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Topological types of isolated singularities defined by weighted homogeneous polynomials

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§ 1. Introduction and Theorem.

It is well-known that the weights of a weighted homogeneous polynomial with an isolated singularity determines the topological type of the germ of the singularity ([4]). Is the converse true? Namely, does the topological type of the germ of isolated singularity defined by a weighted homogeneous polynomial determine the weights of the polynomial?

In what follows, we shall assume that the weights of weighted homogeneous polynomials are greater than zero and less than or equal to 1/2, without loss of generality ([6]).

The above problem in the case of plane curves was treated in [7] and it was proved affirmatively. It is well-known that the problem in the case of three variables is also affirmative ([5]).

Now, let $f(z_1, z_2, \cdots, z_n)$ be a weighted homogeneous polynomial of type $(r_1, r_2, \cdots, r_n)$ with an isolated singularity and let $r_i=a_i/b_i$ be irreducible fractions. Then, J. Milnor and P. Orlik ([2]) proved that the characteristic polynomial $\Delta_f(t)$ of the Milnor fibration of the polynomial $f(z_1, z_2, \cdots, z_n)$ is determined by

$$\operatorname{divisor} \Delta_f(t)=\prod_{i=1}^{n}\left(\frac{1}{a_i}\Lambda_{b_i}-1\right),$$

where Λ_b means the divisor (t^b-1), (see p. 386 [3]).

Lê Dũng Tráng proved that the characteristic polynomial $\Delta_f(t)$ is a topological invariant ([1]).

Our main result in this paper is the following

THEOREM. *Let $f(z_1, z_2, \cdots, z_n)$ (resp. $g(z_1, z_2, \cdots, z_n)$) be a weighted homogeneous polynomial of type $(r_1, r_2, \cdots, r_n)$ (resp. $(k_1, k_2, \cdots, k_n)$) with an isolated singularity and let $r_i=a_i/b_i$ (resp. $k_i=c_i/d_i$) be irreducible fractions. Then, $\Delta_f(t)=\Delta_g(t)$ if and only if the following two conditions are satisfied:*

(1) $\{2, b_1, b_2, \cdots, b_n\}=\{2, d_1, d_2, \cdots, d_n\}$.

(2) *For any* $b\in\{2, b_1, b_2, \cdots, b_n\}$,

$$\prod_{b_i=b}\left(1-\frac{1}{r_i}\right)=\prod_{d_i=b}\left(1-\frac{1}{k_i}\right)$$

where the product over an empty set is assumed to be one.

COROLLARY. *Let $f(z_1, z_2, \cdots, z_n)$, $g(z_1, z_2, \cdots, z_n)$ be polynomials of Brieskorn-Pham type, namely $a_i=c_i=1\,(i=1, 2, \cdots, n)$ in the theorem. Then, the following three assertions are equivalent:*

(1) *The germs $\{f(z)=0\}$ and $\{g(z)=0\}$ at the origin are of the same topological type.*

(2) $\Delta_f(t)=\Delta_g(t)$.

(3) *Let $b_1\leqq b_2\leqq \cdots \leqq b_n$ and $d_1\leqq d_2\leqq \cdots \leqq d_n$, then $b_i=d_i$ for $i=1, 2, \cdots, n$.*

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§2. Proof of Theorem.

LEMMA. *Let n_i, q_j be integers such that $2\leqq n_1<n_2<\cdots<n_k$, $2\leqq q_1<q_2<\cdots<q_l$. Let c_i, r_j be integers and m_i, p_j be positive rational numbers. Then, if*

$$(*)\qquad \prod_{i=1}^{k}(m_i\Lambda_{n_i}+(-1)^{c_i})=\prod_{j=1}^{l}(p_j\Lambda_{q_j}+(-1)^{r_j}),$$

$$\sum_{i=1}^{k}c_i\equiv\sum_{j=1}^{l}r_j \mod 2,$$

we have $k=l$, $n_i=q_i$, $m_i=p_i$ and $c_i\equiv r_i$ mod 2 for $i=1, 2, \cdots, k$.

