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Non-separating incompressible tori in 3-manifolds

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1. Introduction.

In [1] Haken has shown that if a closed, connected 3-manifold M is not irreducible, then there exists such an essential 2-sphere in M that intersects a fixed Heegaard surface of M in a circle. Ochiai [4] has extended this result for a 2-sided projective plane in M. In this direction, we shall show that for a 2-sided, non-separating, incompressible torus and a genus two Heegaard splitting of M, the same result holds.

THEOREM 1. *Let M be a closed, connected (possibly, non-orientable) 3-manifold with a Heegaard splitting $(V_1, V_2; F)$ of genus two. Assume that M contains a 2-sided, non-separating, incompressible torus T. Then there exists a 2-sided, non-separating, incompressible torus T' which intersects F in a circle.*

As an application, we shall show that any orientable, closed 3-manifold which has a Heegaard splitting of genus two and contains a non-separating, incompressible torus is obtained by pasting boundary components of two bridge link space by a certain type of homeomorphism and performing a Dehn surgery along the two meridian loops of this link (Theorem 2).

As a consequence of Theorem 2, we have

COROLLARY. *If an orientable, closed 3-manifold M with a Heegaard splitting of genus two contains a non-separating, incompressible torus, then M is a 2-fold branched covering space of $S^2\times S^1$ branched along a 1-manifold.*

I would like to express my gratitude to Prof. M. Ochiai for helpful conversations.

2. Preliminaries.

Throughout this paper, we will work in the piecewise linear category. A Heegaard splitting of a closed, connected 3-manifold M is a pair $(V_1, V_2; F)$, where V_i $(i=1, 2)$ is a three dimensional orientable or nonorientable handlebody such that $M=V_1\cup V_2$ and $V_1\cap V_2=\partial V_1=\partial V_2=F$. Then F is called a Heegaard surface of M. The first Betti number of V_1 is called the genus of the splitting.

It is known that any closed, connected 3-manifold has a Heegaard splitting.

For the definition of standard terms in three dimensional topology and link theory, we refer to [2], [6]. For the definition of a hierarchy for a 2-manifold and an isotopy of type A, we refer to [3].

3. Proof of Theorem 1.

Let $T_i=T\cap V_i$ $(i=1, 2)$. Since T is incompressible in M, we may suppose that T_i is incompressible in V_i. Furthermore, by moving T by a sequence of isotopies of type A, we may suppose that each component of T_1 is a disk. Then as in [3] we have the hierarchy $(T_2^{(0)}, \alpha_0), \cdots, (T_2^{(m)}, \alpha_m)$ for T_2 which gives rise to a sequence of isotopies of T in M where the first isotopy is of type A at $\alpha_0, \cdots$, and the $(m+1)$-st isotopy is of type A at α_m. In addition, we may suppose that $\alpha_i\cap\alpha_j=\emptyset$ $(i\neq j)$. So, we consider each α_i to be an arc on T_2.

We say that α_i is of type 1 if α_i joins distinct components of ∂T_2, α_i is of type 2 if α_i joins one component S of ∂T_2 and there is an arc β in S such that $\partial\beta=\partial\alpha_i$, $\beta\cup\alpha_i$ bounds a disk on T, α_i is of type 3 if α_i joins one component S of ∂T_2 and there is an arc β in S such that $\partial\beta=\partial\alpha_i$, $\beta\cup\alpha_i$ cuts T into an annulus. We easily see that each α_i must be one of type 1, type 2, or type 3.

We say that α_i is a *d-arc* if α_i is of type 1 and there is a component S of ∂T_2 such that α_i is an only arc that joins S (see Figure 1).

