J. Math. Soc. Japan
Vol. 35, No. 4, 1983

Boolean valued interpretation of Hilbert space theory

By Masanao OZAWA

(Received July 19, 1982)

1. Introduction.

In 1966, D. Scott and R. Solovay reformulated the theory of P. J. Cohen's forcing in terms of Boolean valued models and they also introduced Boolean valued analysis as an application of Boolean valued model theory to analysis. Recently, G. Takeuti developed the Boolean valued analysis extensively in connection with operator theory, harmonic analysis and operator algebras [**6**], [**7**], [**8**], [**9**], [**10**].

In this paper, we study Boolean valued analysis of Hilbert space theory. Let $\mathcal{M}$ be a commutative W^*-algebra and $\mathcal{B}$ the complete Boolean algebra of projections in $\mathcal{M}$. We construct an embedding $H\rightarrow\tilde{H}$ of any non-degenerate normal $*$-representation of $\mathcal{M}$ on a Hilbert space H, which we call a normal $\mathcal{M}$-module H in this paper, in Scott-Solovay's Boolean valued model $V^{(\mathcal{B})}$ of set theory as a Hilbert space $\tilde{H}$ in $V^{(\mathcal{B})}$ and study functorial properties of this embedding. We prove that this embedding is an equivalence between the category of normal $\mathcal{M}$-modules and the category of Hilbert spaces in $V^{(\mathcal{B})}$ and that the multiplicity function of a normal $\mathcal{M}$-module H coincides with the dimension of $\tilde{H}$ in $V^{(\mathcal{B})}$, which is a cardinal in $V^{(\mathcal{B})}$. Thus the Hahn-Hellinger spectral multiplicity theory can be reduced to the Boolean valued interpretation of the simple statement that two Hilbert spaces are isomorphic if and only if they have the same dimension. These results also shed some lights on Takeuti's transfer theorem of von Neumann algebras to factors in $V^{(\mathcal{B})}$ [**10**], where he constructed Hilbert spaces in $V^{(\mathcal{B})}$ by enlarging original Hilbert spaces. In particular, our results improve his machinery in the point that we need not enlarge original Hilbert spaces in order to obtain Hilbert spaces in $V^{(\mathcal{B})}$; only we have to do is to change the truth value of the equality between vectors.

In Section 2, we give necessary preliminaries. In Section 3, it is shown that L^p-spaces on the spectrum of $\mathcal{M}$ is the complex numbers in $V^{(\mathcal{B})}$. Later development depends much on this fact. In Section 4, we construct the embedding $H\rightarrow\tilde{H}$. In Section 5, we prove its functorial properties. In Section 6, the connection with multiplicity theory is established.

The author wishes to express his gratitude to Professor G. Takeuti for the use of his results appearing in his preprint.

2. Preliminaries.

Let $\mathcal{M}$ be an abelian W^*-algebra, i. e. a commutative C^*-algebra which is a dual space as a Banach space, or equivalently which admits a faithful representation as a von Neumann algebra on a Hilbert space. For the basic theorems and terminology of the theory of operator algebras, we refer the reader to Sakai [**4**] and Takesaki [**5**]. Let $\mathcal{B}$ be the set of projections in $\mathcal{M}$. Then $\mathcal{B}$ is a complete Boolean algebra by defining the Boolean operations as $b_1 \vee b_2 = b_1 + b_2 - b_1 b_2$, $b_1 \wedge b_2 = b_1 b_2$ and $\neg b = 1 - b$, whose maximum element is the unit 1 of $\mathcal{M}$ and whose minimum element is 0 of $\mathcal{M}$.

Scott-Solovay's Boolean valued model $V^{(\mathcal{B})}$ of set theory is defined in the following way [**11**; p. 59, p. 121]. For an ordinal α, we define $V_\alpha^{(\mathcal{B})}$ by transfinite induction as follows:

(1) $V_0^{(\mathcal{B})} = \emptyset$,

(2) $V_{\alpha+1}^{(\mathcal{B})} = \{u \mid u : \mathcal{D}(u) \to \mathcal{B} \text{ and } \mathcal{D}(u) \subseteq V_\alpha^{(\mathcal{B})}\}$,

(3) $V_\alpha^{(\mathcal{B})} = \bigcup_{\xi < \alpha} V_\xi^{(\mathcal{B})}$, for a limit ordinal α.

Then we define $V^{(\mathcal{B})} = \bigcup_{\alpha \in On} V_\alpha^{(\mathcal{B})}$, where On is the class of all ordinal numbers. For $u, v \in V^{(\mathcal{B})}$, the truth values $[\![u \in v]\!]$ and $[\![u = v]\!]$ are defined as functions from $V^{(\mathcal{B})} \times V^{(\mathcal{B})}$ to $\mathcal{B}$ satisfying the following properties:

(1) $[\![u \in v]\!] = \sup_{y \in \mathcal{D}(v)} (v(y) \wedge [\![u = y]\!])$,

(2) $[\![u = v]\!] = \inf_{x \in \mathcal{D}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \inf_{y \in \mathcal{D}(v)} (v(y) \Rightarrow [\![y \in u]\!])$.

We also use $\wedge$, $\vee$, $\neg$ and $\Rightarrow$ as logical connectives. Let ϕ be a formula in set theory with predicate symbols $\in$ and $=$. If ϕ contains no free variables and all the constants in ϕ are members in $V^{(\mathcal{B})}$, we define the truth value $[\![\phi]\!]$ of ϕ by the following recursive rules.

(1) $[\![\neg \phi]\!] = \neg [\![\phi]\!]$,

(2) $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \wedge [\![\phi_2]\!]$,

(3) $[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \vee [\![\phi_2]\!]$,

(4) $[\![\forall x \phi(x)]\!] = \inf_{u \in V^{(\mathcal{B})}} [\![\phi(u)]\!]$,

(5) $[\![\exists x \phi(x)]\!] = \sup_{u \in V^{(\mathcal{B})}} [\![\phi(u)]\!]$.

The basic theorem of Scott-Solovay's Boolean valued model theory is the following [**11**].

THEOREM 2.1. *If ϕ is a theorem of ZFC, then $[\![\phi]\!] = 1$ is also a theorem of ZFC.*

The original universe V of ZFC can be embedded in $V^{(\mathcal{B})}$ by the following operation $\check{}$ defined by the $\in$-recursion: For $y \in V$, $\mathcal{D}(\check{y}) = \{\check{x} \mid x \in y\}$ and $\check{y}$ is a constant function whose value is 1 in $\mathscr{B}$. A family $\{b_\alpha\}$ of elements of $\mathscr{B}$ is called a *partition of unity* if $\sup_\alpha b_\alpha = 1$ and $b_\alpha \wedge b_\beta = 0$ for any $\alpha \neq \beta$. Let $\{b_\alpha\}$ be a partition of unity and let $\{u_\alpha\}$ be a family of $\mathscr{B}$-valued sets in $V^{(\mathscr{B})}$. Then there is an element $u \in V^{(\mathscr{B})}$ such that $[\![u = u_\alpha]\!] \geqq b_\alpha$ for any α. Moreover if there is another u' such that $[\![u' = u_\alpha]\!] \geqq b_\alpha$ for any α then $[\![u = u']\!] = 1$. We denote this u by $\sum_\alpha u_\alpha b_\alpha$ or $u_1 b_1 + u_2 b_2 + \cdots + u_n b_n$ if $\alpha = 1, 2, \cdots, n$.

Let $\phi(x)$ be a formula with only x as a free variable and such that there is $v_0 \in V^{(\mathscr{B})}$ with $[\![\phi(v_0)]\!] = 1$. Let $X = \{x \mid \phi(x)\}$. We define the interpretation $X^{(\mathscr{B})}$ of X with respect to $V^{(\mathscr{B})}$ as $X^{(\mathscr{B})} = \{u \in V^{(\mathscr{B})} \mid [\![\phi(u)]\!] = 1\}$. Then it is known [**6**; p. 14] that

$$[\![\forall x \in X \psi(x)]\!] = \inf_{u \in X^{(\mathscr{B})}} [\![\psi(u)]\!] ,$$

$$[\![\exists x \in X \psi(x)]\!] = \sup_{u \in X^{(\mathscr{B})}} [\![\psi(u)]\!] .$$

If X is a set, by choosing a representative from an equivalence class $\{v \in V^{(\mathscr{B})} \mid [\![u = v]\!] = 1\}$, we can consider $X^{(\mathscr{B})}$ as a set [**6**; p. 14, Remark]. Then we have $X^{(\mathscr{B})} \times \{1\} \in V^{(\mathscr{B})}$ and that $[\![X = X^{(\mathscr{B})} \times \{1\}]\!] = 1$.

Our special interest of this paper is the following situation. A lot of established notions in functional analysis are concerned with bounded objects such as Banach spaces of bounded functions or integrable functions and Banach algebras of bounded operators. Nevertheless $X^{(\mathscr{B})}$ is in general larger than such spaces. Thus it is important to find a well known bounded object Y which is considered as a subset of $X^{(\mathscr{B})}$ such that $[\![Y \times \{1\} = X]\!] = 1$. In this case we have again

$$[\![\forall x \in X \psi(x)]\!] = \inf_{u \in Y} [\![\psi(u)]\!] ,$$

$$[\![\exists x \in X \psi(x)]\!] = \sup_{u \in Y} [\![\psi(u)]\!] ,$$

by [**11**; Theorem 13.13, p. 125].

Let $d \subseteqq V^{(\mathscr{B})}$. A function $g: d \to V^{(\mathscr{B})}$ is called *extensional* if for any $x, x' \in d$, $[\![x = x']\!] \leqq [\![g(x) = g(x')]\!]$. A $\mathscr{B}$-valued set $u \in V^{(\mathscr{B})}$ is called *definite* if for any $x \in \mathscr{D}(u)$, $u(x) = 1$ and called *separated* if $[\![x = y]\!] = 1$ implies that $x = y$ for any $x, y \in \mathscr{D}(u)$. Then the following theorem is known [**11**].

THEOREM 2.2. *Let $u, v \in V^{(\mathscr{B})}$ be definite. Then there is a bijective correspondence between $\mathscr{B}$-valued sets f such that $[\![f: u \to v]\!] = 1$ and extensional maps $g: \mathscr{D}(u) \to v^{(\mathscr{B})}$ where $v^{(\mathscr{B})} = \{u \mid [\![u \in v]\!] = 1\}$. The correspondence is given by the relation $[\![f(x) = g(x)]\!] = 1$ for any $x \in \mathscr{D}(u)$.*

Now we have a characterization of extensional maps.

