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Cancellation law for Riemannian direct product

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§ 0. Introduction.

L. S. Charlap showed that there are two compact differentiable manifolds M and N such that $M\times S^1$ is diffeomorphic to $N\times S^1$, while M and N are of different homotopy type (see [1]).

On the other hand, considering a Riemannian analogue of the above problem, we obtained the following result [3]:

Let M and N be connected complete Riemannian manifolds and S a connected compact locally symmetric Riemannian manifold. If $M\times S$ is isometric to $N\times S$, then M is isometric to N.

Later on, H. Takagi obtained the following result [2]:

Let M and N be connected complete Riemannian manifolds and let S be a connected complete Riemannian manifold which is simply connected or has the irreducible restricted homogeneous holonomy group. If $M\times S$ is isometric to $N\times S$, then M is isometric to N.

The purpose of this paper is to give a complete answer to the above problem in Riemannian case.

The main result is the following.

THEOREM. *If $M\times S$ is isometric to $N\times S$, then M is isometric to N, where M, N and S are connected complete Riemannian manifolds.*

In this paper, Riemannian manifolds are always supposed to be connected and complete, and $\cong$ means isometric.

We shall give a brief account of the idea of the proof. Let M, N and S be Riemannian manifolds such that $M\times S$ is isometric to $N\times S$. Then $M\cong X/\Gamma_1$, $N\cong X/\Gamma_2$ and $S\cong Y/\Gamma_3$ where X and Y are simply connected Riemannian manifolds and Γ_1, Γ_2 and Γ_3 are deck transformation groups of M, N and S, respectively. If we could find an isometry $\tilde{g}$ of $X\times Y$ satisfying Conditions 1 and 2 in Lemma 3, then our theorem would be proved. An isometry g of $X\times Y$ which is a natural lift of an isometry from $M\times S$ to $N\times S$ satisfies Condition 1 in Lemma 3. While if X and Y have the Euclidean parts in its de Rham decompositions, then g does not always satisfy Condition 2 in Lemma 3. However, using Lemma 4 and Lemma 5, we can change g into $\tilde{g}$ which satisfies Conditions 1 and 2 in Lemma 3.

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§ 1. Basic lemmas.

In this section we shall refer to five lemmas for the proof of the theorem.

By the uniqueness of the de Rham decomposition of a simply connected Riemannian manifold, we have the following lemma.

LEMMA 1. *Let M, N and S be Riemannian manifolds. If $M\times S$ is isometric to $N\times S$, then the universal Riemannian covering manifold $\tilde{M}$ of M is isometric to $\tilde{N}$, the universal Riemannian covering manifold of N.*

DEFINITION. An *FPDA-group* on a Riemannian manifold is a subgroup of the isometry group of the manifold whose action on the manifold is free and properly discontinuous.

LEMMA 2 ([3]). *Let Γ and Γ' be FPDA-groups on simply connected Riemannian manifolds A and A' respectively. Then A/Γ is isometric to A'/Γ' if and only if there exists an isometry ϕ from A to A' with $\Gamma'=\phi\Gamma\phi^{-1}$.*

For isometries $f_1, \cdots, f_n$ from Riemannian manifolds $A_1, \cdots, A_n$ to Riemannian manifolds $B_1, \cdots, B_n$ respectively, we denote by $f_1\times\cdots\times f_n$ the isometry from $A_1\times\cdots\times A_n$ to $B_1\times\cdots\times B_n$ such that the image of $(a_1, \cdots, a_n)\in A_1\times\cdots\times A_n$ is $(f_1(a_1), \cdots, f_n(a_n))$. We denote the identity map of a Riemannian manifold A by id_A. For FPDA-groups Γ and Λ on Riemannian manifolds A and B respectively, we denote by $\Gamma\times\Lambda$ the group consisting of all the isometries on $A\times B$ of the form $\gamma\times\lambda$ for some $\gamma\in\Gamma$ and $\lambda\in\Lambda$. Then $\Gamma\times\Lambda$ is an FPDA-group on $A\times B$.

For an isometry we have the following fact which is essential for the proof of Lemma 3.

FACT. *Let A, B, C and D be Riemannian manifolds and ϕ an isometry from $A\times B$ onto $C\times D$. If, for some points (a_0, b_0) and $(c_0, d_0)=\phi(a_0, b_0)$, $\phi(A, b_0)=(C, d_0)$, then there are isometries ϕ_1 from A to C and ϕ_2 from B to D such that $\phi=\phi_1\times\phi_2$.*

PROOF. By the assumption $\phi(a_0, B)=(c_0, D)$. So there are isometries ϕ_1 from A to C and ϕ_2 from B to D such that $\phi(a, b_0)=(\phi_1(a), b_0)$ and $\phi(a_0, b)=(c_0, \phi_2(b))$ for any $a\in A$ and $b\in B$. Then $\phi(a_0, b_0)=(\phi_1(a_0), \phi_2(b_0))$ and $d\phi_{(a_0, b_0)}=d\phi_{1a_0}+d\phi_{2b_0}$. Hence $\phi=(\phi_1\times\phi_2)$.

The following lemma is essential in our proof of the theorem. In the following lemma, we regard a set consisting of one element as a zero-dimensional

Riemannian manifold.