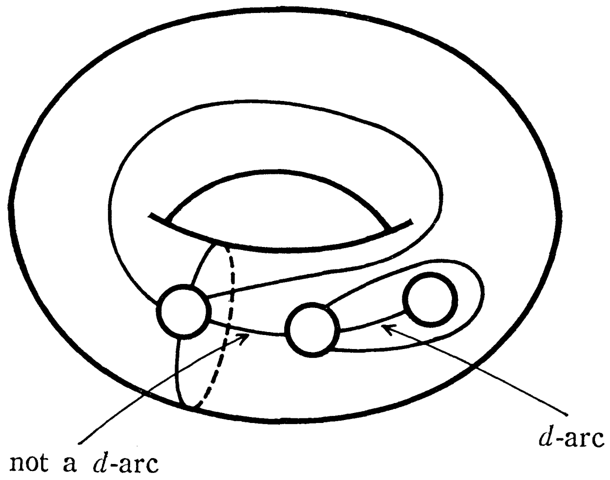


Figure 1.

Now, suppose that T_1 has more than one components.

LEMMA 3.1. *If some α_i is a d-arc, then there is an ambient isotopy h_t $(0\leqq t\leqq 1)$ of M such that each component of $h_1(T)\cap V_1$ is a disk and the number of the components of $h_1(T)\cap V_1$ is less than that of $T\cap V_1$.*

PROOF. This can be proved by using the argument of the inverse operation of an isotopy of type A defined in [4].

LEMMA 3.2. *If α_0 is of type* 1 *or type* 2, *then there is an ambient isotopy* h_t $(0\leqq t\leqq 1)$ *of* M *such that each component of* $h_1(T)\cap V_1$ *is a disk and the number of the components of* $h_1(T)\cap V_1$ *is less than that of* $T\cap V_1$.

PROOF. If α_0 is of type 1, then the conclusion follows immediately by performing an isotopy of type A at α_0. Assume that α_0 is of type 2. Then there is an arc β in ∂T_2 such that $\partial\beta=\partial\alpha_0$, $\alpha_0\cup\beta$ bounds a planar surface P in T_2. Since each α_i is an essential arc of T_2, some α_j on P is a d-arc. Hence, the conclusion follows by Lemma 3.1.

LEMMA 3.3. *Suppose that α_0 is of type* 3 *and one of the following conditions is satisfied:*

(i) α_1 *is of type* 1,

(ii) α_1 *is of type* 2,

(iii) α_1 *is of type* 3 *and* α_1 *intersects the same component of* ∂T_2 *that* α_0 *intersects.*

Then there is an ambient isotopy h_t $(0\leqq t\leqq 1)$ *of* M *such that each component of* $h_1(T)\cap V_1$ *is a disk and the number of the components of* $h_1(T)\cap V_1$ *is less than that of* $T\cap V_1$.

PROOF. If (i) holds, then the Lemma can be proved using the argument of the inverse operation of an isotopy of type A defined in [4]. If (ii) holds, then the conclusion follows by the same argument of the proof of Lemma 3.2. If (iii) holds, then $\alpha_0\cup\alpha_1$ cuts T_2 into two planar surfaces or one planar surface. In either case, we see that some α_i is a d-arc. Hence, the conclusion follows by Lemma 3.1.

REMARK. There exists an example of a hierarchy $(T_2^{(0)}, \alpha_0), \cdots, (T_2^{(m)}, \alpha_m)$ such that each α_i is not a d-arc (see Figure 2).

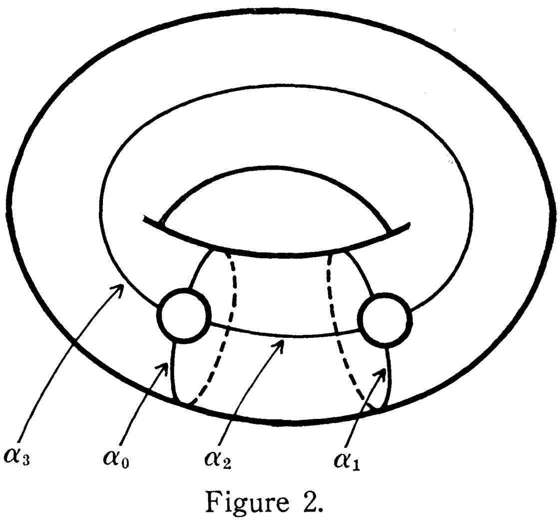


Figure 2.