THEOREM 2.3. *Let $u, v \in V^{(\mathscr{B})}$ be definite and separated, and let $g: \mathscr{D}(u) \to \mathscr{D}(v)$. Then g is extensional if and only if $g(xb + y(\neg b)) = g(x)b + g(y)(\neg b)$ for any $x, y \in \mathscr{D}(u)$, $b \in \mathscr{B}$.*

PROOF. Let $x, y \in \mathscr{D}(u)$ and $b \in \mathscr{B}$. Suppose that g is extensional. Then by

Theorem 2.2, there is some $f \in V^{(\mathscr{B})}$ such that $[\![f: u \to v]\!]=1$ and that $[\![f(x)=g(x)]\!]=1$ for any $x \in \mathscr{D}(u)$. Then we have $b \leqq [\![xb+y(\neg b)=x]\!] \leqq [\![f(xb+y(\neg b))=f(x)]\!]$. Similarly, we have $\neg b \leqq [\![f(xb+y(\neg b))=f(y)]\!]$. It follows that $[\![f(xb+y(\neg b))=f(x)b+f(y)(\neg b)]\!]=1$ so that $g(xb+y(\neg b))=g(x)b+g(y)(\neg b)$. Conversely, suppose that $g(xb+y(\neg b))=g(x)b+g(y)(\neg b)$ and that $b=[\![x=y]\!]$. Then we have $b \leqq [\![x=xb+y(\neg b)]\!]$ and that $b \leqq [\![x=y]\!] \wedge [\![x=xb+y(\neg b)]\!] \leqq [\![y=xb+y(\neg b)]\!]$. Thus we have $[\![y=xb+y(\neg b)]\!]=1$ so that $g(y)=g(x)b+g(y)(\neg b)$. Therefore we have $b \leqq [\![g(y)=g(x)]\!]$. QED

For further results and terminology of Scott-Solovay's Boolean valued model theory, we shall refer the reader to Takeuti and Zaring [**11**].

Now we can construct the set $\boldsymbol{Q}$ of rational numbers in the model $V^{(\mathscr{B})}$ as in [**6**; p. 11], and we have $[\![\boldsymbol{Q}=\check{\boldsymbol{Q}}]\!]=1$. We define a real number to be the lower half line without the end point of a Dedekind cut. Therefore the formal definition of 'a is a real number' is expressed by

$$a \subseteqq \boldsymbol{Q} \wedge \exists s \in \boldsymbol{Q}(s \in a) \wedge \exists s \in \boldsymbol{Q}(s \notin a)$$
$$\wedge \forall s \in \boldsymbol{Q}(s \in a \Leftrightarrow \exists t \in \boldsymbol{Q}(s<t \wedge t \in a)) .$$

We define $\boldsymbol{R}^{(\mathscr{B})}$ to be the interpretation of the set $\boldsymbol{R}$ of real numbers in $V^{(\mathscr{B})}$ and $\boldsymbol{C}^{(\mathscr{B})}$ to be the interpretation of the set $\boldsymbol{C}$ of complex numbers in $V^{(\mathscr{B})}$, i.e.

$$\boldsymbol{R}^{(\mathscr{B})}=\{u \in V^{(\mathscr{B})} \mid [\![u \text{ is a real number}]\!]=1\},$$
$$\boldsymbol{C}^{(\mathscr{B})}=\{u \in V^{(\mathscr{B})} \mid [\![u \text{ is a complex number}]\!]=1\}.$$

For further information about Boolean valued analysis in the case $\mathscr{B}$ is a projection algebra or $\mathscr{B}$ is a measure algebra, we shall refer the reader to Takeuti [**6**].

By a *(left) normal $\mathscr{M}$-module H* we will mean the Hilbert space H of a non-degenerate normal $*$-representation of $\mathscr{M}$ with the corresponding action of $\mathscr{M}$. Usually we write the action of a normal $\mathscr{M}$-module H as $\pi(a)\xi$ for $a \in \mathscr{M}$, $\xi \in H$. If H and K are normal $\mathscr{M}$-modules then we will denote by $\mathrm{Hom}(H, K)$ the Banach space of bounded $\mathscr{M}$-module maps from H to K, in other words the space of all intertwining operators between the representations of $\mathscr{M}$ on H and K. We say that $U \in \mathrm{Hom}(H, K)$ is a *unitary $\mathscr{M}$-module map* if U is a unitary transformation from H onto K. Two normal $\mathscr{M}$-modules H and K are called *unitarily equivalent* if there is a unitary $\mathscr{M}$-module map $U \in \mathrm{Hom}(H, K)$. The classification of all normal $\mathscr{M}$-modules up to unitary equivalence is carried out by the multiplicity theory. In the present context, the *multiplicity function* of a normal $\mathscr{M}$-module is the function m from the class of all cardinals to $\mathscr{B}$ such that if $\alpha \neq \beta$ then $m(\alpha)m(\beta)=0$ and that $\sup_\alpha m(\alpha)=1$. Then to any normal $\mathscr{M}$-module H corresponds a unique multiplicity function m such that $\pi(m(\alpha))H$ has a uniform multiplicity α if $\alpha\neq 0$ and that H is the direct sum of all $\pi(m(\alpha))H$'s. In this case, there is a family $\{\nu_\alpha\}$ of localizable measures on the spectrum Ω of $\mathcal{M}$ with disjoint supports Ω_α such that $[\Omega_\alpha]=m(\alpha)$ and that

$$H\cong\sum_\alpha^\oplus L^2(\Omega_\alpha, \nu_\alpha)\otimes l^2(\alpha)$$

as unitary equivalence of normal $\mathcal{M}$-modules. Thus for any multiplicity function m there is a normal $\mathcal{M}$-module H whose multiplicity function is m and two normal $\mathcal{M}$-modules are unitarily equivalent if and only if they have the same multiplicity function. For the more detail, see [**1**], in particular [**1**; Theorem 64.5, p. 102].

3. Real and complex numbers in the model.

In order to obtain convenient representations of real numbers and complex numbers in $V^{(\mathcal{B})}$, consider the spectrum Ω of $\mathcal{M}$. By the Gelfand representation theorem $\mathcal{M}$ is isomorphic to the commutative C^*-algebra $C(\Omega)$ of all continuous functions on Ω. Since $\mathcal{M}$ is a W^*-algebra, Ω is a hyperstonean space, i. e. Ω is a compact Hausdorff space such that the closure of every open set is open and that it admits sufficiently many positive normal measures [**5**; Theorem 1.18, p. 109]. Then $\mathcal{B}$ is isomorphic to the Boolean algebra of clopen subsets of Ω by the obvious correspondence between projections in $C(\Omega)$ and clopen subsets of Ω. Thus Ω is identical with the Stone representation space of the Boolean algebra $\mathcal{B}$ on account of the uniqueness of the Stone representation space among totally disconnected compact Hausdorff spaces. In the hyperstonean space Ω, every regular open set is clopen and every meager set is nowhere dense [**5**; p. 108]. Let $\mathcal{S}$ be the Borel σ-field of Ω and let $\mathcal{J}$ be the σ-ideal of meager (or nowhere dense) Borel subsets of Ω. Then the countable suprema and countable infima in the quotient Boolean algebra $\mathcal{S}/\mathcal{J}$ are inherited from those in $\mathcal{S}$ and by Loomis' theorem [**2**; Theorem 13, p. 102] $\mathcal{B}$ is σ-isomorphic to $\mathcal{S}/\mathcal{J}$. In the sequel, we denote by $[S]$ for any S in $\mathcal{S}$ the corresponding element in $\mathcal{B}$ by the σ-isomorphism $\mathcal{B}\cong\mathcal{S}/\mathcal{J}$. By the similar argument as in [**6**; pp. 53-54], we can show that there is a correspondence between $\boldsymbol{R}^{(\mathcal{B})}$ and the set $B_{\boldsymbol{R}}(\Omega)$ of all real Borel functions on Ω such that $u\in V^{(\mathcal{B})}$ corresponds to $f\in B_{\boldsymbol{R}}(\Omega)$ if and only if $[\![\check{r}\in u]\!]=[\{\omega\in\Omega \mid r<f(\omega)\}]$ for any $r\in\boldsymbol{Q}$. This correspondence between $\boldsymbol{R}^{(\mathcal{B})}$ and $B_{\boldsymbol{R}}(\Omega)$ is one-to-one in the following sense: If f and g in $B_{\boldsymbol{R}}(\Omega)$ correspond to the same $u\in\boldsymbol{R}^{(\mathcal{B})}$ then f and g are equal except on a nowhere dense set, i. e. $\{\omega\in\Omega \mid f(\omega)\neq g(\omega)\}$ is nowhere dense. If u and v in $\boldsymbol{R}^{(\mathcal{B})}$ correspond to the same $f\in B_{\boldsymbol{R}}(\Omega)$, then $[\![u=v]\!]=1$. Further properties of this correspondence can be shown analogously as in [**6**; Chapter 2, § 2]. Now it is straightforward that the above correspondence can be extended to the analogous correspondence be-

tween $\boldsymbol{C}^{(\mathscr{B})}$ and the set $B(\Omega)$ of all (complex) Borel functions on Ω by the relation $\boldsymbol{C}^{(\mathscr{B})}=\boldsymbol{R}^{(\mathscr{B})}+i\boldsymbol{R}^{(\mathscr{B})}$.

Let $\{\mu_\alpha\}$ be a maximal family of positive normal measures on Ω with disjoint supports Γ_α. Put $\Gamma=\bigcup_\alpha \Gamma_\alpha$ and $\mu=\sum_\alpha \mu_\alpha$. Then Γ is a dense open subset of Ω and μ is a positive Radon measure on Γ such that $C(\Omega)$, as well as $\mathscr{M}$, is isomorphic to the algebra $L^\infty(\Gamma, \mu)$ of all essentially bounded μ-measurable functions on Γ by the unique continuous extension to Ω of bounded continuous representatives in $L^\infty(\Gamma, \mu)$ [**5**; Theorem 1.18, p. 109]. Since a subset S of Ω is nowhere dense if and only if $\mu_\alpha(S)=0$ for any μ_α [**5**; Proposition 1.15, p. 108] and since $\Omega-\Gamma$ is nowhere dense, the correspondence between $\boldsymbol{C}^{(\mathscr{B})}$ and $B(\Omega)$ can be modified as the correspondence between $\boldsymbol{C}^{(\mathscr{B})}$ and the set $L(\Gamma, \mu)$ of μ-measurable functions on Γ such that if f and g in $L(\Gamma, \mu)$ correspond to u and v in $\boldsymbol{C}^{(\mathscr{B})}$ then $\{\omega\in\Gamma \mid f(\omega)\neq g(\omega)\}$ is μ-null if and only if $[\![u=v]\!]=1$. In the sequel, we shall identify $\boldsymbol{C}^{(\mathscr{B})}$ and $L(\Gamma, \mu)$ by the above correspondence, and hence we shall regard $L^\infty(\Gamma, \mu)$ and $L^p(\Gamma, \mu)$ as subsets of $\boldsymbol{C}^{(\mathscr{B})}$ where $L^p(\Gamma, \mu)$ is the space of the p-th power μ-integrable functions on Γ for $1\leqq p<\infty$.

Now we have the following.

THEOREM 3.1. *We have* $[\![\boldsymbol{C}=L^p(\Gamma, \mu)\times\{1\}]\!]=1$ *for* $1\leqq p\leqq\infty$.