LEMMA 3. *Let A and B be Riemannian manifolds, let Π_1 and Π_2 be FPDA-groups on A and let Π_3 be an FPDA-group on B. We assume that there are decompositions $A\cong A_1\times A_2$, $B\cong B_1\times B_2$ and an isometry ϕ of $A\times B$ satisfying the following conditions.*

Condition 1. $\phi(\Pi_1\times\Pi_3)\phi^{-1}=\Pi_2\times\Pi_3$.

Condition 2. *For some isometries η_A and η_B from $A_1\times A_2$ and $B_1\times B_2$ to A and B respectively,* $\phi\circ(\eta_A\times\eta_B)(A_1\times\{a_2\}\times\{b_1\}\times B_2)=A\times\{b\}$ and $\phi\circ(\eta_A\times\eta_B)(\{a_1\}\times A_2\times B_1\times\{b_2\})=\{a\}\times B$ for some $a_1\in A_1$, $a_2\in A_2$, $b_1\in B_1$, $b_2\in B_2$, $a\in A$ *and* $b\in B$.

Then:

(1) *If the dimension of B_1 or the dimension of B_2 is equal to zero, then there is an isometry ψ of A with $\psi\Pi_1\psi^{-1}=\Pi_2$.*

(2) *If the dimension of B_1 and the dimension of B_2 are positive, then $B/\Pi_3\cong B_1/\Lambda_1\times B_2/\Lambda_2$ for some FPDA-groups Λ_1 and Λ_2 on B_1 and B_2 respectively.*

PROOF OF (1). First let the dimension of B_2 be equal to zero. Then B_1 is isometric to B and, by Condition 2, the dimension of A_2 is equal to zero. Hence Condition 2 is as follows; $\phi(A\times\{b\})=A\times\{b^*\}$ and $\phi(\{a\}\times B)=\{a^*\}\times B$ for some a, $a^*\in A$, b and $b^*\in B$. So, by Fact, there are isometries ϕ_A of A and ϕ_B of B such that $\phi=\phi_A\times\phi_B$. By Condition 1, $(\phi_A\times\phi_B)(\Pi_1\times\Pi_3)(\phi_A\times\phi_B)^{-1}=\Pi_2\times\Pi_3$. Hence $(\phi_A\Pi_1\phi_A^{-1})\times(\phi_B\Pi_3\phi_B^{-1})=\Pi_2\times\Pi_3$. Therefore $\phi_A\Pi_1\phi_A^{-1}=\Pi_2$.

Next let the dimension of B_1 be equal to zero. Then B_2 is isometric to B. We consider ϕ as an isometry of $A_1\times A_2\times B$ and also we consider Π_1 and Π_2 as FPDA-groups on $A_1\times A_2$. Then Condition 2 is as follows; $\phi(A_1\times\{a_2^{(0)}\}\times B)=A_1\times A_2\times\{b^{(1)}\}$ and $\phi(\{a_1^{(0)}\}\times A_2\times\{b^{(0)}\})=\{a_1^{(1)}\}\times\{a_2^{(1)}\}\times B$ for some $a_1^{(0)}$, $a_1^{(1)}\in A_1$, $a_2^{(0)}$, $a_2^{(1)}\in A_2$, $b^{(0)}$ and $b^{(1)}\in B$. Hence there is an isometry ϕ_3 from A_2 to B such that $\phi(a_1^{(0)}, a_2, b^{(0)})=(a_1^{(1)}, a_2^{(1)}, \phi_3(a_2))$. Let A_1' and A_2' be submanifolds of $A_1\times A_2$ such that $\phi(A_1\times\{a_2^{(0)}\}\times\{b^{(0)}\})=A_1'\times\{b^{(1)}\}$ and $\phi(\{a_1^{(0)}\}\times\{a_2^{(0)}\}\times B)=A_2'\times\{b^{(1)}\}$. Then there are isometries ϕ_1 and ϕ_2 from A_1 and B to A_1' and A_2', respectively, such that $\phi(a_1, a_2^{(0)}, b^{(0)})=(\phi_1(a_1), b^{(1)})$ and $\phi(a_1^{(0)}, a_2^{(0)}, b)=(\phi_2(b), b^{(1)})$. Since $\phi(A_1\times\{a_2^{(0)}\}\times B)=A_1\times A_2\times\{b^{(1)}\}$, there is an isometry ϕ_0 from $A_1\times B$ to $A_1\times A_2$ such that $\phi(a_1, a_2^{(0)}, b)=(\phi_0(a, b), b^{(1)})$. Let $\eta=(\phi_1\times\phi_2)\circ\phi_0^{-1}$ and $\Pi_2'=\eta\Pi_2\eta^{-1}$. Then η is an isometry from $A_1\times A_2$ to $A_1'\times A_2'$ and Π_2' is an FPDA-group on $A_1'\times A_2'$. Because $\phi(\Pi_1\times\Pi_3)\phi^{-1}=\Pi_2\times\Pi_3$, $(\eta\times\mathrm{id}_B)\circ\phi(\Pi_1\times\Pi_3)\phi^{-1}\circ(\eta\times\mathrm{id}_B)^{-1}=\Pi_2'\times\Pi_3$. Moreover $(\eta\times\mathrm{id}_B)\circ\phi(a_1, a_2^{(0)}, b)=(\eta\times\mathrm{id}_B)(\phi_0(a_1, b), b^{(1)})=(\phi_1(a_1), \phi_2(b), b^{(1)})$ and $(\eta\times\mathrm{id}_B)\circ\phi(a_1^{(0)}, a_2, b^{(0)})=(\eta\times\mathrm{id}_B)(a_1^{(1)}, a_2^{(1)}, \phi_3(a_2))=(\eta(a_1^{(1)}, a_2^{(2)}), \phi_3(a_2))$, where $a_1\in A_1$, $a_2\in A_2$ and $b\in B$. If we show that there exists an isometry ψ from $A_1\times A_2$ to $A_1'\times A_2'$ such that $\Pi_2'=\psi\Pi_1\psi^{-1}$, then $\eta^{-1}\circ\psi$ is an isometry of $A_1\times A_2$ and $\Pi_2=(\eta^{-1}\circ\psi)\Pi_1(\eta^{-1}\circ\psi)^{-1}$. Therefore it is sufficient to show the following assertion:

