otimes_R M, \qquad \mathcal{P}ic(R) \longrightarrow \mathcal{P}ic(S).$$

Let

$$\alpha_{P,Q}: \ddot{\phi}(P+Q) \longrightarrow \ddot{\phi}(P)+\ddot{\phi}(Q),$$
$$\beta_P: \ddot{\phi}(-P) \longrightarrow -\ddot{\phi}(P),$$
$$\gamma: \ddot{\phi}(0_R) \longrightarrow 0_S \qquad (\text{where } 0_R=R,\ 0_S=S)$$

denote the structure of $\ddot{\phi}$, for P, Q in $\mathcal{P}ic(R)$. We define a map in $\overline{\mathcal{P}ic}(S)$

$$\xi_u: \ddot{\phi}(\varepsilon(u)) \longrightarrow \varepsilon(\bar{\phi}(u))$$

for $u\in F(I_R)$ as follows:

i) $\xi_{u+v}=(\xi_u+\xi_v)\circ\alpha_{\varepsilon(u),\varepsilon(v)}$,
ii) $\xi_{-u}=(-\xi_u)\circ\beta_{\varepsilon(u)}$,
iii) $\xi_0=\gamma$,
iv) $\xi_M: S\otimes_R M(=\ddot{\phi}(M)) \longrightarrow S\cdot\phi(M)(=\bar{\phi}(M))$ is the canonical isomorphism if $M\in I_R$.

Since $F(I_R)$ is the free group-like set on I_R, there is a unique family of maps $\{\xi_u\}_{u\in F(I_R)}$ satisfying i)$\sim$iv).

1.5. LEMMA. *We can make* $\bar{\phi}: F(I_R)\to F(I_S)$ *into a functor* $\bar{\phi}: \overline{\mathcal{P}ic}(R)\to \overline{\mathcal{P}ic}(S)$ *in such a way that*

$$\xi: \ddot{\phi}\varepsilon \longrightarrow \varepsilon\bar{\phi}$$

becomes a natural isomorphism. Then the functor $\bar{\phi}$ *becomes a strict homomorphism, and* ξ *is an isomorphism of homomorphisms. In particular,* $\bar{\phi}$ *preserves reduced maps.*

PROOF. Let $g: u\to v$ be a map in $\overline{\mathcal{P}ic}(R)$. Since ε is an equivalence, there is a unique map $g': \bar{\phi}(u)\to\bar{\phi}(v)$ such that $\varepsilon(g')\circ\xi_u=\xi_v\circ\ddot{\phi}(\varepsilon(g))$. We put $g'=\bar{\phi}(g)$. Then $\bar{\phi}$ becomes a functor $\overline{\mathcal{P}ic}(R)\to\overline{\mathcal{P}ic}(S)$. Now conditions i)-iii) mean that ξ is already an isomorphism of homomorphisms if we take the identities as the structure of $\varepsilon\bar{\phi}$. It follows from $\ddot{\phi}\varepsilon$ being a homomorphism that $\varepsilon\bar{\phi}$ is indeed a homomorphism with identities as the structure. Thus $\varepsilon\bar{\phi}$ is a strict homomorphism. Since ε is an equivalence, so is $\bar{\phi}$. Q.E.D.

We will define maps $A(\phi)$ and $B(\phi)$ to make the next commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(R) & \longrightarrow & A(R) & \longrightarrow & B(R) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & 0 \\ & & \downarrow U(\phi) & & \downarrow A(\phi) & & \downarrow B(\phi) & & \downarrow \mathrm{Pic}(\phi) & & \\ 0 & \longrightarrow & U(S) & \longrightarrow & A(S) & \longrightarrow & B(S) & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & 0 \end{array} \tag{1.6}$$

where both rows are (1.4).

It follows from Lemma 1.5 that the functor $\bar{\phi}: \overline{\mathcal{P}ic}(R)\rightarrow\overline{\mathcal{P}ic}(S)$ preserves reduced maps. Hence $u\sim v$ implies $\bar{\phi}(u)\sim\bar{\phi}(v)$ for $u, v\in F(I_R)$, and $(u, a)\sim(v, b)$ implies $(\bar{\phi}(u), \bar{\phi}(a))\sim(\bar{\phi}(v), \bar{\phi}(b))$ for (u, a), (v, b) in $\Lambda(R)$. Hence the maps

$$B(\phi)[u]=[\bar{\phi}(u)], \qquad A(\phi)[u, a]=[\bar{\phi}(u), \bar{\phi}(a)]$$

are well-defined, and seen to be homomorphisms to make diagram (1.6) commute.