Now, let T' be a 2-sided, non-separating, incompressible torus in M such that among all 2-sided, non-separating, incompressible tori in M which intersect V_1 in disks the number of the components of $T'\cap V_1$ is minimum.

Let $T'_i=T'\cap V_i$ $(i=1, 2)$. Assume that T'_1 has more than one component. Then $T'_1=D_1\cup\cdots\cup D_n$ $(n\geqq 2)$, where $D_1, \cdots, D_n$ are mutually disjoint, properly embedded disks in V_1. We have the hierarchy $(T_2'^{(0)}, \alpha'_0), \cdots, (T_2'^{(l)}, \alpha'_l)$ for T'_2 as above. By Lemmas 3.2 and 3.3, α'_0 and α'_1 are of type 3 and we may suppose that α'_0 joins points on ∂D_1 and α'_1 joins points on ∂D_2.

LEMMA 3.4. *For any i, j $(1\leqq i<j\leqq n)$, $\{D_i, D_j\}$ is not a complete system of meridian disks of V_1.*

PROOF. Suppose that for some i, j, $\{D_i, D_j\}$ is a complete system of meridian disks of V_1 (i.e. $D_i\cup D_j$ cuts V_1 into a 3-cell D^3). Let $T^{(1)}$ be the image of T' after an isotopy of type A at α'_0. Then $T^{(1)}\cap V_1=A_1\cup D_2\cup\cdots\cup D_n$, where A_1 is an annulus properly embedded in V_1.

We claim that A_1 can be pushed into D^3. By the definition of an isotopy of type A, there is a disk D in V_2 such that $D\cap T_2=\alpha'_0$, $D\cap\partial V_2=\beta$ where $\alpha'_0\cap\beta=\partial\alpha'_0=\partial\beta\subset\partial D_1$, $\alpha'_0\cup\beta=\partial D$. Then β must join D_1 from one side of D_1, for otherwise there exists a simple loop which is contained in a regular neighborhood of A_1 in V_1 and intersects A_1 transversely in a single point (see Figure 3) and this contradicts the fact that $T^{(1)}$ is 2-sided in M. Hence, by moving $T^{(1)}$ by a small isotopy we may suppose that $A_1\cap D_k=\emptyset$ $(1\leqq k\leqq n)$ and this establishes the claim.

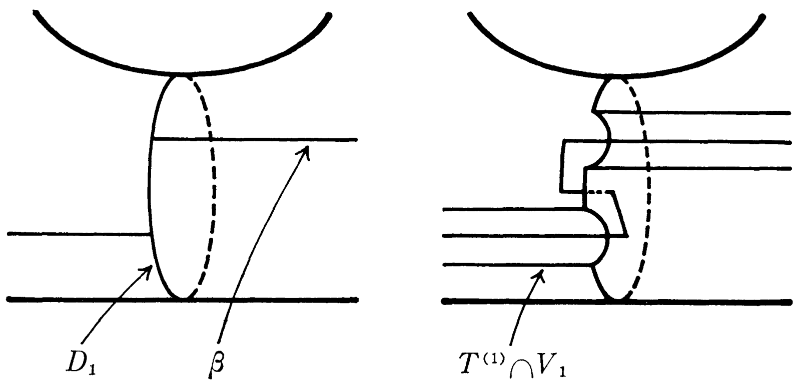


Figure 3.

On the other hand A_1 is incompressible in V_1 for α'_0 is of type 3. This is a contradiction.

By Lemma 3.4 there are following three possible cases.

Case 1. $\{D_1, \cdots, D_n\}$ has only one parallel class in V_1, and each D_i cuts V_1 into two solid tori.

Case 2. $\{D_1, \cdots, D_n\}$ has two parallel classes in V_1, and one of them is represented by a meridian disk of V_1, the other is represented by a disk which cuts V_1 into two solid tori.

Case 3. $\{D_1, \cdots, D_n\}$ has only one parallel class in V_1, and each D_i is a meridian disk of V_1.