Let ϕ be an isometry from $A_1\times A_2\times B$ to $A_1'\times A_2'\times B$, and Π_1, Π_2 and Π_3

FPDA-groups on $A_1\times A_2$, $A_1'\times A_2'$ and B respectively. We assume the following two conditions;

(1) $\phi(\Pi_1\times\Pi_3)\phi^{-1}=\Pi_2\times\Pi_3$,

(2) There are isometries ϕ_1, ϕ_2 and ϕ_3 from A_1, B and A_2 to A_1', A_2' and B, respectively, such that $\phi(a_1, a_2^{(0)}, b)=(\phi_1(a_1), \phi_2(b), b^{(1)})$ and $\phi(a_1^{(0)}, a_2, b^{(0)})=(a_1^{(1)}, a_2^{(1)}, \phi_3(a_2))$ for some $a_1^{(0)}\in A_1$, $a_2^{(0)}\in A_2$, $a_1^{(1)}\in A_1'$, $a_2^{(1)}\in A_2'$, $b^{(0)}$ and $b^{(1)}\in B$, where $a_1\in A_1$, $a_2\in A_2$ and $b\in B$ are arbitrary. Then there is an isometry ψ from $A_1\times A_2$ to $A_1'\times A_2'$ such that $\Pi_2=\psi\Pi_1\psi^{-1}$.

Let ν be an isometry from $A_1\times A_2\times B$ to $A_1\times B\times A_2$ such that $\nu(a_1, a_2, b)=(a_1, b, a_2)$ for $a_1\in A_1$, $a_2\in A_2$ and $b\in B$. Then, by Fact, $\phi\circ\nu^{-1}=\phi_1\times\phi_2\times\phi_3$, that is $\phi=(\phi_1\times\phi_2\times\phi_3)\circ\nu$. Hence each isometry of $\phi(\{\mathrm{id}_{A_1\times A_2}\}\times\Pi_3)\phi^{-1}$ is of the form $\mathrm{id}_{A_1'}\times\sigma\times\mathrm{id}_B$, where σ is some isometry of A_2'. So, by the condition, $\phi(\{\mathrm{id}_{A_1\times A_2}\}\times\Pi_3)\phi^{-1}\subset\Pi_2\times\{\mathrm{id}_B\}$. Similarly $\phi^{-1}(\{\mathrm{id}_{A_1'\times A_2'}\}\times\Pi_3)\phi\subset\Pi_1\times\{\mathrm{id}_B\}$. Therefore $\Pi_2\times\{\mathrm{id}_B\}=((\Pi_2\times\{\mathrm{id}_B\})\cap\phi(\Pi_1\times\{\mathrm{id}_B\})\phi^{-1})\cdot(\phi(\{\mathrm{id}_{A_1\times A_2}\}\times\Pi_3)\phi^{-1})$ and $\Pi_1\times\{\mathrm{id}_B\}=(\phi^{-1}(\Pi_2\times\{\mathrm{id}_B\})\phi\cap(\Pi_1\times\{\mathrm{id}_B\}))\cdot(\phi^{-1}(\{\mathrm{id}_{A_1'\times A_2'}\}\times\Pi_3)\phi)$. Let $\psi=(\phi_1\times\phi_2)\circ(\mathrm{id}_{A_1}\times\phi_3)$. Since each isometry of $\phi^{-1}(\Pi_2\times\{\mathrm{id}_B\})\phi\cap(\Pi_1\times\{\mathrm{id}_B\})$ is of the form $\sigma\times\mathrm{id}_{A_2}\times\mathrm{id}_B$ for some isometry σ of A_1, $(\psi\times\mathrm{id}_B)(\phi^{-1}(\Pi_2\times\{\mathrm{id}_B\})\phi\cap(\Pi_1\times\{\mathrm{id}_B\}))(\psi\times\mathrm{id}_B)^{-1}=\phi(\phi^{-1}(\Pi_2\times\{\mathrm{id}_B\})\phi\cap(\Pi_1\times\{\mathrm{id}_B\}))\phi^{-1}=(\Pi_2\cap\{\mathrm{id}_B\})\cap\phi(\Pi_1\times\{\mathrm{id}_B\})\phi^{-1}$. Similarly $(\psi\times\mathrm{id}_B)(\phi^{-1}(\{\mathrm{id}_{A_1'\times A_2'}\}\times\Pi_3)\phi)(\psi\times\mathrm{id}_B)^{-1}=\phi(\{\mathrm{id}_{A_1\times A_2}\}\times\Pi_3)\phi^{-1}$. Hence $(\psi\times\mathrm{id}_B)(\Pi_2\times\{\mathrm{id}_B\})(\psi\times\mathrm{id}_B)^{-1}=\Pi_2\times\{\mathrm{id}_B\}$. Therefore $\psi\Pi_1\psi^{-1}=\Pi_2$.