Let $\psi: S\rightarrow T$ be another homomorphism of commutative rings. It is easy to see

$$\bar{\psi}\circ\bar{\phi}=\overline{\psi\phi}$$

as functors: $\overline{\mathcal{P}ic}(R)\rightarrow\overline{\mathcal{P}ic}(T)$, (while $\ddot{\psi}\circ\ddot{\phi}$ is different from $\ddot{\psi\phi}$). It follows that $A(\psi\circ\phi)=A(\psi)\circ A(\phi)$ and $B(\psi\circ\phi)=B(\psi)\circ B(\phi)$.

If $1: R\rightarrow R$ denote the identity, then $\bar{1}: \overline{\mathcal{P}ic}(R)\rightarrow\overline{\mathcal{P}ic}(R)$ is the identity. Hence $A(1)$ and $B(1)$ are identities.

Thus we get an exact sequence of abelian group functors on commutative rings

$$0\longrightarrow U\xrightarrow{i} A\xrightarrow{f} B\xrightarrow{\pi}\mathrm{Pic}\longrightarrow 0. \tag{1.7}$$

§ 2. Identification.

We will identify the Amitsur or Galois mapping cone sequence obtained from (1.7) with the Amitsur or Galois Pic-U sequence.

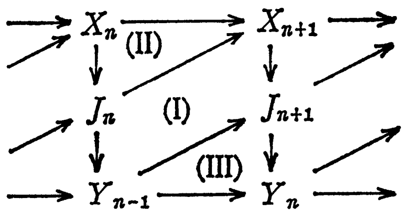
Let

$$X:\ \cdots\longrightarrow X_n\longrightarrow X_{n+1}\longrightarrow\cdots,$$

$$Y:\ \cdots\longrightarrow Y_n\longrightarrow Y_{n+1}\longrightarrow\cdots$$

be complexes of abelian groups.

A diagram of abelian groups

(2.1) 

where complexes X and Y appear as two rows, is called a V-Z *system* [**12**, p. 37] if the following conditions are fulfilled.

(a) The composite along each diagonal is zero:

$$Y_{n-2} \longrightarrow J_n \longrightarrow X_{n+1}.$$

(b) The parallelograms (I) anticommute.

(c) The triangles (II), (III) commute.

(d) The five term, crank-shaped sequences are exact:

$$X_{n-1} \longrightarrow X_n \longrightarrow J_n \longrightarrow Y_{n-1} \longrightarrow Y_n.$$

We can associate a long exact sequence

$$\cdots \longrightarrow H^n(X) \longrightarrow H^n(J) \longrightarrow H^{n-1}(Y) \longrightarrow H^{n+1}(X) \longrightarrow \cdots \tag{2.2}$$

with each V-Z system (2.1) [**12**, p. 39]. $H^n(J)$ means $\mathrm{Ker}(J_n \to X_{n+1})/\mathrm{Im}(Y_{n-2} \to J_n)$. $H^n(X) \to H^n(J) \to H^{n-1}(Y)$ are induced from $X_n \to J_n \to Y_{n-1}$. If $y \in \mathrm{Ker}(Y_{n-1} \to Y_n)$, y comes from some $z \in J_n$. Let $x \in X_{n+1}$ be the image of z by $J_n \to X_{n+1}$. Then $H^{n-1}(Y) \to H^{n+1}(X)$ is induced by (class of y)$\mapsto$(class of x).

Isomorphisms between two V-Z systems are defined obviously. Isomorphic V-Z systems have isomorphic sequences.