PROOF. Since the family $\{\mu_\alpha\}$ of finite measures has disjoint supports $\{\Gamma_\alpha\}$, we have a partition $\{[\Gamma_\alpha]\}$ of unity of $\mathscr{B}$. Then by [**6**; Proposition 2.7] we can assume that μ is a finite measure. Then we have $L^\infty(\Gamma, \mu)\subseteqq L^p(\Gamma, \mu)$ for $1\leqq p<\infty$ so that we have only to show the case $p=\infty$. Let f be an element in $\boldsymbol{C}^{(\mathscr{B})}$. For $n=0, 1, 2, \cdots$, put $b_n=[\{\omega\in\Gamma \mid n\leqq|f(\omega)|<n+1\}]$. Then $\{b_n\}$ is a partition of unity in $\mathscr{B}$. For any n, define a function g_n in $L^\infty(\Gamma, \mu)$ by $g_n(\omega)=f(\omega)$ if $n\leqq|f(\omega)|<n+1$ and by $g_n(\omega)=0$ otherwise. Then we have

$$[\![f\in L^\infty(\Gamma, \mu)\times\{1\}]\!]=\sup_{g\in L^\infty(\Gamma,\mu)}[\![f=g]\!]$$
$$\geqq[\![f=g_n]\!]=[\{\omega\in\Gamma \mid f(\omega)=g_n(\omega)\}]\geqq b_n\,,$$

for every n. Thus we have $[\![f\in L^\infty(\Gamma, \mu)\times\{1\}]\!]=1$ and consequently

$$[\![\boldsymbol{C}\subseteqq L^\infty(\Gamma, \mu)\times\{1\}]\!]=\inf_{f\in C^{(\mathscr{B})}}[\![f\in L^\infty(\Gamma, \mu)\times\{1\}]\!]=1\,.$$

Since $[\![L^\infty(\Gamma, \mu)\times\{1\}\subseteqq\boldsymbol{C}]\!]=1$ is obvious, we complete the proof. QED

The above theorem has the following technical usefulness. Let $\phi(x)$ be a formula with only x as a free variable. It is known [**11**; Theorem 13.13, p. 125] that for any $u\in V^{(\mathscr{B})}$

$$[\![\forall x\in u\phi(x)]\!]=\inf_{x\in\mathscr{D}(u)}(u(x)\Rightarrow[\![\phi(x)]\!])\,.$$

Then by the above theorem, we have

$$[\![\forall x\in\boldsymbol{C}\phi(x)]\!]=[\![(\forall x\in L^p(\Gamma, \mu)\times\{1\})\phi(x)]\!]=\inf_{x\in L^p(\Gamma,\mu)}[\![\phi(x)]\!]\,,$$

since if $u=L^p(\Gamma, \mu)\times\{1\}$ then $\mathscr{D}(u)=L^p(\Gamma, \mu)$ and $u(x)=1$ for all $x\in\mathscr{D}(u)$. Similarly, we have

$$[\![\exists x\in \boldsymbol{C}\phi(x)]\!]=\sup_{x\in L^p(\Gamma,\mu)}[\![\phi(x)]\!] .$$

Furthermore, by the maximum principle [**11**; Theorem 16.2, p. 148], if $\sup_{x\in L^p(\Gamma,\mu)}[\![\phi(x)]\!]=1$ then there is some $x\in L(\Gamma, \mu)$ such that $[\![\phi(x)]\!]=1$. This fact will be seen as a useful technique in analysis, if we can clarify the meaning of $[\![\phi(x)]\!]=1$.

In the following sections, we identify two functions on Γ which are equal almost everywhere and also identify two elements u and v in $\boldsymbol{C}^{(\mathscr{B})}$ such that $[\![u=v]\!]=1$. Formally, this can be justified by regarding $L(\Gamma, \mu)$ as the quotient space of all μ-measurable functions by the set of all μ-null functions and by extracting one element from $\{u\mid [\![u=v]\!]=1\}$ by a method mentioned in [**6**; p. 14]. Thus we can say that $f=g$ if and only if $u=v$ when f and g in $L(\Gamma, \mu)$ correspond to u and v in $\boldsymbol{C}^{(\mathscr{B})}$ respectively, and so we shall again identify $L(\Gamma, \mu)$ and $\boldsymbol{C}^{(\mathscr{B})}$. We shall also identify $\mathscr{M}$ and $L^\infty(\Gamma, \mu)$ by the correspondence mentioned above.

4. The construction of Hilbert spaces in the model.

Let $\langle H, \pi\rangle$ be a normal $\mathscr{M}$-module. Then $\pi(\mathscr{M})$ is a von Neumann algebra on H. For any ξ, η in H, the function $B\to(\pi([B])\xi\mid\eta)$ on $\mathcal{S}$ is a finite signed measure absolutely continuous with respect to μ, so that by the Radon-Nikodym theorem there is a μ-integrable function $F(\xi, \eta)$ on Γ uniquely up to almost everywhere such that

$$(\pi(a)\xi\mid\eta)=\int_\Gamma a(\omega)F(\xi, \eta)(\omega)\mu(d\omega) ,$$

for every a in $L^\infty(\Gamma, \mu)$. Then it is easy to see that for any a, $b\in L^\infty(\Gamma, \mu)$ and ξ, η, $\zeta\in H$,

(F1) $F(\pi(a)\xi+\pi(b)\eta, \zeta)(\omega)=a(\omega)F(\xi, \zeta)(\omega)+b(\omega)F(\eta, \zeta)(\omega)$ a. e.,

(F2) $\overline{F(\xi, \eta)(\omega)}=F(\eta, \xi)(\omega)$ a. e.,

(F3) $F(\xi, \xi)(\omega)\geqq 0$ a. e.,

(F4) $\xi=0$ if and only if $F(\xi, \xi)(\omega)=0$ a. e.

Roughly speaking, this shows that F has the properties similar to the inner product if we consider $\boldsymbol{C}^{(\mathscr{B})}$ as the scalars. Now we proceed to construct a Hilbert space in $V^{(\mathscr{B})}$ by embedding H and F into $V^{(\mathscr{B})}$.

For this purpose we use the following lemmas analogous with the Schwarz inequality and with the Riesz representation theorem.

LEMMA 4.1. *For any ξ, $\eta\in H$, we have*

(1) $|F(\xi, \eta)(\omega)| \leqq F(\xi, \xi)(\omega)^{1/2} F(\eta, \eta)(\omega)^{1/2}$ *a.e.*,

(2) $F(\xi-\eta, \xi-\eta)(\omega)^{1/2} \leqq F(\xi, \xi)(\omega)^{1/2} + F(\eta, \eta)(\omega)^{1/2}$ *a.e.*

PROOF. Let t be a real number. Define a bounded μ-measurable function a on Γ by the relation $a(\omega)=t|F(\xi, \eta)(\omega)|/F(\xi, \eta)(\omega)$, if $F(\xi, \eta)(\omega)\neq 0$ and $a(\omega)=t$, if $F(\xi, \eta)(\omega)=0$. Put $B=\{\omega\in\Gamma | F(\xi, \eta)(\omega)\neq 0\}$. Then since $F(\pi(a)\xi+\eta, \pi(a)\xi+\eta)(\omega)\geqq 0$ a. e., the routine computations leads that

$$F(\xi, \xi)(\omega)t^2+2|F(\xi, \eta)(\omega)|t+F(\eta, \eta)(\omega)\geqq 0$$

almost everywhere on B. Thus the relation (1) follows immediately. Now the relation (2) follows from the routine computations using the relation (1). QED

LEMMA 4.2. *Let $G: H\to L(\Gamma, \mu)$ be a function with the following properties:*

(1) *For any $a\in L^\infty(\Gamma, \mu)$ and $\xi\in H$, $G(\pi(a)\xi)(\omega)=a(\omega)G(\xi)(\omega)$ a.e.*

(2) *For any $\xi, \eta\in H$, $G(\xi+\eta)(\omega)=G(\xi)(\omega)+G(\eta)(\omega)$ a.e.*

(3) *There is some $g\in L^2(\Gamma, \mu)$ such that for any $\xi\in H$.*

$$|G(\xi)(\omega)|\leqq g(\omega)F(\xi, \xi)(\omega)^{1/2} \quad a.e.$$

Then there is some $\zeta\in H$ such that for any $\xi\in H$,

$$G(\xi)(\omega)=F(\xi, \zeta)(\omega) \quad a.e.$$

PROOF. Since $F(\xi, \xi)\in L^1(\Gamma, \mu)$, by the condition (3) we have $G(\xi)\in L^1(\Gamma, \mu)$. Define a function $\phi: H\to \boldsymbol{C}$ by the relation

$$\phi(\xi)=\int_\Gamma G(\xi)(\omega)\mu(d\omega),$$

for any $\xi\in H$. Then ϕ is a linear functional on H. Let $g\in L^2(\Gamma, \mu)$ be as in (3). Then we have

$$\begin{aligned}|\phi(\xi)| &\leqq \int_\Gamma |G(\xi)(\omega)|\mu(d\omega) \leqq \int_\Gamma g(\omega)F(\xi, \xi)(\omega)^{1/2}\mu(d\omega) \\ &\leqq \left(\int_\Gamma g(\omega)^2\mu(d\omega)\right)^{1/2}\left(\int_\Gamma F(\xi, \xi)(\omega)\mu(d\omega)\right)^{1/2} \\ &= \left(\int_\Gamma g(\omega)^2\mu(d\omega)\right)^{1/2}\|\xi\|.\end{aligned}$$

Thus ϕ is bounded and hence by the Riesz representation theorem there is some $\zeta\in H$ such that for any $\xi\in H$, $\phi(\xi)=(\xi|\zeta)$. Then we have for any $a\in L^\infty(\Gamma, \mu)$,

$$\begin{aligned}\int_\Gamma a(\omega)G(\xi)(\omega)\mu(d\omega) &= \int_\Gamma G(\pi(a)\xi)(\omega)\mu(d\omega) \\ &= \phi(\pi(a)\xi)=(\pi(a)\xi|\zeta)=\int_\Gamma a(\omega)F(\xi, \zeta)(\omega)\mu(d\omega).\end{aligned}$$

Therefore we have $G(\xi)(\omega)=F(\xi, \zeta)(\omega)$ a. e. for any $\xi\in H$. QED

THEOREM 4.3. *For any $\xi\in H$, define a $\mathcal{B}$-valued set $\tilde{\xi}\in V^{(\mathcal{B})}$ by the relation*

$$\mathcal{D}(\tilde{\xi})=\{\check{\xi}\mid\xi\in H\} \quad and \quad \tilde{\xi}(\check{\eta})=[\![F(\xi-\eta, \xi-\eta)=0]\!],$$

for any $\eta\in H$. Let $\tilde{H}\in V^{(\mathcal{B})}$ be a $\mathcal{B}$-valued set such that

$$\tilde{H}=\{\tilde{\xi}\mid\xi\in H\}\times\{1\}.$$

Then $\tilde{H}$ is a Hilbert space in $V^{(\mathcal{B})}$ with addition $\tilde{\xi}+\tilde{\eta}=(\xi+\eta)^{\sim}$, scalar multiplication $a\tilde{\xi}=(\pi(a)\xi)^{\sim}$ and inner product $(\tilde{\xi}\mid\tilde{\eta})=F(\xi, \eta)^{\sim}$. Precisely, there are $\mathcal{B}$-valued sets A, S, and $I\in V^{(\mathcal{B})}$ such that

$$\begin{aligned}&[\![A: \tilde{H}\times\tilde{H}\to\tilde{H}\wedge S: \boldsymbol{C}\times\tilde{H}\to\tilde{H}\wedge I: \tilde{H}\times\tilde{H}\to\boldsymbol{C}\\ &\wedge\langle\tilde{H}, A, S, I\rangle \text{ is a Hilbert space with addition } A,\\ &\text{scalar multiplication } S \text{ and inner product } I]\!]=1,\end{aligned}$$

and that for any ξ, $\eta\in H$ and $a\in\mathcal{M}$,

$$[\![A(\tilde{\xi}, \tilde{\eta})=(\xi+\eta)^{\sim}\wedge S(a, \tilde{\xi})=(\pi(a)\xi)^{\sim}\wedge I(\tilde{\xi}, \tilde{\eta})=F(\xi, \eta)]\!]=1.$$

The correspondence $\xi\to\tilde{\xi}$ is bijective in the sense that $\xi=\eta$ if and only if $[\![\tilde{\xi}=\tilde{\eta}]\!]=1$ for any ξ, $\eta\in H$.