PROOF OF (2). By the same way as the proof of (1), it is sufficient to prove the existence of Λ_1 and Λ_2 in the following situation:

Let ϕ be an isometry from $A_1\times A_2\times B_1\times B_2$ to $A_1'\times A_2'\times B_1'\times B_2'$, and Π_1, Π_3, Π_2 and Π_4 FPDA-groups on $A_1\times A_2$, $B_1\times B_2$, $A_1'\times A_2'$ and $B_1'\times B_2'$ respectively. We assume the following two conditions;

(1) $\phi(\Pi_1\times\Pi_3)\phi^{-1}=\Pi_2\times\Pi_4$,

(2) There are isometries ϕ_1, ϕ_2, ϕ_3 and ϕ_4 from A_1, B_2, B_1 and A_2 to A_1', A_2', B_1' and B_2', respectively, such that $\phi(a_1, a_2^{(0)}, b_1^{(0)}, b_2)=(\phi_1(a_1), \phi_2(b_2), b_1^{(1)}, b_2^{(1)})$ and $\phi(a_1^{(0)}, a_2, b_1, b_2^{(0)})=(a_1^{(1)}, a_2^{(1)}, \phi_3(b_1), \phi_4(a_2))$ for some $a_1^{(0)}\in A_1$, $a_1^{(1)}\in A_1'$, $a_2^{(0)}\in A_2$, $a_2^{(1)}\in A_2'$, $b_1^{(0)}\in B_1$, $b_1^{(1)}\in B_1'$, $b_2^{(0)}\in B_2$ and $b_2^{(1)}\in B_2'$, where $a_1\in A_1$, $a_2\in A_2$, $b_1\in B_1$ and $b_2\in B_2$ are arbitrary.

Let ν be an isometry from $A_1\times A_2\times B_1\times B_2$ to $A_1\times B_2\times B_1\times A_2$ such that $\nu(a_1, a_2, b_1, b_2)=(a_1, b_2, b_1, a_2)$ for $a_1\in A_1$, $a_2\in A_2$, $b_1\in B_1$ and $b_2\in B_2$. Then, by Fact, $\phi\circ\nu^{-1}=\phi_1\times\phi_2\times\phi_3\times\phi_4$ that is $\phi=(\phi_1\times\phi_2\times\phi_3\times\phi_4)\circ\nu$. By the condition, for $\sigma\in\Pi_3$, there are $\sigma_2\in\Pi_2$ and $\sigma_4\in\Pi_4$ such that $\mathrm{id}_{A_1\times A_2}\times\sigma=\phi^{-1}\circ(\sigma_2\times\mathrm{id}_{B_1'\times B_2'})\circ(\mathrm{id}_{A_1'\times A_2'}\times\sigma_4)\circ\phi$. Hence $\nu\circ(\mathrm{id}_{A_1\times A_2}\times\sigma)\circ\nu^{-1}=((\phi_1^{-1}\times\phi_2^{-1})\circ\sigma_2\circ(\phi_1\times\phi_2))\times((\phi_3^{-1}\times\phi_4^{-1})\circ\sigma_4\circ(\phi_3\times\phi_4))$. So, by Fact, $(\phi_1^{-1}\times\phi_2^{-1})\circ\sigma_2\circ(\phi_1\times\phi_2)=\mathrm{id}_{A_1}\times\tau_2$ and $(\phi_3^{-1}\times\phi_4^{-1})\circ\sigma_4\circ(\phi_3\times\phi_4)=\tau_1\times\mathrm{id}_{A_2}$ for some isometry τ_1 and τ_2 of B_1 and B_2 respectively. Hence $\phi^{-1}\circ(\sigma_2\times\mathrm{id}_{B_1'\times B_2'})\circ\phi$ and $\phi^{-1}\circ(\mathrm{id}_{A_1'\times A_2'}\times\sigma_4)\circ\phi$ are in $\{\mathrm{id}_{A_1\times A_2}\}\times\Pi_3$. Therefore $\{\mathrm{id}_{A_1\times A_2}\}\times\Pi_3=((\{\mathrm{id}_{A_1\times A_2}\}\times\Pi_3)\cap\phi^{-1}(\Pi_2\times\{\mathrm{id}_{B_1'\times B_2'}\})\phi)\cdot((\{\mathrm{id}_{A_1\times A_2}\}\times\Pi_3)\cap$