Let

$$0 \longrightarrow X \longrightarrow C \overset{f}{\longrightarrow} D \longrightarrow Y \longrightarrow 0 \tag{2.3}$$

be an exact sequence of complexes. We can associate to it some V-Z system containing X and Y as two rows. The sequence (2.3) contains square diagrams

$$\begin{array}{ccc} C_{n-1} & \longrightarrow & D_{n-1} \\ \partial \downarrow & & \downarrow \partial \\ C_n & \longrightarrow & D_n \end{array}$$

with coboundary operator ∂. Let J_n be the *center* of the square, i.e.,

$$J_n = (C_n \times_{D_n} D_{n-1})/\mathrm{Im}(C_{n-1} \longrightarrow C_n \times_{D_n} D_{n-1}).$$

We denote by $[c, d] \in J_n$ the image of element (c, d) in the fiber product, and by $\bar{d} \in Y_n$ the image of $d \in D_n$. With well-defined maps

$$X_n \longrightarrow J_n, \qquad x \longmapsto [x, 0],$$
$$J_n \longrightarrow Y_{n-1}, \qquad [c, d] \longmapsto \bar{d},$$
$$J_n \longrightarrow X_{n+1}, \qquad [c, d] \longmapsto \partial(c),$$
$$Y_{n-1} \longrightarrow J_{n+1}, \qquad \bar{d} \longmapsto [0, \partial(d)]$$

we have a V-Z system as is easily checked.

Next we review complexes of categories introduced in [**10**].

2.4. DEFINITION. A sequence of homomorphisms of categories with abelian group structure

$$\cdots \longrightarrow \mathcal{C}_n \xrightarrow{\partial} \mathcal{C}_{n+1} \longrightarrow \cdots$$

together with isomorphisms of homomorphisms

$$\chi : \partial^2 \xrightarrow{\sim} 0$$

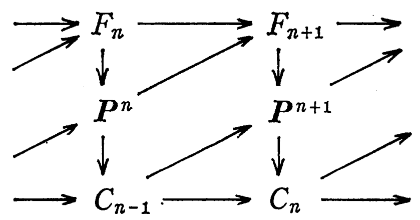
where $0 : \mathcal{C}_n \to \mathcal{C}_{n+2}$ denotes the constant homomorphism, is called a *coherent complex of categories* if

$$\chi\partial : \partial^3 \xrightarrow{\partial\chi} \partial 0 \xrightarrow{\text{cano}} 0 .$$

Strictly speaking, some coherence conditions for $\mathcal{C}_n$ as asserted in [**10**, Lemma 1.2] are necessary to assume. But they are fulfilled for $\mathcal{P}ic(R)$ or their direct products. Coherent complexes of categories are special cases of $\mathbb{X}$-systems of [**9**].

In [**10**], Ulbrich constructs a V-Z system

(2.5)



with each coherent complex of categories $\{\mathcal{C}_n, \partial\}$, where maps are defined:

$$\boldsymbol{P}^n \longrightarrow F_{n+2} \text{ [10, Proposition 2.5]}, \qquad C_{n-1} \longrightarrow \boldsymbol{P}^{n+1} \text{ [10, } \uparrow 1\text{, p. 133]},$$
$$\boldsymbol{P}^{n+1} \longrightarrow C_n \text{ [10, (19), p. 134]}, \qquad F_n \longrightarrow \boldsymbol{P}^n \text{ [10, (21), p. 134]}.$$

(We lower the dimension of $\boldsymbol{P}$. $\boldsymbol{P}^n$ here means $\boldsymbol{P}^{n-1}$ in [**10**].)

He defines two coherent complexes of categories corresponding to the Amitsur and the Galois cases:

(2.6) $$\mathcal{P}ic(S) \xrightarrow{\partial} \mathcal{P}ic(S^2) \longrightarrow \cdots \longrightarrow \mathcal{P}ic(S^n) \xrightarrow{\partial} \mathcal{P}ic(S^{n+1}) \longrightarrow \cdots$$

for a commutative ring extension S/R [**10**, (32), p. 137] and

$$(2.7)\qquad \mathcal{P}ic(R) \xrightarrow{\partial} (G, \mathcal{P}ic(R)) \longrightarrow \cdots \longrightarrow (G^{n-1}, \mathcal{P}ic(R)) \xrightarrow{\partial} (G^n, \mathcal{P}ic(R)) \longrightarrow \cdots$$

for a group G acting on a commutative ring R [**10**, (31), p. 137]. In (2.6), $\mathcal{P}ic(S^n)$ is of degree $n-1$. In (2.7), $(G^n, \mathcal{P}ic(R))$ means the direct product of $\mathcal{P}ic(R)$ indexed by G^n. He shows that the V-Z system (2.5) associated with complex (2.6) (respectively (2.7)) has the Amitsur Pic-U sequence [**12**], [**3**]

$$(2.8)\qquad \cdots \longrightarrow H^n(S/R, U) \longrightarrow H^n(J) \longrightarrow H^{n-1}(S/R, \mathrm{Pic}) \longrightarrow H^{n+1}(S/R, U) \longrightarrow \cdots$$