PROOF. Since the addition on H is an internal operation, it is easy to see that $[\![\langle\check{H}, \check{+}\rangle$ is an abelian group$]\!]=1$ and that $[\![\check{\xi}\check{+}\check{\eta}=(\xi+\eta)^{\vee}]\!]=1$ for any ξ, $\eta\in H$. In the sequel, we shall also write $\check{\xi}+\check{\eta}$ instead of $\check{\xi}\check{+}\check{\eta}$ and identify $\check{H}\times\check{H}$ with $(H\times H)^{\vee}$. Since the function $(\check{\xi}, \check{\eta})\to F(\xi, \eta)$ on $\mathcal{D}(\check{H})\times\mathcal{D}(\check{H})$ into $\boldsymbol{C}^{(\mathcal{B})}$ is obviously extensional, it follows from Theorem 2.2 that there is some $I_0\in V^{(\mathcal{B})}$ such that $[\![I_0: \check{H}\times\check{H}\to\boldsymbol{C}]\!]=1$ and that $[\![I_0(\check{\xi}, \check{\eta})=F(\xi, \eta)]\!]=1$ for any ξ, $\eta\in H$. Now, it is obvious that I_0 inherits the properties (F1)-(F3) of F. Let $N\in V^{(\mathcal{B})}$ be such that $N=\{\check{\xi}\in\check{H}\mid I_0(\check{\xi}, \check{\xi})=0\}\times\{1\}$. By the relation (2) of Lemma 4.1, for any ξ, $\eta\in H$,

$$\begin{aligned}&[\{\omega\in\Gamma\mid F(\xi, \xi)(\omega)=0\}]\wedge[\{\omega\in\Gamma\mid F(\eta, \eta)=0\}]\\ &\leqq[\{\omega\in\Gamma\mid F(\xi-\eta, \xi-\eta)(\omega)=0\}].\end{aligned}$$

Thus we have

$$[\![\forall u, \forall v\in\check{H}(I_0(u, u)=0\wedge I_0(v, v)=0\Rightarrow I_0(u-v, u-v)=0)]\!]=1,$$

so that $[\![N$ is a subgroup of $\check{H}]\!]=1$. Therefore we have

$$\begin{aligned}&[\![(\check{H}/N \text{ is an abelian group})\\ &\wedge\forall u, \forall v\in\check{H}(u+N=v+N\Leftrightarrow I_0(u-v, u-v)=0)]\!]=1.\end{aligned}$$

Next we shall define a scalar multiplication on $\check{H}/N$. For any a, $b\in\mathcal{M}$ and ξ,

$\eta\in H$ we have that

$$\begin{aligned}
&[\![a=b]\!][\![\check{\xi}+N=\check{\eta}+N]\!]\\
&\quad=[\![a=b]\!][\![I_0(\check{\xi}-\check{\eta},\ \check{\xi}-\check{\eta})=0]\!]\\
&\quad=[\![a=b]\!][\![F(\xi-\eta,\ \xi-\eta)=0]\!]\\
&\quad\leqq[\![a=b]\!][\![F(\xi,\ \xi)-F(\xi,\ \eta)-F(\eta,\ \xi)+F(\eta,\ \eta)=0]\!]\\
&\quad\leqq[\![aa^*F(\xi,\ \xi)-ab^*F(\xi,\ \eta)-ba^*F(\eta,\ \xi)+bb^*F(\eta,\ \eta)=0]\!]\\
&\quad=[\![F(\pi(a)\xi,\ \pi(a)\xi)-F(\pi(a)\xi,\ \pi(b)\eta)\\
&\qquad-F(\pi(b)\eta,\ \pi(a)\xi)+F(\pi(b)\eta,\ \pi(b)\eta)=0]\!]\\
&\quad=[\![F(\pi(a)\xi-\pi(b)\eta,\ \pi(a)\xi-\pi(b)\eta)=0]\!]\\
&\quad=[\![(\pi(a)\xi)^{\vee}+N=(\pi(b)\eta)^{\vee}+N]\!]\,.
\end{aligned}$$

Since $[\![\boldsymbol{C}=\mathcal{M}\times\{1\}]\!]=1$ by Theorem 3.1, we can conclude that there is some $S\in V^{(\mathcal{B})}$ such that $[\![S:\boldsymbol{C}\times\check{H}/N\to\check{H}/N]\!]=1$ and that $[\![S(a,\ \check{\xi}+N)=(\pi(a)\xi)^{\vee}+N]\!]=1$ for any $a\in\mathcal{M}$ and $\xi\in H$, by the argument similar to the proof of [**11**; Theorem 16.8, p. 151]. Then it is easy to see that S has the properties of scalar multiplication on $\check{H}/N$ in $V^{(\mathcal{B})}$. Denote by A and I the quotient maps $\check{+}/N$ and I_0/N in $V^{(\mathcal{B})}$ induced from $\check{+}$ and I_0. Then we can now easily check that

$[\![\langle\check{H}/N,\ A,\ S,\ I\rangle$ is a pre-Hilbert space with addition A,

scalar multiplication S and inner product $I]\!]=1$.

In the following, we shall also write $u+v$ and au instead of $A(u,\ v)$ and $S(a,\ u)$, respectively. Next we shall show the completeness of $\check{H}/S$. For this purpose, we use a theorem in ZFC such that every pre-Hilbert space which satisfies the Riesz representation theorem is a Hilbert space. Suppose that $[\![f$ is a bounded linear functional in $\check{H}/N]\!]=1$. Then it is easy to see that there is some $g\in V^{(\mathcal{B})}$ such that $[\![g:\check{H}\to\boldsymbol{C}]\!]=1$ and that $[\![g(\check{\xi})=f(\check{\xi}+N)]\!]=1$ for any $\xi\in H$. Then by Theorem 2.2, there is a function $G: H\to\boldsymbol{C}^{(\mathcal{B})}$ such that $[\![G(\xi)=f(\check{\xi}+N)]\!]=1$ for any $\xi\in H$. Then for any $a\in\mathcal{M}$ and $\xi\in H$, we have $[\![G(\pi(a)\xi)=f((\pi(a)\xi)^{\vee}+N)$ $=f(a(\check{\xi}+N))=af(\check{\xi}+N)=aG(\xi)]\!]=1$, whence $G(\pi(a)\xi)(\omega)=a(\omega)G(\xi)(\omega)$ a. e. for any $a\in L^{\infty}(\Gamma,\ \mu)$ and $\xi\in H$. Similarly, we have $G(\xi+\eta)(\omega)=G(\xi)(\omega)+G(\eta)(\omega)$ a. e. for any $\xi,\ \eta\in H$. Since $[\![f$ is bounded$]\!]=1$, we have $[\![\exists h\in\boldsymbol{C},\ \forall u\in\check{H}/N,$ $|f(u)|\leqq hI(u,\ u)^{1/2}]\!]=1$, so that there is some $h\in L(\Gamma,\ \mu)$ such that for any $\xi\in H$, $|G(\xi)(\omega)|\leqq h(\omega)F(\xi,\ \xi)(\omega)^{1/2}$ a. e. By the similar argument as in the proof of Theorem 3.1, we can show that there is a partition $\{B_\alpha\}$ of Γ such that $\chi_{B_\alpha}h$ is in $L^2(\Gamma,\ \mu)$, for any α. Let $G_\alpha: H\to L(\Gamma,\ \mu)$ be such that

$$G_\alpha(\xi)(\omega)=\chi_{B_\alpha}(\omega)h(\omega)G(\xi)(\omega)\,.$$

Then it is easy to see that G_α satisfies the assumptions of Lemma 4.2. Thus there is a vector $\zeta_\alpha \in H$ such that

$$G_\alpha(\xi)(\omega)=F(\xi,\ \zeta_\alpha)(\omega) \quad \text{a. e.}$$

and hence we have that

$$\begin{aligned}[B_\alpha] &\leqq [\{\omega\in\Gamma \mid G(\xi)(\omega)=F(\xi,\ \zeta_\alpha)(\omega)\}] \\ &=[\![f(\check{\xi}+N)=I(\check{\xi}+N,\ \check{\zeta}_\alpha+N)]\!]\,,\end{aligned}$$

for any α and $\xi\in H$. It follows that

$$\begin{aligned}&[\![\exists y\in\check{H},\ \forall x\in\check{H}\ f(x+N)=I(x+N,\ y+N)]\!] \\ &\quad\geqq \sup_\alpha[\![\forall x\in\check{H}\ f(x+N)=I(x+N,\ \check{\zeta}_\alpha+N)]\!] \\ &\quad=\sup_\alpha\inf_{\xi\in H}[\![f(\check{\xi}+N)=I(\check{\xi}+N,\ \check{\zeta}_\alpha+N)]\!] \\ &\quad\geqq \sup_\alpha[B_\alpha]=1\,.\end{aligned}$$

Therefore we have that

$$[\![\exists v\in\check{H}/N,\ \forall u\in\check{H}/N\ f(u)=I(u,\ v)]\!]=1\,.$$