$\phi^{-1}(\{id_{A'_1\times A'_2}\}\times\Pi_4)\phi)$ and each element of $(\{id_{A_1\times A_2}\}\times\Pi_3)\cap\phi^{-1}(\Pi_2\times\{id_{B'_1\times B'_2}\})\phi$ and $(\{id_{A_1\times A_2}\}\times\Pi_3)\cap\phi^{-1}(\{id_{A'_1\times A'_2}\}\times\Pi_4)\phi$ are of the form $id_{A_1\times A_2}\times id_{B_1}\times\tau_2$ and $id_{A_1\times A_2}\times\tau_1\times id_{B_2}$, respectively, where τ_1 and τ_2 are isometries of B_1 and B_2 respectively. Let $\Lambda_1=\{\tau:\tau$ is an isometry of B_1 such that $id_{A_1\times A_2}\times\tau\times id_{B_2}$ is in $(\{id_{A_1\times A_2}\}\times\Pi_3)\cap\phi^{-1}(\{id_{A'_1\times A'_2}\}\times\Pi_4)\phi\}$ and $\Lambda_2=\{\tau:\tau$ is an isometry of B_2 such that $id_{A_1\times A_2}\times id_{B_1}\times\tau$ is in $(\{id_{A_1\times A_2}\}\times\Pi_3)\cap\phi^{-1}(\Pi_2\times\{id_{B'_1\times B'_2}\})\phi\}$. Then $(B_1\times B_2)/\Pi_2\cong B_1/\Lambda_1\times B_2/\Lambda_2$.

Concerning a group consisting of isometries of a Euclidean space, we have the following.

LEMMA 4. *Let V be an n-dimensional real vector space with a Euclidean metric and Π a group consisting of isometries of V. For $v\in V$, let V_v be the linear subspace spanned by $\{\sigma v-v:\sigma\in\Pi\}$, $V_0=\sum_{v\in V}V_v$ and V_1 the orthogonal complement of V_0. Let us choose an orthonormal basis $\{e_1,\cdots,e_n\}$ of V such that e_i $(1\leqq i\leqq k)$ are in V_0 and e_j $(k+1\leqq j\leqq n)$ are in V_1, where k=dimension of V_0. Then, by the canonical coordinate with respect to $e_1,\cdots,e_n$, any element $\sigma\in\Pi$ has the following form,*

$$\begin{array}{c} \\ 1 \\ \vdots \\ k \\ k+1 \\ \vdots \\ n \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} 1\cdots\cdots k & k+1\cdots\cdots n & \end{array} \\ \left(\begin{array}{c:c:c} A & 0 & \xi \\ \hdashline 0 & I_{n-k} & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right)\end{array}$$

where A is a (k, k)-orthogonal matrix, I_{n-k} is the identity matrix of $(n-k, n-k)$-type and ξ is a k-dimensional real vector. Hence for a linear map ϕ of the following form,

$$\begin{array}{c} \\ 1 \\ \vdots \\ k \\ k+1 \\ \vdots \\ n \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} 1\cdots\cdots k & k+1\cdots\cdots n & \end{array} \\ \left(\begin{array}{c:c:c} I_k & 0 & 0 \\ \hdashline 0 & T & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right)\end{array}$$

we have $\phi\circ\sigma\circ\phi^{-1}=\sigma$ for any $\sigma\in\Pi$, where I_k is the identity matrix of (k, k)-type and T is a non-singular matrix.

PROOF. For $v\in V$ and $\sigma\in\Pi$ we write

$$\sigma v=A(\sigma)v+a(\sigma),$$

where $a(\sigma)\in V$ and $A(\sigma)$ is an orthogonal transformation of the vector space V. Denoting the origin by $\boldsymbol{O}$, we have

$$a(\sigma)\in V_{\boldsymbol{O}}\subset V_0\,.$$

Also by $\sigma(\sigma'v-v)=(\sigma\sigma')v-v-(\sigma v-v)+a(\sigma)\in V_0$, for σ, $\sigma'\in\Pi$, we have

$$\sigma V_0=V_0 \quad\text{and}\quad A(\sigma)V_0=V_0\,.$$

Since $A(\sigma)$ is orthogonal, it follows

$$A(\sigma)V_1=V_1\,.$$

Next for $\sigma\in\Pi$ and $w\in V_1$,

$$(A(\sigma)-1)w=(\sigma w-w)-a(\sigma)\in V_0\cap V_1=\{0\}\,,$$

and we have

$$A(\sigma)=\text{identity on } V_1\,.$$

The last assertion is obvious.

Let Π be an FPDA-group on a Riemannian manifold A and d a distance function on A. A geodesic from a to b is called, by definition, *minimal* if its length is equal to $d(a, b)$. In this paper a geodesic is always parametrized by the arc length from the starting point. For a fixed $a_0\in A$, we consider the set $\Omega(a_0)$ consisting of all minimal geodesics each of which issues from a_0 and ends at $\sigma(a_0)$ for some $\sigma\in\Pi-\{\text{id}\}$, where id is the identity. Since Π acts on A properly discontinuously, the subset $\{\sigma(a_0):\sigma\in\Pi-\{\text{id}\}$ and $d(a_0,\sigma(a_0))\leqq r\}$ of A is a finite set where r is a positive number. So any subset of $\{d(a_0,\sigma(a_0)):\sigma\in\Pi-\{\text{id}\}\}$ has a minimal element because any element of the set is positive. So we have the following lemma.