(respectively the Galois Pic-U sequence [**2**]

$$(2.9)\qquad \cdots \longrightarrow H^n(G, U(R)) \longrightarrow \boldsymbol{H}^n(R, G) \longrightarrow H^{n-1}(G, \mathrm{Pic}(R)) \longrightarrow H^{n+1}(G, U(R)) \longrightarrow \cdots).$$

2.10. THEOREM. (a) *Let S/R be a commutative ring extension. Let*

$$(2.11)\qquad 0 \longrightarrow U(S^{\cdot}) \longrightarrow A(S^{\cdot}) \longrightarrow B(S^{\cdot}) \longrightarrow \mathrm{Pic}(S^{\cdot}) \longrightarrow 0$$

be the exact sequence of complexes obtained by applying sequence (1.7) *to the Amitsur semi-simplicial complex*

$$S \rightrightarrows S\otimes_R S \rightrightarrows S\otimes_R S\otimes_R S \rightrightarrows \cdots.$$

There is a natural isomorphism between the V-Z system associated with (2.11) *and the V-Z system associated with complex* (2.6).

(b) *Let G be a group acting on a commutative ring R as automorphisms. Let C be the non-homogeneous standard complex of G, which is a free $\boldsymbol{Z}[G]$-resolution of the trivial G-module $\boldsymbol{Z}$. Let*

$$(2.12)\quad 0 \to \mathrm{Hom}_G(C, U(R)) \to \mathrm{Hom}_G(C, A(R)) \to \mathrm{Hom}_G(C, B(R)) \to \mathrm{Hom}_G(C, \mathrm{Pic}(R)) \to 0$$

be the exact sequence obtained by the exact sequence (1.4) *of G-modules. There is a natural isomorphism between the V-Z system associated with the sequence of complexes* (2.12) *and the V-Z system associated with complex* (2.7).

2.13. COROLLARY. *The V-Z system associated with complex exact sequence* (2.11) (*respectively* (2.12)) *has the Amitsur* (*respectively Galois*) Pic-U *sequence* (2.8) (*respectively* (2.9)).

PROOF. (a) Recall the definition of (2.6). S^n is the n-fold tensor product of S over R and the functor

$$\partial : \mathcal{P}ic(S^n) \longrightarrow \mathcal{P}ic(S^{n+1})$$

maps an object P to $\partial P = (\cdots((\ddot{\varepsilon}_0 P + (-1)\ddot{\varepsilon}_1 P) + (-1)^2\ddot{\varepsilon}_2 P) + \cdots) + (-1)^n \ddot{\varepsilon}_n P$, where

$$\varepsilon_i : S^n \longrightarrow S^{n+1},\ a_1\otimes\cdots\otimes a_n \longmapsto a_1\otimes\cdots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\cdots\otimes a_n$$

for $0 \leqq i \leqq n$. Using $\bar{\varepsilon}_i$ instead of $\ddot{\varepsilon}_i$, we get a coherent complex of categories

$$\overline{\mathcal{P}ic}(S) \xrightarrow{\bar{\partial}} \overline{\mathcal{P}ic}(S^2) \longrightarrow \cdots \longrightarrow \overline{\mathcal{P}ic}(S^n) \xrightarrow{\bar{\partial}} \overline{\mathcal{P}ic}(S^{n+1}) \longrightarrow \cdots, \tag{2.14}$$

with structure $\bar{\chi}: \bar{\partial}^2 \to 0$, and we have a diagram of homomorphisms