PROOF. We shall prove the lemma by the induction. Firstly, we shall prove the following

(1) $n_1=q_1$, $m_1=p_1$ and $c_1\equiv r_1$ mod 2.

Proof of (1). Using the formula:

$$\Lambda_a\Lambda_b=(a, b)\Lambda_{[a,b]}$$

where (a, b) and $[a, b]$ are the greatest common divisor and the least common multiple of a, b respectively, we can express both sides of $(*)$ as a linear combination

$$\alpha_1\Lambda_{\beta_1}+\alpha_2\Lambda_{\beta_2}+\cdots+\alpha_s\Lambda_{\beta_s}\qquad(\beta_1<\beta_2<\cdots<\beta_s).$$

This expression is unique. By the left side expression of $(*)$, we have $\beta_1=n_1$ and $\alpha_1=(-1)^{c_2+\cdots+c_k}m_1$. By the right side expression, we get $\beta_1=q_1$, $\alpha_1=(-1)^{r_2+\cdots+r_l}p_1$. Thus, we have:

$$n_1=q_1,\qquad (-1)^{\sum_{i\neq1}c_i}m_1=(-1)^{\sum_{j\neq1}r_j}p_1.$$

This implies the assertion (1).

Secondly, we shall prove the following (2) under the hypothesis of the induction: $n_i=q_i$, $m_i=p_i$ and $c_i\equiv r_i \bmod 2$ for $i\leqq i_0-1$.

(2) $n_{i_0}=q_{i_0}$, $m_{i_0}=p_{i_0}$ and $c_{i_0}\equiv r_{i_0} \bmod 2$.

Proof of (2). By the hypothesis of the induction, the coefficient of Λ_α in $\prod_{i=1}^{k}(m_i\Lambda_{n_i}+(-1)^{c_i})$ equals the coefficient of Λ_α in $\prod_{j=1}^{l}(p_j\Lambda_{q_j}+(-1)^{r_j})$ for each $\alpha<n_{i_0}$ (or q_{i_0}). So, by the assumption of the lemma, we get $n_{i_0}=q_{i_0}$.

Now, the terms of the divisor $\Lambda_{n_{i_0}}$ in $\prod_{i=1}^{k}(m_i\Lambda_{n_i}+(-1)^{c_i})$ is

$$(-1)^\alpha m_{i_0}\Lambda_{n_{i_0}}\prod_{\substack{n_i|n_{i_0}\\ n_i\neq n_{i_0}}}(m_i\Lambda_{n_i}+(-1)^{c_i})+\sum_{\substack{[n_{i_1},n_{i_2},\cdots,n_{i_q}]=n_{i_0}\\ n_{i_1}<n_{i_2}<\cdots<n_{i_q}<n_{i_0}}}(-1)^{\alpha_I}m_{i_1}m_{i_2}\cdots m_{i_q}(n_{i_1},n_{i_2},\cdots,n_{i_q})\Lambda_{n_{i_0}},$$

where $\alpha=\sum_{n_i\nmid n_{i_0}}c_i$, $\alpha_I=\sum_{i\notin I}c_i$, $I=\{i_1, i_2, \cdots, i_q\}$ and the integer $(n_{i_1}, n_{i_2}, \cdots, n_{i_q})$ is the greatest common divisor of the integers $n_{i_1}, n_{i_2}, \cdots, n_{i_q}$.

There is the same sum in the terms of $\Lambda_{q_{i_0}}=\Lambda_{n_{i_0}}$ in $\prod_{j=1}^{l}(p_j\Lambda_{q_j}+(-1)^{r_j})$ corresponding to the second sum in the above terms. Hence, we have:

$$(-1)^\alpha m_{i_0}\prod_{\substack{n_i|n_{i_0}\\ n_i\neq n_{i_0}}}(m_i\Lambda_{n_i}+(-1)^{c_i})=(-1)^\beta p_{i_0}\prod_{\substack{p_i|p_{i_0}\\ p_i\neq p_{i_0}}}(p_i\Lambda_{q_i}+(-1)^{r_i})$$

where $\beta=\sum_{q_i\nmid q_{i_0}}r_i$.