LEMMA 5. *Let Π_1 and Π_2 be FPDA-groups on a Riemannian manifold A. For $a\in A$, let*

$$\Omega_1(a)=\left\{\begin{array}{c} c : c \text{ is a minimal geodesic segment from } a \text{ to } \sigma(a) \\ \text{for some } \sigma\in\Pi_1-\{\text{id}\}\end{array}\right\}$$

and

$$\Omega_2(a)=\left\{\begin{array}{c} c : c \text{ is a minimal geodesic segment from } a \text{ to } \sigma(a) \\ \text{for some } \sigma\in\Pi_2-\{\text{id}\}\end{array}\right\}.$$

Then there are the shortest elements in any subset of $\Omega_1(a)$ and in any subset of $\Omega_2(a)$. Further, if there is an isometry ϕ of A with $\phi\Pi_1\phi^{-1}=\Pi_2$, then $\phi(\Omega_1(a))=\Omega_2(\phi(a))$ holds, where $(\phi(c))(t)=\phi(c(t))$ for $c\in\Omega_1(a)$.

§ 2. Proof of Theorem.

Let M, N and S be Riemannian manifolds such that $M\times S$ is isometric to

$N\times S$. By Lemma 1, the universal Riemannian covering manifold of M is isometric to the one of N. So let X be the universal Riemannian covering manifold of M and N, and let Y be the one of S. Let Γ_1, Γ_2 and Γ_3 be the deck transformation groups of M, N and S respectively. Then Γ_1 and Γ_2 are subgroups of the isometry group of X, Γ_3 is a subgroup of the isometry group of Y, and $\Gamma_1\times\Gamma_3$ and $\Gamma_2\times\Gamma_3$ are regarded naturally as the deck transformation groups of $M\times S$ and $N\times S$ respectively. By Lemma 2 and by the assumption, there is an isometry g of $X\times Y$ such that $g(\Gamma_1\times\Gamma_3)g^{-1}=\Gamma_2\times\Gamma_3$. To prove the theorem, it is sufficient to prove the theorem for a Riemannian manifold S which is never isometric to the Riemannian direct product of any two Riemannian manifolds of positive dimension. So we assume that S is never isometric to the Riemannian direct product of any two Riemannian manifolds of positive dimension. If an isometry $\tilde{g}$ of $X\times Y$ satisfies Conditions 1 and 2 in Lemma 3, then by the above assumption, it is the case (1) in Lemma 3. Therefore by Lemma 2, M is isometric to N.

We shall construct such an isometry $\tilde{g}$. Let X_0 and Y_0 be the Euclidean parts, and X' and Y' the non-Euclidean parts of the de Rham decompositions of X and Y, respectively. Then $X_0\times Y_0$ is the Euclidean part and $X'\times Y'$ is the non-Euclidean part of $X\times Y$. Identifying an isometry of $X\times Y$ with an isometry of $(X_0\times Y_0)\times(X'\times Y')$, an isometry σ of $X\times Y$ is $\sigma_0\times\sigma'$ where σ_0 is an isometry of $X_0\times Y_0$ and σ' is an isometry of $X'\times Y'$. We call σ_0 the Euclidean component of σ. Let $g=g_0\times g'$ and

$$\Gamma_i^*=\{\sigma_0 : \sigma_0 \text{ is the Euclidean component of some } \sigma\in\Gamma_i\}\ ,$$

for $i=1, 2, 3$. Then by the assumption, $g_0(\Gamma_1^*\times\Gamma_3^*)g_0^{-1}=\Gamma_2^*\times\Gamma_3^*$ holds.

Let $X'=A_1\times\cdots\times A_s$ and $Y'=A_{s+1}\times\cdots\times A_{s+t}$ be the de Rham decompositions of X' and Y' respectively. Then there are a permutation τ of $\{1, \cdots, s+t\}$ and isometries ϕ_i from A_i to $A_{\tau(i)}$ $(i=1, \cdots, s+t)$ such that

$$g'(a_1, \cdots, a_s, a_{s+1}, \cdots, a_{s+t})=(\phi_{\tau^{-1}(1)}(a_{\tau^{-1}(1)}), \cdots, \phi_{\tau^{-1}(s+t)}(a_{\tau^{-1}(s+t)}))\ .$$

Let

$$X_1'=\prod_{\substack{1\leq i\leq s\\ 1\leq\tau(i)\leq s}} A_i,\qquad X_2'=\prod_{\substack{1\leq i\leq s\\ s+1\leq\tau(i)\leq s+t}} A_i\,,$$

$$Y_1'=\prod_{\substack{s+1\leq i\leq s+t\\ s+1\leq\tau(i)\leq s+t}} A_i,\qquad Y_2'=\prod_{\substack{s+1\leq i\leq s+t\\ 1\leq\tau(i)\leq s}} A_i\,.$$

Then

$$X'=X_1'\times X_2' \quad\text{and}\quad Y'=Y_1'\times Y_2'\,,$$

and these decompositions satisfy Condition 2 in Lemma 3 for the isometry g' of $X'\times Y'$. So if there is an isometry $\tilde{g}_0$ of $X_0\times Y_0$ which satisfies Condition 2 in Lemma 3 as well as an equality $\tilde{g}_0\circ\sigma_0\circ\tilde{g}_0^{-1}=g_0\circ\sigma_0\circ g_0^{-1}$ for any $\sigma_0\in\Gamma_1^*\times\Gamma_3^*$, then the theorem holds. In fact, the isometry $\tilde{g}=\tilde{g}_0\times g'$ of $X\times Y$ satisfies Conditions 1 and 2 in Lemma 3. We shall construct such an isometry $\tilde{g}_0$ in the following manner.