This concludes that $[\![\check{H}/N$ is a Hilbert space$]\!]=1$. Since it is easy to check that $\xi=\eta$ if and only if $[\![\check{\xi}+N=\check{\eta}+N]\!]=1$, in order to complete the proof we have only to show that $[\![\tilde{\xi}=\check{\xi}+N]\!]=1$ for any $\xi\in H$ and that $[\![\tilde{H}=\check{H}/N]\!]=1$. Since $\mathscr{D}(\tilde{\xi})=\mathscr{D}(\check{H})$, any $\tilde{\xi}$ is extensional. For any $\xi\in H$, we have that

$$\begin{aligned}&[\![\tilde{\xi}\subseteqq\check{\xi}+N]\!]=[\![\forall u\in\tilde{\xi}(u\in\check{\xi}+N)]\!] \\ &\quad=\inf_{u\in\mathscr{D}(\tilde{\xi})}\tilde{\xi}(u)\Rightarrow[\![u\in\check{\xi}+N]\!] \\ &\quad=\inf_{\eta\in H}\tilde{\xi}(\check{\eta})\Rightarrow[\![\check{\eta}+N=\check{\xi}+N]\!] \\ &\quad=\inf_{\eta\in H}[\![F(\xi-\eta,\ \xi-\eta)=0]\!]\Rightarrow[\![F(\eta-\xi,\ \eta-\xi)=0]\!] \\ &\quad=1\,,\end{aligned}$$

and that

$$\begin{aligned}&[\![\check{\xi}+N\subseteqq\tilde{\xi}]\!]=[\![\forall u\in\check{\xi}+N(u\in\tilde{\xi})]\!] \\ &\quad=[\![\forall u\in\check{H}(I_0(u-\check{\xi},\ u-\check{\xi})=0\Rightarrow u\in\tilde{\xi})]\!] \\ &\quad=\inf_{\eta\in H}[\![F(\eta-\xi,\ \eta-\xi)=0]\!]\Rightarrow[\![\check{\eta}\in\tilde{\xi}]\!] \\ &\quad=\inf_{\eta\in H}[\![F(\eta-\xi,\ \eta-\xi)=0]\!]\Rightarrow\tilde{\xi}(\check{\eta}) \\ &\quad=1\,.\end{aligned}$$

Thus we have $[\![\tilde{\xi}=\check{\xi}+N]\!]=1$ for any $\xi\in H$. From this, we can verify that $[\![\tilde{H}=\check{H}/N]\!]=1$ by routine computations. QED

DEFINITION 4.4. Let H be a normal $\mathcal{M}$-module. The Hilbert space $\tilde{H}$ in $V^{(\mathcal{B})}$ constructed by Theorem 4.3 is called the *Boolean embedding* of H in $V^{(\mathcal{B})}$.

5. Interpretation of Hilbert spaces in $V^{(\mathcal{B})}$.

Let X be a Hilbert space in $V^{(\mathcal{B})}$, i.e. $[\![X$ is a Hilbert space$]\!]=1$. Denote by $(\cdot|\cdot)_{\mathcal{B}}$ the inner product on X in $V^{(\mathcal{B})}$ and by $\|\cdot\|_{\mathcal{B}}$ the norm on X in $V^{(\mathcal{B})}$. Let $X^{(\mathcal{B})}$ be the interpretation of X, i.e. $X^{(\mathcal{B})}=\{\dot{\xi}\in V^{(\mathcal{B})}\,|\,[\![\xi\in X]\!]=1\}$, where $\dot{\xi}$ is some representative from $\{\eta\in V^{(\mathcal{B})}\,|\,[\![\xi=\eta]\!]=1\}$. Let $\xi,\ \eta\in X^{(\mathcal{B})}$. Then $(\xi|\eta)_{\mathcal{B}}\in \boldsymbol{C}^{(\mathcal{B})}$ and $\|\xi\|_{\mathcal{B}}\in\boldsymbol{C}^{(\mathcal{B})}$. By the identification $\boldsymbol{C}^{(\mathcal{B})}\cong L(\Gamma,\mu)$, $(\xi|\eta)_{\mathcal{B}}$ and $\|\xi\|_{\mathcal{B}}$ can be identified with μ-measurable functions $(\xi|\eta)_{\mathcal{B}}(\omega)$ and $\|\xi\|_{\mathcal{B}}(\omega)$ on Γ respectively and we have $\|\xi\|_{\mathcal{B}}(\omega)^2=(\xi|\xi)_{\mathcal{B}}(\omega)$ a.e. Thus $X^{(\mathcal{B})}$ is an $L(\Gamma,\mu)$-module with $L(\Gamma,\mu)$-valued inner product. Now let

$$X_2^{(\mathcal{B})}=\{\xi\in X^{(\mathcal{B})}\,|\,\|\xi\|_{\mathcal{B}}\in L^2(\Gamma,\mu)\}.$$

Then it is shown in Takeuti [**10**] that $X_2^{(\mathcal{B})}$ is a normal $\mathcal{M}$-module by defining the scalar multiplication $\alpha\xi=\check{\alpha}\xi$, the inner product $(\xi|\eta)=\int_{\Gamma}(\xi|\eta)_{\mathcal{B}}(\omega)\mu(d\omega)$ and the action $\pi(a)\xi=a\xi$ for any $\xi,\ \eta\in X_2^{(\mathcal{B})}$, $\alpha\in\boldsymbol{C}$, $a\in\mathcal{M}$.

THEOREM 5.1. *Let X be a Hilbert space in $V^{(\mathcal{B})}$. Then we have* $[\![X_2^{(\mathcal{B})}\times\{1\}=X]\!]=1$.

PROOF. Let $\xi\in X^{(\mathcal{B})}$. Then it is easy to see that there is a partition $\{B_\alpha\}$ of Γ such that $\|[B_\alpha]\xi\|_{\mathcal{B}}\in L^2(\Gamma,\mu)$ for every α. Thus it follows from the piecewise argument as in the proof of Theorem 3.1 that $[\![X^{(\mathcal{B})}\times\{1\}\subseteqq X_2^{(\mathcal{B})}\times\{1\}]\!]=1$. Thus the conclusion follows immediately. QED

Let X and Y be two Hilbert spaces in $V^{(\mathcal{B})}$. Let f be a mapping from X into Y in $V^{(\mathcal{B})}$, i.e. $[\![f: X\to Y]\!]=1$. Then there corresponds a unique extensional map $f^{(\mathcal{B})}: X^{(\mathcal{B})}\to Y^{(\mathcal{B})}$ such that $[\![f^{(\mathcal{B})}(\xi)=f(\xi)]\!]=1$ for every $\xi\in X^{(\mathcal{B})}$. Let

$$\mathcal{L}^{(\mathcal{B})}(X,Y)=\{f^{(\mathcal{B})}\,|\ [\![f \text{ is a bounded linear map from } X \text{ into } Y]\!]=1\}.$$

THEOREM 5.2. *Let T be a map from $X^{(\mathcal{B})}$ into $Y^{(\mathcal{B})}$. Then $T\in\mathcal{L}^{(\mathcal{B})}(X,Y)$ if and only if T satisfies the following conditions* (L1)-(L2).

(L1) *T is an $\mathcal{M}$-module map.*

(L2) *There is some $a\in L(\Gamma,\mu)$ such that for every $\xi\in X_2^{(\mathcal{B})}$*

$$\|T\xi\|_{\mathcal{B}}(\omega)\leqq a(\omega)\|\xi\|_{\mathcal{B}}(\omega)\quad a.e.$$

PROOF. The necessity of the conditions (L1)-(L2) is obvious. We prove sufficiency. Let $\xi,\ \eta\in X^{(\mathcal{B})}$ and let $b\in\mathcal{B}$. Since $\mathcal{B}\subseteqq\mathcal{M}$, we can consider that $b\in\mathcal{M}\subseteqq\boldsymbol{C}^{(\mathcal{B})}$. Then we have $[\![b=1]\!]=b$ so that $b=[\![b=1]\!]\leqq[\![b\xi+(1-b)\eta=\xi]\!]$.

Similarly, we have $\neg b \leqq [\![b\xi+(1-b)\eta=\eta]\!]$. It follows that $[\![b\xi+(1-b)\eta=\xi b+\eta(\neg b)]\!]=1$ with respect to the partition $\{b, \neg b\}$ of unity. Since T is an $\mathcal{M}$-module map, we have $T(b\xi+(1-b)\eta)=bT(\xi)+(1-b)T(\eta)$. Thus by Theorem 2.3, T is extensional, and hence there is some $f\in V^{(\mathcal{B})}$ such that $[\![f: X\to Y]\!]=1$ and that $f^{(\mathcal{B})}=T$. Since $[\![\mathcal{M}\times\{1\}=\boldsymbol{C}]\!]=1$ and since $[\![X_2^{(\mathcal{B})}\times\{1\}=X]\!]=1$, it is easy to see that (L1) implies the linearity of f in $V^{(\mathcal{B})}$ and that (L2) implies the boundedness of f in $V^{(\mathcal{B})}$. QED

Let $f^{(\mathcal{B})}\in\mathcal{L}^{(\mathcal{B})}(X, Y)$ and let $\|f\|_{\mathcal{B}}\in\boldsymbol{R}^{(\mathcal{B})}$ be the bound of f in $V^{(\mathcal{B})}$. Then it is easy to see that

$$\|f\|_{\mathcal{B}}=\inf\{a\in\boldsymbol{R}^{(\mathcal{B})} \mid \|f(\xi)\|_{\mathcal{B}}(\omega)\leqq a(\omega)\|\xi\|_{\mathcal{B}}(\omega) \text{ a. e. for any } \xi\in X^{(\mathcal{B})}\},$$

where the infimum is taken with respect to the ordering on $\boldsymbol{R}^{(\mathcal{B})}$ such that $a\leqq b$ if and only if $a(\omega)\leqq b(\omega)$ a. e. Now define two subsets of $\mathcal{L}^{(\mathcal{B})}(X, Y)$ as follows:

$$\mathcal{L}_\infty^{(\mathcal{B})}(X, Y)=\{f^{(\mathcal{B})}\in\mathcal{L}^{(\mathcal{B})}(X, Y) \mid \|f\|_{\mathcal{B}}\in L^\infty(\Gamma, \mu)\},$$

$$\mathcal{L}_2^{(\mathcal{B})}(X, Y)=\{f^{(\mathcal{B})}\in\mathcal{L}^{(\mathcal{B})}(X, Y) \mid \|f\|_{\mathcal{B}}\in L^2(\Gamma, \mu)\}.$$

Recall that $\mathrm{Hom}(X_2^{(\mathcal{B})}, Y_2^{(\mathcal{B})})$ is the set of bounded $\mathcal{M}$-module maps from a normal $\mathcal{M}$-module $X_2^{(\mathcal{B})}$ into a normal $\mathcal{M}$-module $Y_2^{(\mathcal{B})}$.

THEOREM 5.3. *For any $f^{(\mathcal{B})}\in\mathcal{L}_\infty^{(\mathcal{B})}(X, Y)$, the restriction $f^{(\mathcal{B})}|_{X_2^{(\mathcal{B})}}$ of $f^{(\mathcal{B})}$ on $X_2^{(\mathcal{B})}$ is a bounded $\mathcal{M}$-module map from $X_2^{(\mathcal{B})}$ to $Y_2^{(\mathcal{B})}$. The correspondence $f^{(\mathcal{B})}\to f^{(\mathcal{B})}|_{X_2^{(\mathcal{B})}}$ is a bijection from $\mathcal{L}_\infty^{(\mathcal{B})}(X, Y)$ onto* $\mathrm{Hom}(X_2^{(\mathcal{B})}, Y_2^{(\mathcal{B})})$.