This implies $(-1)^\alpha m_{i_0}=(-1)^\beta p_{i_0}$ since $m_i=p_i$, $n_i=q_i$, $c_i\equiv r_i \pmod 2$ for $i<i_0$ by the hypothesis of the induction. So, the assertion (2) is proved. This completes the proof of the lemma.

Now, we shall prove the theorem.

PROOF OF THEOREM. Let

$$\{b_1, \cdots, b_n\}=\{n_1, \cdots, n_k\}, \quad n_1<\cdots<n_k$$

and let $c_i=\#\{j\,|\,b_j=n_i\}$.

Then

$$\prod_{b_j=n_i}\left(\frac{1}{a_j}\Lambda_{b_j}-1\right)=m_i\Lambda_{n_i}+(-1)^{c_i}$$

where $m_i=\dfrac{1}{n_i}\left\{\prod_{b_j=n_i}\left(\dfrac{b_j}{a_j}-1\right)-(-1)^{c_i}\right\}$.

Thus,

$$\text{divisor}\,\Delta_f(t)=\prod_{i=1}^{k}(m_i\Lambda_{n_i}+(-1)^{c_i}).$$

Noting that $m_i=0$ if and only if $n_i=2$ and $c_i=$even, the theorem follows immediately the above lemma.

PROOF OF COROLLARY.

(1) ⇒ (2). This was proved by Lê Dũng Tráng [1].

(2) ⇒ (3). Let us put: $r_i=1/b_i$ and $k_i=1/d_i$ in the theorem. So, we have:

$$\prod_{b_i=b}(1-b_j)=\prod_{d_i=b}(1-d_i)$$

for any $b\in\{2, b_1, \cdots, b_n\}$ by the theorem.

Here, let $b\geqq 3$, then the above equation implies

$$\#\{i\,|\,b_i=b\}=\#\{i\,|\,d_i=b\}.$$

These imply

$$\#\{i\,|\,b_i=2\}=\#\{i\,|\,d_i=2\}.$$

Thus, the assertion (3) is proved.

(3) ⇒ (1). This was proved by M. Oka [4].

This completes the proof of the corollary.

§ 3. Examples.

It is clear that the topology (not the topological type) of $S_\varepsilon^3\cap\{f(z_1, z_2)=0\}$ does not determine the weights of the weighted homogeneous polynomial $f(z_1, z_2)$ where S_ε^3 is a three dimensional sphere of an ε radius (ε small enough) centered at the origin. On the other hand, we can determine the weights of a weighted homogeneous polynomial $f(z_1, z_2, z_3)$ knowing only the topology of $S_\varepsilon^5\cap\{f(z_1, z_2, z_3)=0\}$ ([5]).

By the next two examples, we can see that the topology (not the topological type) of the isolated singularity defined by a weighted homogeneous polynomial does not determine the weights of the polynomial for each $n(\geqq 4)$, in general. So, we know that the case of three variables is the only exceptional case.

EXAMPLE 1. Let $f(z_1, \cdots, z_n, w_1, w_2)=z_1^2+\cdots+z_n^2+w_1^3+w_2^{2p}$, a weighted homogeneous polynomial of type $(1/2, \cdots, 1/2, 1/3, 1/2p)$ with an isolated singularity at the origin. Let $n\geqq 0$, even and $(3, p)=1$.

Then, we have:

$$\text{divisor } \Delta_f(t)=(\Lambda_3-1)(\Lambda_{2p}-1)$$
$$=\Lambda_{6p}-\Lambda_{2p}-\Lambda_3+1,$$

so

$$\Delta_f(t)\doteq\frac{(t-1)(t^{6p}-1)}{(t^3-1)(t^{2p}-1)}=\frac{t^{4p}+t^{2p}+1}{t^2+t+1}.$$

Hence, $\Delta_f(1)=1$. By Theorem 8.5 in [2], $K_f=S_\varepsilon\cap\{f(z, w)=0\}$ is a topological sphere. Thus all K_f for all p, $(3, p)=1$ are homeomorphic each other though $f(z, w)$ are of the different weighted homogeneous types for all p.