We can consider X_0 and Y_0 as vector spaces V_1 and V_2 for some fixed origins respectively. Let W_1 and W_2 be vector spaces whose underlying spaces are X_0 and Y_0, and whose origins are the images of the origins of V_1 and V_2 under g_0 respectively. Then g_0 is a linear isometry from $V=V_1+V_2$ to $W=W_1+W_2$. If there are a basis $\{e_1, \cdots, e_n\}$ of V and a linear isometry $\tilde{g}_0$ from V to W satisfying the following conditions,

$$(*)\qquad \begin{cases} (0)\quad \tilde{g}_0\circ\sigma_0\circ\tilde{g}_0^{-1}=g_0\circ\sigma_0\circ g_0^{-1} \qquad \text{for any} \quad \sigma_0\in\Gamma_1^*\times\Gamma_3^*, \\ (1)\quad \text{each } e_i \text{ is in } V_1 \text{ or } V_2, \\ (2)\quad \text{each } \tilde{g}(e_i) \text{ is in } W_1 \text{ or } W_2, \end{cases}$$

then $\tilde{g}_0$ satisfies Conditions 1 and 2 in Lemma 3.

Next we shall construct such a basis and such a linear map. By the method of Lemma 4, we obtain linear subspaces V_v, V_0 of V for $\Gamma_1^*\times\Gamma_3^*$, and linear subspaces W_w, W_0 of W for $\Gamma_2^*\times\Gamma_3^*$. Since $g_0(\Gamma_1^*\times\Gamma_3^*)g_0^{-1}=\Gamma_2^*\times\Gamma_3^*$, $g_0(V_v)=W_{g_0(v)}$ and $g_0(V_0)=W_0$. If we choose a basis $\{\tilde{e}_1, \cdots, \tilde{e}_m\}$ of V_v such that, for each i with $1\leqq i\leqq m=\dim V_v$,

(1) $\tilde{e}_i$ is in V_1 or V_2,

(2) $g_0(\tilde{e}_i)$ is in W_1 or W_2,

then it is easy to choose a basis of V_0 such that it satisfies the above condition. Let $\{e_1, \cdots, e_k\}$ be such a basis of V_0, let V' be the orthogonal complement of V_0 in V and let W' be the orthogonal complement of W_0 in W. Since $\{e_1, \cdots, e_k\}$ is a basis of V_0 such that for each i $(1\leqq i\leqq k)$, e_i is in V_1 or V_2, we can choose an orthonormal basis $\{e_{k+1}, \cdots, e_n\}$ of V' such that e_i is in V_1 or V_2 for each i $(k+1\leqq i\leqq n)$. Similarly we can choose an orthonormal basis $\{f_{k+1}, \cdots, f_n\}$ of W' such that f_i is in W_1 or W_2 for each i $(k+1\leqq i\leqq n)$. We define a linear isometry $\tilde{g}_0$ from V to W as follows,

$$\begin{cases} \tilde{g}_0|_{V_0}=g_0, \\ \tilde{g}_0(e_i)=f_i \qquad (i=k+1, \cdots, n). \end{cases}$$

Then by Lemma 4, $\tilde{g}_0$ satisfies (0) of $(*)$. So the basis $\{e_1, \cdots, e_n\}$ and the isometry $\tilde{g}_0$ satisfy the condition $(*)$.

Lastly we shall find out a basis $\{\tilde{e}_1, \cdots, \tilde{e}_m\}$ of V_v such that, for each i $(1\leqq i\leqq m$; m is the dimension of $V_v)$,

(1) $\tilde{e}_i$ is in V_1 or V_2,

(2) $g_0(\tilde{e}_i)$ is in W_1 or W_2.

Let (x, y) be a point of $X\times Y$ whose Euclidean component with respect to the de Rham decomposition $(X_0\times Y_0)\times(X'\times Y')$ is v. Let $(x', y')=g(x, y)$. By the method in Lemma 5, we obtain $\Omega_1(x, y)$ and $\Omega_2(x', y')$ for $\Pi_1=\Gamma_1\times\Gamma_3$ and $\Pi_2=$

$\Gamma_2\times\Gamma_3$ respectively. Then $g(\Omega_1(x, y))=\Omega_2(x', y')$ because $g(\Gamma_1\times\Gamma_3)g^{-1}=\Gamma_2\times\Gamma_3$. By Lemma 5, we can carry out the following method.