Let $T^{(1)}$, A_1 be those defined in the proof of Lemma 3.4. A_1 is an incompressible annulus properly embedded in V_1. By Haken's theorem (see Corollary II. 10 of [3]) and the fact that M has a Heegaard splitting of genus two, we see that M is irreducible. Hence, each 2-sided, nonseparating torus in M is incompressible.

Now, we shall derive a contradiction in each of above cases. Then we complete the proof of Theorem 1.

Case 1. In this case, there is an annulus A_1' in F such that $A_1'\cap T^{(1)}=A_1'\cap A_1=\partial A_1'=\partial A_1$ (see Figure 4). We get a 2-sided, non-separating torus $\bar{T}$ by exchanging A_1 on $T^{(1)}$ for A_1'. We can move $\bar{T}$ by a small isotopy so that each component of $\bar{T}\cap V_1$ is a disk and the number of components of $\bar{T}\cap V_1$ is less than that of $T'\cap V_1$. This contradicts the minimality of T'.

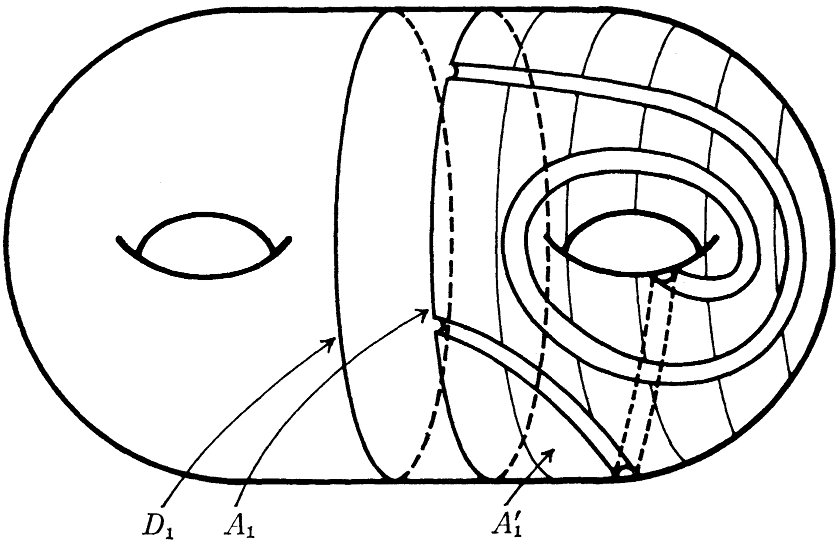


Figure 4.

Case 2. Since A_1 is an incompressible annulus in V_1, D_1 must separate V_1 into two solid tori. So we have a contradiction as in Case 1.

Case 3. Let $T^{(2)}$ be the image of $T^{(1)}$ after an isotopy of type A at α_1'. Then $T^{(2)}\cap V_1=A_1\cup A_2\cup D_3\cup\cdots\cup D_n$, where A_2 is an incompressible annulus properly embedded in V_1. We have two subcases.

Case 3.1. A_1 and A_2 are parallel in V_1.

In this case, there is an annulus A such that A is contained in the interior of V_1, $A\cap T^{(2)}=\partial A$ and one component of ∂A is in A_1 and the other is in A_2

(see Figure 5). The annulus A cuts $T^{(2)}$ into two annuli A^1 and A^2. By pasting A^i ($i=1$, 2) and A along its boundary, we get a 2-sided torus T^i in M. Then either T^1 or T^2, say T^1, is nonseparating in M. Then $T^1 \cap V_1 = A^1 \cup D_{i1} \cup \cdots \cup D_{ik}$, where A^1 is an annulus and $\{D_{i1}, \cdots, D_{ik}\}$ $(k \leqq n-2)$ is a subset of $\{D_3, \cdots, D_n\}$. We easily see that by moving T^1 by an isotopy of type A there is a 2-sided, non-separating torus which intersects V_1 in $k+1$ disks. This contradicts the minimality of T'.