$$\begin{array}{ccccccc} \cdots \longrightarrow & \overline{\mathcal{P}ic}(S^n) & \xrightarrow{\bar{\partial}} & \overline{\mathcal{P}ic}(S^{n+1}) & \longrightarrow \cdots \\ & \downarrow \varepsilon & & \downarrow \varepsilon & \\ \cdots \longrightarrow & \mathcal{P}ic(S^n) & \xrightarrow{\partial} & \mathcal{P}ic(S^{n+1}) & \longrightarrow \cdots \end{array}$$

where (2.14) and (2.6) appear as two rows. It follows from Lemma 1.5 that there is a natural isomorphism

$$\xi : \partial\varepsilon \xrightarrow{\sim} \varepsilon\bar{\partial}$$

such that

$$\begin{array}{ccccc} \partial^2\varepsilon & \xrightarrow{\partial\xi} & \partial\varepsilon\bar{\partial} & \xrightarrow{\xi\bar{\partial}} & \varepsilon\bar{\partial}^2 \\ \downarrow \chi\varepsilon & & & & \downarrow \varepsilon\bar{\chi} \\ 0\varepsilon & = & 0 & = & \varepsilon 0 \end{array}$$

commutes. Since ε is an equivalence, it follows that the V-Z systems corresponding to (2.14) and (2.6) are isomorphic. Let (2.5) be the V-Z system associated with (2.14). By definition, we can identify $F_n = U(S^{n+1})$ and $C_n = \mathrm{Pic}(S^{n+1})$. $\boldsymbol{P}^n$ is the quotient set of the set of all pairs (u, a) with $u \in \mathrm{Ob}(\overline{\mathcal{P}ic}(S^n))$ and $a : \bar{\partial}(u) \to 0$ in $\overline{\mathcal{P}ic}(S^{n+1})$ by the equivalence relation: $(u, a) \sim (v, b)$ if there is a map $c: u \to v$ in $\overline{\mathcal{P}ic}(S^n)$ such that $b \circ \bar{\partial}(c) = a$. Denote by $\{u, a\}$ the equivalence class of (u, a). Next, let (2.1) be the V-Z system associated with (2.11). We can also identify $X_n = U(S^{n+1})$ and $Y_n = \mathrm{Pic}(S^{n+1})$. Recall that J_n is the center of square

$$\begin{array}{ccc} A(S^n) & \xrightarrow{f} & B(S^n) \\ \downarrow & & \downarrow \\ A(S^{n+1}) & \xrightarrow{f} & B(S^{n+1}) . \end{array}$$

If $\{u, a\} \in \boldsymbol{P}^n$, we have $[u] \in B(S^n)$, $[\bar{\partial}(u), a] \in A(S^{n+1})$, and $([\bar{\partial}(u), a], [u])$ is in the fiber product. Assume $\{u, a\} = \{v, b\}$ in $\boldsymbol{P}^n$ with $c: u \to v$ in $\overline{\mathcal{P}ic}(S^n)$. Put

$$e : u + (-v) \xrightarrow{c+I} v + (-v) \xrightarrow{\text{reduced map}} 0 .$$

Then $[u + (-v), e] \in A(S^n)$ and we have

$$([\bar{\partial}(u), a], [u]) = ([\bar{\partial}(v), b], [v]) + \Delta[u + (-v), e]$$

with diagonal map $\Delta: A(S^n) \to A(S^{n+1}) \times B(S^n)$. Hence the map

$$\{u, a\} \longmapsto [[\partial(u), a], [u]], \quad \boldsymbol{P}^n \longrightarrow J_n$$

is well-defined and seen to be a homomorphism. It is very easy to check that this homomorphism gives rise to a homomorphism of the V-Z system associated with (2.14) to the V-Z system associated with (2.11), together with identities $F_n \to X_n$ and $C_n \to Y_n$. In particular $\boldsymbol{P}^n \to J_n$ is an isomorphism by (d) below (2.1). This proves (a). (b) is proved similarly. Q.E.D.

The final step is to identify the sequence (2.2) of the V-Z system associated to (2.3) with the mapping cone sequence. We review the definition of the mapping cone sequence [4, Theorem 1.3], [8], [7, p. 46].