PROOF. Let $f^{(\mathcal{B})}\in\mathcal{L}_\infty^{(\mathcal{B})}(X, Y)$ and let T be $f^{(\mathcal{B})}|_{X_2^{(\mathcal{B})}}$. Then obviously T is an $\mathcal{M}$-module map. Let $\xi\in X_2^{(\mathcal{B})}$. Then we have $\|T\xi\|_{\mathcal{B}}(\omega)=\|f(\xi)\|_{\mathcal{B}}(\omega)\leqq \|f\|_{\mathcal{B}}(\omega)\|\xi\|_{\mathcal{B}}(\omega)$ a. e. Since $\|f\|_{\mathcal{B}}\in L^\infty(\Gamma, \mu)$ and since $\|\xi\|_{\mathcal{B}}\in L^2(\Gamma, \mu)$, we have $\|f(\xi)\|_{\mathcal{B}}\in L^2(\Gamma, \mu)$ so that $T\xi\in Y_2^{(\mathcal{B})}$. Furthermore we have

$$\begin{aligned}\|T\xi\|^2&=\int_\Gamma\|T\xi\|_{\mathcal{B}}(\omega)^2\mu(d\omega)\\&\leqq\int_\Gamma\|f\|_{\mathcal{B}}(\omega)^2\|\xi\|_{\mathcal{B}}(\omega)^2\mu(d\omega)\\&\leqq\|\,\|f\|_{\mathcal{B}}\|_\infty^2\int_\Gamma\|\xi\|_{\mathcal{B}}(\omega)^2\mu(d\omega)\\&=\|\,\|f\|_{\mathcal{B}}\|_\infty^2\|\xi\|^2,\end{aligned}$$

where $\|\,\|f\|_{\mathcal{B}}\|_\infty$ is the essential supremum norm of $\|f\|_{\mathcal{B}}\in L^\infty(\Gamma, \mu)$. Thus T is a bounded $\mathcal{M}$-module map from $X_2^{(\mathcal{B})}$ into $Y_2^{(\mathcal{B})}$. Suppose that $f, g\in\mathcal{L}_\infty^{(\mathcal{B})}(X, Y)$ and that $f^{(\mathcal{B})}(\xi)=g^{(\mathcal{B})}(\xi)$ for every $\xi\in X_2^{(\mathcal{B})}$. Then it follows from $[\![X_2^{(\mathcal{B})}\times\{1\}=X]\!]=1$ that $[\![f=g]\!]=1$, so that $f^{(\mathcal{B})}=g^{(\mathcal{B})}$. Thus the correspondence $f^{(\mathcal{B})}\to f^{(\mathcal{B})}|_{X_2^{(\mathcal{B})}}$ is injective. Now we have only to show that the correspondence is surjective. Let $T\in\mathrm{Hom}(X_2^{(\mathcal{B})}, Y_2^{(\mathcal{B})})$. By the similar argument as in the proof of

Theorem 5.2, T is extensional. Thus by Theorem 2.2 and Theorem 5.1, there is some $f\in V^{(\mathcal{B})}$ such that $[\![f: X\to Y]\!]=1$ and that $[\![f(\xi)=T\xi]\!]=1$ for any $\xi\in X_2^{(\mathcal{B})}$. In order to show that $\|f\|_{\mathcal{B}}\in L^{\infty}(\Gamma, \mu)$, let $\xi\in X_2^{(\mathcal{B})}$ and let B be a Borel set in $\|f\|_{\mathcal{B}}\in L^{\infty}(\Gamma, \mu)$. Since $T\xi\in Y_2^{(\mathcal{B})}$, we have $\|T\xi\|_{\mathcal{B}}\in L^2(\Gamma, \mu)$ and hence

$$\begin{aligned}
&\int_B \|f(\xi)\|_{\mathcal{B}}(\omega)^2\mu(d\omega)\\
&\qquad=\int_\Gamma \|[B]f(\xi)\|_{\mathcal{B}}(\omega)^2\mu(d\omega)\\
&\qquad=\|\pi([B])T\xi\|^2\\
&\qquad=\|T\pi([B])\xi\|^2\\
&\qquad\leqq\|T\|^2\|\pi([B])\xi\|^2\\
&\qquad=\|T\|^2\int_B\|\xi\|_{\mathcal{B}}(\omega)^2\mu(d\omega)\\
&\qquad=\int_B\|T\|^2\|\xi\|_{\mathcal{B}}(\omega)^2\mu(d\omega)\,.
\end{aligned}$$

It follows that $\|f(\xi)\|_{\mathcal{B}}(\omega)\leqq\|T\|\|\xi\|_{\mathcal{B}}(\omega)$ a. e., for any $\xi\in X_2^{(\mathcal{B})}$. Since $[\![X_2^{(\mathcal{B})}\times\{1\}=X]\!]=1$ and since $[\![T\xi=f(\xi)]\!]=1$ for any $\xi\in X_2^{(\mathcal{B})}$, we have $[\![\forall x\in X\|f(x)\|_{\mathcal{B}}\leqq\|T\|^{\vee}\|x\|_{\mathcal{B}}]\!]=1$. Thus $f^{(\mathcal{B})}\in\mathcal{L}_{\infty}^{(\mathcal{B})}(X, Y)$. Therefore, the correspondence $f^{(\mathcal{B})}\to f^{(\mathcal{B})}|_{X_2^{(\mathcal{B})}}$ is surjective. QED

We shall now turn to the Hilbert space $\tilde{H}$ in $V^{(\mathcal{B})}$ which is the Boolean embedding of a normal $\mathcal{M}$-module H. Then $\tilde{H}_2^{(\mathcal{B})}$ is another normal $\mathcal{M}$-module. First we shall show how these two are connected each other.

THEOREM 5.4. *Let H be a normal $\mathcal{M}$-module and let $\tilde{H}$ be the Boolean embedding of H in $V^{(\mathcal{B})}$. Then the relation $U_H\xi=\dot{\tilde{\xi}}$ for any $\xi\in H$ defines a unitary $\mathcal{M}$-module map U_H from H onto $\tilde{H}_2^{(\mathcal{B})}$.*

PROOF. Let $\xi\in H$. Then $F(\xi, \xi)\in L^1(\Gamma, \mu)$ and $\|U_H\xi\|_{\mathcal{B}}=\|\tilde{\xi}\|_{\mathcal{B}}=(\tilde{\xi}|\tilde{\xi})_{\mathcal{B}}^{1/2}=F(\xi, \xi)^{1/2}$. It follows that $U_H\xi\in\tilde{H}_2^{(\mathcal{B})}$ and that $\|U_H\xi\|=\|\xi\|$. Since U_H is obviously an $\mathcal{M}$-module map, we have only to show that U_H is surjective. Let $\eta\in\tilde{H}_2^{(\mathcal{B})}$. Then $\|\eta\|_{\mathcal{B}}\in L^2(\Gamma, \mu)$. Define a mapping $f: H\to L(\Gamma, \mu)$ by the relation $f(\xi)=(\tilde{\xi}|\eta)_{\mathcal{B}}$ for any $\xi\in H$. Then by the Schwarz inequality in $V^{(\mathcal{B})}$ we have for any $\xi\in H$,

$$\begin{aligned}
|f(\xi)(\omega)|=|(\tilde{\xi}|\eta)_{\mathcal{B}}(\omega)|&\leqq\|\tilde{\xi}\|_{\mathcal{B}}(\omega)\|\eta\|_{\mathcal{B}}(\omega)\\
&=\|\eta\|_{\mathcal{B}}(\omega)F(\xi, \xi)(\omega)^{1/2}\quad\text{a. e.},
\end{aligned}$$

so that f satisfies the assumptions of Lemma 4.2. Thus there is some $\zeta\in H$ such that for any $\xi\in H$, $f(\xi)=F(\xi, \zeta)$. It follows that $[\![(\tilde{\xi}|\eta)_{\mathcal{B}}=(\tilde{\xi}|\tilde{\zeta})_{\mathcal{B}}]\!]=1$ for any $\xi\in H$, so that $[\![\eta=\tilde{\zeta}]\!]=1$. Since $\eta\in\tilde{H}^{(\mathcal{B})}$, we have $\eta=U_H\zeta$. Therefore, U_H is surjective. QED

In general, the set $\tilde{H}_2^{(\mathscr{B})}$ depends on the selection of the representatives from $\{v \in V^{(\mathscr{B})} \mid [\![u=v]\!]=1\}$. However by the proof of the above theorem, we can choose such a selection as $\tilde{H}_2^{(\mathscr{B})}=\mathscr{D}(\tilde{H})$, that is, for any $u \in \tilde{H}_2^{(\mathscr{B})}$ there is exactly one $\xi \in H$ such that $[\![\tilde{\xi}=u]\!]=1$.

Let H and K be two normal $\mathscr{M}$-modules and denote $\tilde{H}$ and $\tilde{K}$ be their Boolean embeddings, respectively. By the above remark, we can assume $\mathscr{D}(\tilde{H})=\tilde{H}_2^{(\mathscr{B})}$ and $\mathscr{D}(\tilde{K})=\tilde{K}_2^{(\mathscr{B})}$. Let $T \in \mathrm{Hom}(H, K)$ and define a map $T_0: \mathscr{D}(\tilde{H}) \to \mathscr{D}(\tilde{K})$ by the relation $T_0\tilde{\xi}=(T\xi)^{\sim}$ for any $\xi \in H$. Then $T_0=U_K T U_H{}^{-1}$ and that $T_0 \in \mathrm{Hom}(\tilde{H}_2^{(\mathscr{B})}, \tilde{K}_2^{(\mathscr{B})})$ by Theorem 5.4. Therefore from Theorem 4.3 there is a $\mathscr{B}$-valued set $\tilde{T} \in V^{(\mathscr{B})}$ uniquely in $V^{(\mathscr{B})}$ such that $[\![\tilde{T}$ is a bounded linear map$]\!]=1$, that $[\![\tilde{T}\xi=(T\xi)^{\sim}]\!]=1$ for any $\xi \in H$ and that $\|\tilde{T}\|_{\mathscr{B}} \in \mathscr{M}$. We call this $\tilde{T}$ the *Boolean embedding* of T.

Now we can summarize the functorial properties of the Boolean embedding $H \to \tilde{H}$ and $T \to \tilde{T}$. Let $\mathbf{Hilbert}_\infty^{(\mathscr{B})}$ be the category of Hilbert spaces in $V^{(\mathscr{B})}$ and bounded linear maps f in $V^{(\mathscr{B})}$ such that $\|f\|_{\mathscr{B}} \in \mathscr{M}$, and let **Normod-**$\mathscr{M}$ be the category of normal $\mathscr{M}$-modules and bounded $\mathscr{M}$-module maps. Then the following theorem can be verified without any difficulties (see, [**3**; Theorem IV. 1, p. 91]).