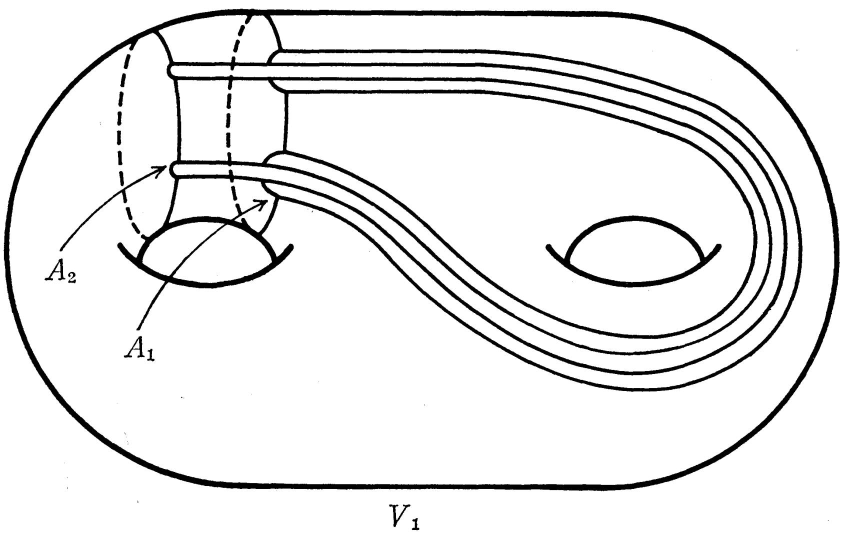


Figure 5.

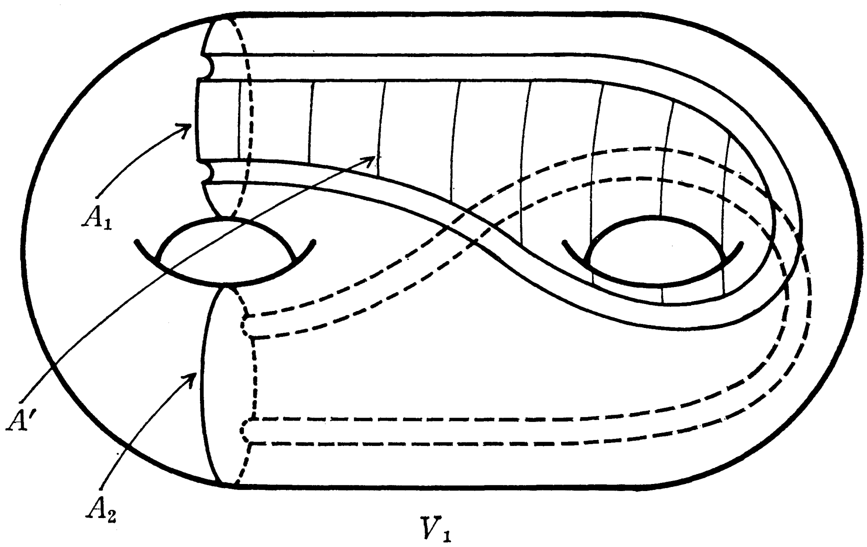


Figure 6.

Case 3.2. A_1 and A_2 are not parallel in V_1.

In this case, there is an annulus A' in F such that $A'\cap T^{(2)}=A'\cap(A_1\cup A_2)=\partial A'$ (see Figure 6). Then by pushing A' slightly into the interior of V_1, we have an annulus which has the same property as A in Case 3.1. So we have a contradiction as in Case 3.1.

4. Statement and proof of Theorem 2.

We can construct a closed, connected, orientable 3-manifold M as follows.

(*) Let $L=k_1\cup k_2$ be a two bridge link in S^3, $M(L)$ be the manifold obtained by removing the interior of the regular neighborhood of L from S^3. Let $\bar{m}_i$ ($i=1, 2$) be a meridian of the regular neighborhood of k_i, m_i' be a simple loop obtained by pushing $\bar{m}_i$ slightly into $M(L)$. Let M_1 be a closed, orientable 3-manifold obtained by pasting the two boundary components of $M(L)$ by a homeomorphism which takes $\bar{m}_1$ to $\bar{m}_2$, m_i ($i=1, 2$) be the image of m_i' in M_1. Then we get M by performing Dehn surgery on M_1 along $m_1\cup m_2$.