The mapping cone $M(f)$ of (2.3) is defined by:

$$M(f) = \{M_n, \partial\}, \quad M_n = C_n \times D_{n-1},$$

$$\partial(x, y) = (-\partial x, fx + \partial y).$$

(In [4], M_n is given degree $n-1$.) There is a long exact sequence

(2.15) $$\cdots \longrightarrow H^n(X) \xrightarrow{\alpha} H^n(M(f)) \xrightarrow{\beta} H^{n-1}(Y) \xrightarrow{\gamma} H^{n+1}(X) \longrightarrow \cdots$$

where

$$\alpha: (\text{class of } x \in X_n) \longmapsto (\text{class of } (x, 0)),$$

$$\beta: (\text{class of } (x, y) \in M_n) \longmapsto (\text{class of } -\bar{y}),$$

$$\gamma: (\text{class of } \bar{y} \in Y_{n-1} \text{ with } \partial y = fx) \longmapsto (\text{class of } \partial x).$$

Here we denote by $\bar{y} \in Y_{n-1}$ the image of $y \in D_{n-1}$. (The last map γ is $-\gamma$ with the notation of [4].)

If $(x, y) \in M_n$ is an n-cocycle, then $\partial x = 0$ and $fx + \partial y = 0$. Hence $(x, -y) \in J_n$. The homomorphism

$$\theta: (\text{class of } (x, y)) \longmapsto (\text{class of } [x, -y]), \quad H^n(M(f)) \longrightarrow H^n(J)$$

is well-defined. It is easy to prove:

2.16. PROPOSITION. *We have a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^n(X) & \longrightarrow & H^n(M(f)) & \longrightarrow & H^{n-1}(Y) & \longrightarrow & H^{n+1}(X) & \longrightarrow \cdots \\
& \| & & \downarrow \theta & & \| & & \| & \\
\cdots \longrightarrow & H^n(X) & \longrightarrow & H^n(J) & \longrightarrow & H^{n-1}(Y) & \longrightarrow & H^{n+1}(X) & \longrightarrow \cdots
\end{array}$$

where the first row is the mapping cone sequence (2.15), *and the second row is the sequence of the V-Z system associated with* (2.3). *Especially, θ is an isomorphism.*

Combining (2.16) and (2.13), we have:

2.17. THEOREM. (a) *Let S/R be a commutative ring extension. There is an isomorphism of sequences*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^n(S/R, U) & \longrightarrow & H^n(J) & \longrightarrow & H^{n-1}(S/R, \mathrm{Pic}) & \longrightarrow & H^{n+1}(S/R, U) & \longrightarrow \cdots \\
& \| & & \wr\downarrow & & \| & & \| & \\
\cdots \longrightarrow & H^n(S/R, U) & \longrightarrow & H^n(M(f)) & \longrightarrow & H^{n-1}(S/R, \mathrm{Pic}) & \longrightarrow & H^{n+1}(S/R, U) & \longrightarrow \cdots
\end{array}$$

where the first row is the Amitsur Pic-U *sequence* (2.8) *and the second row is the mapping cone sequence of the sequence* (2.11).

(b) *Let G be a group acting on a commutative ring R. There is an isomorphism of sequences*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^n(G, U(R)) & \longrightarrow & \boldsymbol{H}^n(R, G) & \longrightarrow & H^{n-1}(G, \mathrm{Pic}(R)) & \longrightarrow & H^{n+1}(G, U(R)) & \longrightarrow \cdots \\
& \| & & \wr\downarrow & & \| & & \| & \\
\cdots \longrightarrow & H^n(G, U(R)) & \longrightarrow & H^n(G, f) & \longrightarrow & H^{n-1}(G, \mathrm{Pic}(R)) & \longrightarrow & H^{n+1}(G, U(R)) & \longrightarrow \cdots
\end{array}$$

where the first row is the Galois Pic-U *sequence* (2.9) *and the second row is the mapping cone sequence of the sequence* (2.12).

Note that the second row of (b) is obtained by applying [4, Proposition 2.1] to the sequence of G-modules

$$0 \longrightarrow U(R) \longrightarrow A(R) \xrightarrow{f_R} B(R) \longrightarrow \mathrm{Pic}(R) \longrightarrow 0 .$$

($H^n(G, f)$ in the above means $H^{n-1}(G, f)$ of [4].)

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Mitsuhiro TAKEUCHI
Department of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun
Ibaraki-ken 305
Japan