The topological types of the germs of isolated singularities defined by $f(z, w)$ are of the different types for all p, by [1] since the characteristic polynomials $\Delta_f(t)$ are different each other.

EXAMPLE 2. Let $f(z_1, \cdots, z_n, w_1, w_2)=z_1^2+\cdots+z_n^2+w_1^3+w_2^p$, a weighted homogeneous polynomial of type $(1/2, \cdots, 1/2, 1/3, 1/p)$ with an isolated singularity at the origin. Let $n\geqq 1$, odd and $(2, p)=(3, p)=1$.

Then, we have:

$$\text{divisor } \Delta_f(t)=(\Lambda_2-1)(\Lambda_3-1)(\Lambda_p-1)$$
$$=\Lambda_{6p}-\Lambda_{3p}-\Lambda_{2p}-\Lambda_6+\Lambda_p+\Lambda_3+\Lambda_2-1$$

so

$$\Delta_f(t)=\frac{(t^2-1)(t^3-1)(t^p-1)(t^{6p}-1)}{(t-1)(t^6-1)(t^{2p}-1)(t^{3p}-1)}$$
$$=\frac{(t+1)(t^{3p}+1)}{(t^3+1)(t^p+1)}.$$

Hence, $\Delta_f(1)=1$ and K_f is a topological shere for $n\geqq 3$. Thus, all K_f for all p, $(2, p)=(3, p)=1$ are homeomorphic each other, though weighted homogeneous polynomials $f(z, w)$ are of the different types for all p.

The topological types of isolated singularities defined by $f(z, w)$ are of the different types.

Our last example is the following

EXAMPLE 3. Let us put:

$$f(x, y, z, w)=x^2+y^2+z^2+w^{13} \quad \text{of type} \quad (1/2, 1/2, 1, 2, 1/13),$$

$$g(x, y, z, w)=x^2+y^3z+z^2w+yw^2 \quad \text{of type} \quad (1/2, 3/13, 4/13, 5/13).$$

Each one is a weighted homogeneous polynomial with an isolated singularity at the origin. A polynomial $f(x, y, z, w)$ is of Brieskorn-Pham type and a polynomial $g(x, y, z, w)$ is not so.

Now, we have:

$$\text{divisor } \Delta_f(t)=(\Lambda_2-1)(\Lambda_{13}-1)$$
$$=\Lambda_{26}-\Lambda_{13}-\Lambda_2+1,$$
$$\text{divisor } \Delta_g(t)=(\Lambda_2-1)\left(\frac{1}{3}\Lambda_{13}-1\right)\left(\frac{1}{4}\Lambda_{13}-1\right)\left(\frac{1}{5}\Lambda_{13}-1\right)$$
$$=(\Lambda_2-1)\left(\frac{1}{3}\Lambda_{13}-1\right)\left\{\left(\frac{13}{20}-\frac{1}{5}-\frac{1}{4}\right)\Lambda_{13}+1\right\}$$
$$=(\Lambda_2-1)\left\{\left(\frac{13}{15}-\frac{1}{5}+\frac{1}{3}\right)\Lambda_{13}-1\right\}$$

$$=(\Lambda_2-1)(\Lambda_{13}-1)$$
$$=\Lambda_{26}-\Lambda_{13}-\Lambda_2+1\,.$$

So, the characteristic polynomials

$$\Delta_f(t)=\Delta_g(t)=\frac{(t-1)(t^{26}-1)}{(t^2-1)(t^{13}-1)}=\frac{t^{13}+1}{t+1}\,.$$

Hence, $\Delta_f(1)=\Delta_g(1)=1$ and K_f, K_g are topological spheres by Theorem 8.5 in [2].

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