THEOREM 5.5. *The Boolean embedding* $E: H \to \tilde{H}$, $E: T \to \tilde{T}$ *is a functor* **Normod-**$\mathscr{M} \to \mathbf{Hilbert}_\infty^{(\mathscr{B})}$ *which is an equivalence of the two categories. Its adjoint functor is* $R: X \to X_2^{(\mathscr{B})}$, $R: f \to f^{(\mathscr{B})}|_{X_2^{(\mathscr{B})}}$ *obtained in Theorem* 5.3. *The natural isomorphism* $RE \cong 1$ *on* **Normod-**$\mathscr{M}$ *is* $\{U_H \mid H \in$ **Normod-**$\mathscr{M}\}$ *obtained in Theorem* 5.4.

6. Boolean valued interpretation of Hilbert space theory.

With any Hilbert space X in $V^{(\mathscr{B})}$, we can associate a cardinal in $V^{(\mathscr{B})}$, i. e. the dimension $\dim(X)$ of X in $V^{(\mathscr{B})}$. Let X and Y be two Hilbert spaces in $V^{(\mathscr{B})}$. Denote by $X \cong Y$ the relation that X and Y are isomorphic as Hilbert spaces. Then by the interpretation in the usual Hilbert space theory we have $[\![X \cong Y]\!]=[\![\dim(X)=\dim(Y)]\!]$.

Now we have the following.

THEOREM 6.1. *Let H and K be two normal $\mathscr{M}$-modules. Then the following conditions are equivalent.*

(1) *H and K are unitarily equivalent.*

(2) $[\![\tilde{H} \cong \tilde{K}]\!]=1$.

(3) $[\![\dim(\tilde{H})=\dim(\tilde{K})]\!]=1$.

PROOF. We have only to show the equivalence of (1) and (2). By the functorial properties of the functor $H \to \tilde{H}$, it suffices to show that if $T: H \to K$ is a unitary $\mathscr{M}$-module map then $\tilde{T}: \tilde{H} \to \tilde{K}$ is a unitary in $V^{(\mathscr{B})}$. This follows from the relations

$$\int_{\Gamma} a(\omega)(\tilde{T}\tilde{\xi},\ \tilde{T}\tilde{\xi})_{\mathcal{B}}(\omega)\mu(d\omega)=(\pi(a)T\xi,\ T\xi)$$

$$=(T\pi(a)\xi,\ T\xi)=(\pi(a)\xi,\ \xi)=\int_{\Gamma} a(\omega)(\tilde{\xi},\ \tilde{\xi})_{\mathcal{B}}(\omega)\mu(d\omega)$$

for any $\xi\in H,\ a\in\mathcal{M}$. QED

Recall that μ is a sum of finite measures μ_i with disjoint supports Γ_i such that $\bigcup_i\Gamma_i=\Gamma$. Let α be a cardinal. We have $[\![\mathrm{Card}(\check{\alpha})]\!]\geqq[\Gamma_i]$ for any i, since the Boolean algebra $[\Gamma_i]\mathcal{B}$ satisfies the countable chain condition [**11**; Corollary 17.5, p. 162]. It follows that $[\![\mathrm{Card}(\check{\alpha})]\!]=1$, that is, cardinals are absolute in $V^{(\mathcal{B})}$. Denote by $l^2(\check{\alpha})$ the l^2-space over $\check{\alpha}$ in $V^{(\mathcal{B})}$. Then $[\![\dim(l^2(\check{\alpha}))=\check{\alpha}]\!]=1$ by the interpretation of the usual theory. Let $b_\alpha=[\![\dim(X)=\check{\alpha}]\!]$. Then $b_\alpha=[\![X\cong l^2(\check{\alpha})]\!]$. Thus $[\![\dim(X)=\sum_\alpha\check{\alpha}b_\alpha]\!]=1$ and $[\![X\cong\sum_\alpha l^2(\check{\alpha})b_\alpha]\!]=1$. In order to clarify the above decomposition of X in $V^{(\mathcal{B})}$, we examine the l^2-spaces in $V^{(\mathcal{B})}$.

THEOREM 6.2. *Let α be a cardinal. Then we have*

$$l^2(\check{\alpha})_2^{(\mathcal{B})}\cong L^2(\Gamma,\ \mu)\otimes l^2(\alpha)$$

as unitary equivalence of normal $\mathcal{M}$-modules.

PROOF. By the definition of l^2-spaces we have $[\![f\in l^2(\check{\alpha})]\!]=[\![f:\check{\alpha}\to\boldsymbol{C}\wedge\sum_{\beta<\check{\alpha}}|f^{(\mathcal{B})}(\beta)|^2<\infty]\!]$. Since every function from $\mathcal{D}(\check{\alpha})$ is extensional, there is a bijective correspondence between $\mathcal{B}$-valued sets $f\in V^{(\mathcal{B})}$ such that $[\![f\in l^2(\check{\alpha})]\!]=1$ and functions $f^{(\mathcal{B})}:\alpha\to\boldsymbol{C}^{(\mathcal{B})}$ such that there is some $g\in L(\Gamma,\ \mu)$ for which $\sum_{i=1}^n|f^{(\mathcal{B})}(\beta_i)(\omega)|^2\leqq g(\omega)$ a. e. for any finite sequences $\beta_1,\ \beta_2,\ \cdots,\ \beta_n<\alpha$. In the above correspondence, we have $\|f\|_{\mathcal{B}}=\sup\{(\sum_{i=1}^n|f^{(\mathcal{B})}(\beta_i)|^2)^{1/2}\mid n\in\omega,\ \beta_1,\ \beta_2,\ \cdots,\ \beta_n\in\alpha\}$, where the supremum is taken with respect to the ordering $\leqq$ a. e. in $L(\Gamma,\ \mu)$. Let $f\in l^2(\check{\alpha})_2^{(\mathcal{B})}$. Then $\sum_{i=1}^n|f^{(\mathcal{B})}(\beta_i)|^2\leqq\|f\|_{\mathcal{B}}{}^2\in L^1(\Gamma,\ \mu)$ for any $\beta_1,\ \beta_2,\ \cdots,\ \beta_n\in\alpha$. Thus we have

$$\sum_{i=1}^n\|f^{(\mathcal{B})}(\beta_i)\|_2^2=\sum_{i=1}^n\int_{\Gamma}|f^{(\mathcal{B})}(\beta_i)(\omega)|^2\mu(d\omega)$$

$$=\int_{\Gamma}\sum_{i=1}^n|f^{(\mathcal{B})}(\beta_i)(\omega)|^2\mu(d\omega)\leqq\int_{\Gamma}\|f\|_{\mathcal{B}}(\omega)^2\mu(d\omega).$$

It follows that $f^{(\mathcal{B})}\in l^2(\alpha;\ L^2(\Gamma,\ \mu))\cong l^2(\alpha)\otimes L^2(\Gamma,\ \mu)$. Conversely, let $f^{(\mathcal{B})}\in l^2(\alpha;\ L^2(\Gamma,\ \mu))$. Then

$$\sum_{\beta<\alpha}\int_{\Gamma}|f^{(\mathcal{B})}(\beta)(\omega)|^2\mu(d\omega)<\infty,$$

so that by Fubini's theorem $\sum_{\beta<\alpha}|f^{(\mathcal{B})}(\beta)(\omega)|^2\in L^1(\Gamma,\ \mu)$. Since $\sum_{i=1}^n|f^{(\mathcal{B})}(\beta_i)(\omega)|^2\leqq\sum_{\beta<\alpha}|f^{(\mathcal{B})}(\beta)(\omega)|^2$ a. e. for any $\beta_1,\ \beta_2,\ \cdots,\ \beta_n$, we have that $\|f\|_{\mathcal{B}}^2\leqq\sum_{\beta<\alpha}|f^{(\mathcal{B})}(\beta)(\omega)|^2$. It follows that $f\in l^2(\check{\alpha})_2^{(\mathcal{B})}$. Now the conclusion can be checked in a straightforward way by the obvious correspondence $l^2(\alpha; L^2(\Gamma, \mu))\cong L^2(\Gamma, \mu)\otimes l^2(\alpha)$. QED

Now we have the following.

THEOREM 6.3. *Let X be a Hilbert space in $V^{(\mathcal{B})}$ with $[\![\dim(X)=\sum_\alpha \check{\alpha} b_\alpha]\!]=1$. Then there is a family $\{\nu_\alpha\}$ of measures on Ω with disjoint supports Ω_α such that $[\Omega_\alpha]=b_\alpha$ and that*

$$X_2^{(\mathcal{B})}\cong\sum_\alpha^\oplus L^2(\Omega_\alpha, \nu_\alpha)\otimes l^2(\alpha)$$

as unitary equivalence of normal $\mathcal{M}$-modules.

PROOF. Let Ω_α be the clopen subsets of Ω corresponding to b_α by the Stone representation. Then the family $\{\Omega_\alpha\}$ is disjoint, $[\Omega_\alpha]=b_\alpha$ and $(\bigcup_\alpha \Omega_\alpha)^-=\Omega$. Let ν_α be the restriction of μ on Ω_α. Then $b_\alpha\mathcal{M}\cong L^\infty(\Omega_\alpha, \nu_\alpha)$. Since $[\![\dim(X)=\check{\alpha}]\!]=b_\alpha$, by Theorem 6.2 we have $[\![l^2(\check{\alpha})\cong L^2(\Omega_\alpha, \nu_\alpha; l^2(\alpha))^\sim]\!]=b_\alpha$. Obviously we can embed $L^2(\Omega_\alpha, \nu_\alpha; l^2(\alpha))$ in $\sum_\alpha^\oplus L^2(\Omega_\alpha, \nu_\alpha)\otimes l^2(\alpha)$ as a closed submodule by putting $f(\omega, \beta)=0$ for $\omega\in\Omega-\Omega_\alpha$ and $\beta<\alpha$. Write $H=\sum_\alpha^\oplus L^2(\Omega_\alpha, \nu_\alpha)\otimes l^2(\alpha)$. For any $a\in\mathcal{M}$ and $f, g\in H$, we have

$$(\pi(a)f\,|\,g)=\sum_\alpha\int_{\Omega_\alpha}\sum_{\beta<\alpha}a(\omega)f(\omega, \beta)\overline{g(\omega, \beta)}\nu_\alpha(d\omega)$$

$$=\int_\Omega a(\omega)\sum_\beta f(\omega, \beta)\overline{g(\omega, \beta)}\mu(d\omega),$$

where $\sum_\beta$ means $\sum_{\beta<\alpha}$ if $\omega\in\Omega_\alpha$. Thus $(f\,|\,g)_{\mathcal{B}}(\omega)=\sum_\beta f(\omega, \beta)\overline{g(\omega, \beta)}$ for any $f, g\in H$. Let $f\in H$ and let $g(\omega, \beta)=f(\omega, \beta)$ if $\omega\in\Omega_\alpha$ and $g(\omega, \beta)=0$ if $\omega\notin\Omega_\alpha$ for a fixed α. Then $g\in L^2(\Omega_\alpha, \nu_\alpha; l^2(\alpha))\subseteq H$ and that $(\tilde{f}-\tilde{g}\,|\,\tilde{f}-\tilde{g})_{\mathcal{B}}(\omega)=0$ for any $\omega\in\Omega_\alpha$. It follows that $[\![\tilde{H}\subseteq L^2(\Omega_\alpha, \nu_\alpha; l^2(\alpha))^\sim]\!]\geqq[\Omega_\alpha]$. Since obviously $[\![\tilde{H}\supseteq L^2(\Omega_\alpha, \nu_\alpha; l^2(\alpha))^\sim]\!]=[\Omega_\alpha]$ so that $[\![X\cong\tilde{H}]\!]=b_\alpha$ for any α. Thus $[\![X\cong\tilde{H}]\!]=1$. Therefore $X_2^{(\mathcal{B})}\cong\sum_\alpha^\oplus L^2(\Omega_\alpha, \nu_\alpha)\otimes l^2(\alpha)$ by Theorem 5.4. QED

The following corollary is an immediate consequence of Theorem 6.1 and Theorem 6.3 (for the multiplicity functions see Section 2).