Let c_1 be one of the shortest elements of $\Omega_1(x, y)$. Then c_1 is a minimal geodesic from (x, y) to $(\gamma_1(x), \gamma_3(y))$ for some $\gamma_1\in\Gamma_1$ and $\gamma_3\in\Gamma_3$. If neither γ_1 nor γ_3 is the identity, then a minimal geodesic from (x, y) to $(\gamma_1(x), y)$ is in $\Omega_1(x, y)$ and whose length is smaller than c_1, a contradiction. Hence γ_1 or γ_3 is the identity, i.e. $\dot{c}_1(0)$ is tangent to $X\times\{y\}$ or $\{x\}\times Y$. Next we consider the subset $\Omega_1^{(1)}$ of $\Omega_1(x, y)$ consisting of all elements whose initial vectors are not contained in the linear subspace spanned by $\dot{c}_1(0)$. Let c_2 be one of the shortest elements of $\Omega_1^{(1)}$. Then, similarly to c_1, $\dot{c}_2(0)$ is tangent to $X\times\{y\}$ or $\{x\}\times Y$. When we have chosen $c_1, \cdots, c_k$, let c_{k+1} be one of the shortest elements of $\Omega_1^{(k)}$, where $\Omega_1^{(k)}$ is the subset of $\Omega_1(x, y)$ consisting of all elements whose initial vectors are not contained in the linear subspace spanned by $\dot{c}_1(0), \cdots, \dot{c}_{k-1}(0)$ and $\dot{c}_k(0)$. Then similarly to c_1, $\dot{c}_{k+1}(0)$ is tangent to $X\times\{y\}$ or $\{x\}\times Y$. Let $T\Omega_1(x, y)$ and $T\Omega_2(x', y')$ be the linear subspaces spanned by $\{\dot{c}(0); c\in\Omega_1(x, y)\}$ and $\{\dot{d}(0); d\in\Omega_2(x', y')\}$ respectively. Let $d_i=g(c_i)$ for $i=1, \cdots, l$, where l is the dimension of $T\Omega_1(x, y)$. Then, because g is an isometry, $d_1, \cdots, d_l$ are ones which are chosen by the above method, i.e. d_1 is an element of $\Omega_2(x', y')$ whose length is minimum in $\Omega_2(x', y')$ and so on. Hence for $i=1, \cdots, l$, $\dot{d}_i(0)$ is tangent to $X\times\{y'\}$ or $\{x'\}\times Y$ at (x', y') and $\{\dot{d}_1(0), \cdots, \dot{d}_l(0)\}$ is a basis of $T\Omega_2(x', y')$.

For a differentiable manifold A and a point $a\in A$, let T_aA denote the tangent space to A at a. Then $T_{(x,y)}(X\times Y)$ is naturally identified with $T_v(X_0\times Y_0)+T_p(X'\times Y')$ where p is the component of the non-Euclidean part of (x, y). Let $\pi_{(x,y)}$ be the projection from $T_{(x,y)}(X\times Y)$ to $T_v(X_0\times Y_0)$ and $\pi_{(x',y')}$ the projection from $T_{(x',y')}(X\times Y)$ to $T_{g_0(v)}(X_0\times Y_0)$ with respect to the above identifications. The tangent space at a point of a vector space is naturally identified with the original vector space, so $T_v(X_0\times Y_0)$ and $T_{g_0(v)}(X_0\times Y_0)$ are naturally identified with V and W respectively. Then, under this identification,

(1) for $\xi\in T_v(X_0\times Y_0)$, $dg_0\xi=g_0\xi$,

(2) an element ξ of $T_v(X_0\times Y_0)$ tangents to X_0 or Y_0 if and only if ξ is contained in V_1 or V_2 respectively,

(3) an element η of $T_{g_0(v)}(X_0\times Y_0)$ tangents to X_0 or Y_0 if and only if η is contained in W_1 or W_2 respectively,

(4) $V_v=\pi_{(x,y)}T\Omega_1(x, y)$ and $W_{g_0(v)}=\pi_{(x',y')}T\Omega_2(x', y')$.

Let $\xi_i=\pi_{(x,y)}\dot{c}_i(0)$ and $\eta_i=\pi_{(x',y')}\dot{d}_i(0)$ for $i=1, \cdots, l$. Then $\eta_i=dg_0(\xi_i)$ because $dg(\dot{c}_i(0))=\dot{d}_i(0)$. Let $\{\xi_{i_1}, \cdots, \xi_{i_m}\}$ be a maximal subset of $\{\xi_1, \cdots, \xi_l\}$ such that $\xi_{i_1}, \cdots, \xi_{i_m}$ are linearly independent. Then $\{\eta_{i_1}, \cdots, \eta_{i_m}\}$ is so because of $\eta_i=dg_0(\xi_i)$. We denote $\tilde{e}_j=\xi_{i_j}$ and $\tilde{f}_j=\eta_{i_j}$ for $j=1, \cdots, m$. Since $V_v=\pi_{(x,y)}T\Omega_1(x, y)$ and $W_{g_0(v)}=\pi_{(x',y')}T\Omega_2(x', y')$, $\{\tilde{e}_1, \cdots, \tilde{e}_m\}$ is a basis of V_v and $\{\tilde{f}_1, \cdots, \tilde{f}_m\}$

is a basis of $W_{g_0(v)}$. By the fact that, for $i=1, \cdots, l$, $\dot{c}_i(0)$ is tangent to $X\times\{y\}$ or $\{x\}\times Y$ and $\dot{d}_i(0)$ is tangent to $X\times\{y\}$ or $\{x\}\times Y$, $\tilde{e}_i$ is in V_1 or V_2 and $\tilde{f}_j$ is in W_1 or W_2 for $j=1, \cdots, m$. This completes the proof.

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