COROLLARY 6.4. *Let H be a normal $\mathcal{M}$-module with the multiplicity function m. Then $[\![\dim(\tilde{H})=\sum_\alpha\check{\alpha}m(\alpha)]\!]=1$.*

Appendix.

In the following, we shall consider the relation between Takeuti's construction of Hilbert spaces in $V^{(\mathcal{B})}$ obtained in [**10**; Section 1] and ours. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, $\mathcal{Z}_0$ a von Neumann algebra included in its center and $\mathcal{B}$ the complete Boolean algebra of projections in $\mathcal{Z}_0$. Let Ω be the spectrum of $\mathcal{Z}_0$ and assume that $\mathcal{Z}_0\cong L^\infty(\Omega, \mu)$ where μ is a Radon measure on Ω. Then we have $\boldsymbol{C}^{(\mathcal{B})}\cong L(\Omega, \mu)$. In view of our construction in Section 4, we can present Takeuti's construction in the following manner. Let

$(\boldsymbol{C}\times\check{\mathscr{H}})^{(\mathscr{B})}$ be the set of all $\langle f, \check{\xi}\rangle^{(\mathscr{B})}\in V^{(\mathscr{B})}$ where $f\in\boldsymbol{C}^{(\mathscr{B})}$ and $\xi\in\mathscr{H}$; in the sequel, $\langle f, \check{\xi}\rangle^{(\mathscr{B})}$ is denoted by $f\oplus\check{\xi}$. It is known [**11**; p. 135] that $[\![(\boldsymbol{C}\times\check{\mathscr{H}})^{(\mathscr{B})}\times\{1\}=\boldsymbol{C}\times\check{\mathscr{H}}]\!]=1$. For any $\xi, \eta\in\mathscr{H}$, we can define $F(\xi, \eta)\in L^1(\Omega, \mu)$ as in Section 4 for the normal $\mathscr{Z}_0$-module $\mathscr{H}$. Obviously, $[\![\boldsymbol{C}\times\check{\mathscr{H}}$ is an abelian group$]\!]=1$. It is shown that there is some $I_0\in V^{(\mathscr{B})}$ such that $[\![I_0\colon(\boldsymbol{C}\times\check{\mathscr{H}})\times(\boldsymbol{C}\times\check{\mathscr{H}})\to\boldsymbol{C}]\!]=1$ and that $[\![I_0(f\oplus\check{\xi}, g\oplus\check{\eta})=f\bar{g}F(\xi, \eta)]\!]=1$ for any $f, g\in\boldsymbol{C}^{(\mathscr{B})}$ and $\xi, \eta\in\mathscr{H}$. Let $N\in V^{(\mathscr{B})}$ be such that

$$N=\{f\oplus\check{\xi}\in(\boldsymbol{C}\times\check{\mathscr{H}})^{(\mathscr{B})}\mid I_0(f\oplus\check{\xi}, g\oplus\check{\eta})=0\}\times\{1\}.$$

Then it is shown in [**10**] that

$$[\![(\boldsymbol{C}\times\check{\mathscr{H}})^{(\mathscr{B})}\times\{1\}/N \text{ is a Hilbert space with inner product } I_0/N]\!]=1\,.$$

In this case, we have an equivalence relation $\equiv$ on $(\boldsymbol{C}\times\check{\mathscr{H}})^{(\mathscr{B})}$ such that $f\oplus\check{\xi}\equiv g\oplus\check{\eta}$ if and only if $[\![(f\oplus\check{\xi})+N=(g\oplus\check{\eta})+N]\!]=1$. Then it is easy to see that $f\oplus\check{\xi}\equiv g\oplus\check{\eta}$ if and only if

$$f(\omega)F(\xi, \zeta)(\omega)=g(\omega)F(\eta, \zeta)(\omega) \qquad \text{a.e.}$$

for any $\zeta\in\mathscr{H}$. Denote by $f\check{\xi}$ the equivalence class of $f\oplus\check{\xi}$ and by $\boldsymbol{C}^{(\mathscr{B})}\check{\mathscr{H}}$ the quotient space $(\boldsymbol{C}\times\check{\mathscr{H}})^{(\mathscr{B})}/\equiv$. Now we can state a precise construction of $(\boldsymbol{C}\times\check{\mathscr{H}})^{(\mathscr{B})}\times\{1\}/N$. For any $f\check{\xi}\in\boldsymbol{C}^{(\mathscr{B})}\check{\mathscr{H}}$, define a $\mathscr{B}$-valued set $\widetilde{f\xi}\in V^{(\mathscr{B})}$ by $\mathscr{D}(\widetilde{f\xi})=\{\check{\xi}\mid\xi\in\mathscr{H}\}$ and

$$\widetilde{f\xi}(\check{\zeta})=[\![I_0(f\oplus\check{\xi}-1\oplus\check{\zeta},\ f\oplus\check{\xi}-1\oplus\check{\zeta})=0]\!]\,,$$

for any $\zeta\in\mathscr{H}$. Let $\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}}\in V^{(\mathscr{B})}$ be a $\mathscr{B}$-valued set such that

$$\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}}=\{\widetilde{f\xi}\mid f\in\boldsymbol{C}^{(\mathscr{B})},\ \xi\in\mathscr{H}\}\times\{1\}.$$

Then the correspondence $f\check{\xi}\to\widetilde{f\xi}$ is one-to-one and that $[\![(\boldsymbol{C}\times\check{\mathscr{H}})^{(\mathscr{B})}\times\{1\}/N=\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}}]\!]=1$. The proof of the above facts is similar to the proof of Theorem 4.3. Thus the definition of $\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}}$ is a precise expression of Takeuti's construction of Hilbert spaces in $V^{(\mathscr{B})}$, although the definition of $\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}}$ is tacit in [**10**].

Let $\tilde{\mathscr{H}}$ be the Boolean embedding of the $\mathscr{Z}_0$-module $\mathscr{H}$ in $V^{(\mathscr{B})}$ defined as Definition 4.4. Then the relation of Takeuti's construction $\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}}$ and our construction $\tilde{\mathscr{H}}$ is as follows.

THEOREM A.1. *By a suitable selection of the representatives from* $\{v\in V^{(\mathscr{B})}\mid [\![u=v]\!]=1\}$, *we have*

$$\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}}=\tilde{\mathscr{H}}^{(\mathscr{B})}\times\{1\}.$$

PROOF. By the similar argument as in the proof of Theorem 5.4, for any $u\in(\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}})_2^{(\mathscr{B})}$ there is a unique $\xi\in\mathscr{H}$ such that $[\![\widetilde{1\xi}=u]\!]=1$. By definition we have $\widetilde{1\xi}=\tilde{\xi}$, where $\tilde{\xi}$ is defined in Theorem 4.3. Thus we have $(\boldsymbol{C}^{(\mathscr{B})}\tilde{\mathscr{H}})_2^{(\mathscr{B})}=$

$\mathscr{D}(\tilde{\mathscr{H}})$. Then by Theorem 5.1, we have $[\![C^{(\mathscr{B})}\tilde{\mathscr{H}}=\tilde{\mathscr{H}}]\!]=1$ and hence we can assume that $\mathscr{D}(C^{(\mathscr{B})}\tilde{\mathscr{H}})\subseteq\tilde{\mathscr{H}}^{(\mathscr{B})}$. Let $u\in\tilde{\mathscr{H}}^{(\mathscr{B})}$. Then $\|u\|_{\mathscr{B}}\in L(\Omega, \mu)$. Let $v\in\tilde{\mathscr{H}}^{(\mathscr{B})}$ be such that $[\![v=(1/1+\|u\|_{\mathscr{B}})u]\!]=1$. Then $\|v\|_{\mathscr{B}}\in L^{\infty}(\Omega, \mu)$ and hence $\|v\|_{\mathscr{B}}\in L^{2}(\Omega, \mu)$ since Ω is compact. It follows that there is some $\xi\in H$ such that $[\![v=\tilde{\xi}]\!]=1$. Let $f=1+\|u\|_{\mathscr{B}}$. Then we have $[\![u=f\tilde{\xi}]\!]=1$. Now it is straightforward to prove that $[\![\widetilde{f\xi}=f\tilde{\xi}]\!]=1$. Therefore, we have $\mathscr{D}(C^{(\mathscr{B})}\tilde{\mathscr{H}})=\tilde{\mathscr{H}}^{(\mathscr{B})}$ so that $C^{(\mathscr{B})}\tilde{\mathscr{H}}=\tilde{\mathscr{H}}^{(\mathscr{B})}\times\{1\}$. QED

References

[1] P.R. Halmos, Introduction to Hilbert space and the theory of spectral multiplicity, Chelsea, New York, 1951.

[2] P.R. Halmos, Lectures on Boolean algebras, Van Nostrand, New York, 1963.

[3] S. Mac Lane, Categories for the working mathematician, Springer, New York, 1971.

[4] S. Sakai, C^*-Algebras and W^*-algebras, Springer, Berlin, 1971.

[5] M. Takesaki, Theory of operator algebras I, Springer, New York, 1979.

[6] G. Takeuti, Two applications of logic to mathematics, Iwanami and Princeton University Press, Tokyo and Princeton, 1978.

[7] G. Takeuti, Boolean valued analysis, Applications of Sheaves, Ed. by Fourman, Mulvey and Scott, 714-731, Lecture Notes in Math., **753**, Springer, Berlin, 1979.

[8] G. Takeuti, A transfer principle in harmonic analysis, J. Symbolic Logic, **44** (1979), 417-440.

[9] G. Takeuti, Boolean completion and m-convergence, Categorical aspects of topology and analysis, Ed. by B. Banaschewski, 333-350, Lecture Notes in Math., **915**, Springer, Berlin, 1982.

[10] G. Takeuti, Von Neumann algebras and Boolean valued analysis, J. Math. Soc. Japan, **35** (1983), 1-21.

[11] G. Takeuti and W.M. Zaring, Axiomatic set theory, Springer, Heidelberg, 1973.

Masanao OZAWA

Department of Information Sciences
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo 152, Japan