Then we have

LEMMA 4.1. *If M is obtained by the construction* (*) *then M has a Heegaard splitting of genus two.*

PROOF. We use the notations and symbols in the construction (*). Since L is a two bridge link, L is obtained as the union of the two trivial tangles. This is shown for the two bridge knot in 115p of [**6**]. And the argument holds for the two bridge link. Hence there is a disk with three holes B properly embedded in $M(L)$ such that each component of ∂B is homotopic to $\bar{m}_i$ ($i=1$ or 2) in $\partial M(L)$ and such that the closures of the components of $M(L)-B$ are two handlebodies. We denote them by V_1' and V_2'. Then $V_i'\cap\partial M(L)$ consists of two annuli A_1^i, A_2^i ($i=1, 2$). Since $\bar{m}_1$ and $\bar{m}_2$ are identified in M, we may suppose that A_1^i and A_2^i are identified in M. Moreover, we may suppose that $m_i'\subset V_i'$. If we identify A_1^i with A_2^i on $\partial V_i'$ then we get a genus two handlebody V_i''. And if we perform a Dehn surgery on V_i'' along $\bar{m}_i$ then we get a genus two handlebody V_i. Then $M=V_1\cup V_2$ and $V_1\cap V_2=\partial V_1=\partial V_2$.

This completes the proof of Lemma 4.1.

Let θ_1 be the collection of all 3-manifolds which are given by the construction (*), and θ_2 be the collection of all orientable 3-manifolds which have Heegaard splittings of genus two and contain non-separating, incompressible tori. Then

THEOREM 2. *Let θ_1 and θ_2 be as above, then θ_2 is a subcollection of θ_1. Moreover, the element of θ_1 that is not an element of θ_2 is homeomorphic to $S^2\times S^1$ or $S^2\times S^1\# S^2\times S^1$ or $S^2\times S^1\# Ln$, where $\#$ denotes a connected sum and S^n, Ln denote an n-sphere, a three dimensional lens space, respectively.*

Now, we shall prove Theorem 2. Let M be an element of θ_2, $(V_1, V_2; F)$

be a Heegaard splitting of genus two of M, T be a non-separating, incompressible torus in M. By Theorem 1, we may suppose that $T\cap V_1$ is a disk.

LEMMA 4.2. *$T\cap V_1$ is a meridian disk of V_1.*

PROOF. Since T is incompressible in M, $T\cap V_1$ is not parallel to a disk in ∂V_1. So assume that $T\cap V_1$ cuts V_1 into two solid tori. Since T is non-separating, there exists such a simple closed curve l in M that intersects T transversely in a single point p. We may suppose that p is contained in the interior of $T\cap V_1$. Let l' be an arc on l such that p is contained in the interior of l' and l' is contained in the interior of V_1. There is an ambient isotopy h_t $(0\leqq t\leqq 1)$ of M such that $h_1(l)$ is contained in V_1, $h_t|_{l'}=\mathrm{id}_{l'}$ $(0\leqq t\leqq 1)$ and $h_1(l)$ is in general position with respect to T. Since $l-l'$ does not intersect T and $\partial(l-l')$ is fixed by h_t, $h_1(l-l')$ intersects T even number of times. This contradicts the fact that $h_1(l)$ is contained in V_1 and $T\cap V_1$ separates V_1.

As in [3] we have a hierarchy $(T_2^{(0)}, \alpha_0)$, $(T_2^{(1)}, \alpha_1)$ for $T\cap V_2$ which gives rise to a sequence of isotopies of type A at α_0 and α_1. Note that α_i $(i=0, 1)$ is of type 3 defined in Section 3. Let T' be the image of T after an isotopy of type A at α_0, and $A_i=T'\cap V_i$ $(i=1, 2)$. Then A_i is a non-separating, incompressible annulus properly embedded in V_i. By cutting V_i along A_i we get a genus two solid torus V_i' (see Figure 7). Let A_i' and A_i'' be the copies of A_i on $\partial V_i'$.

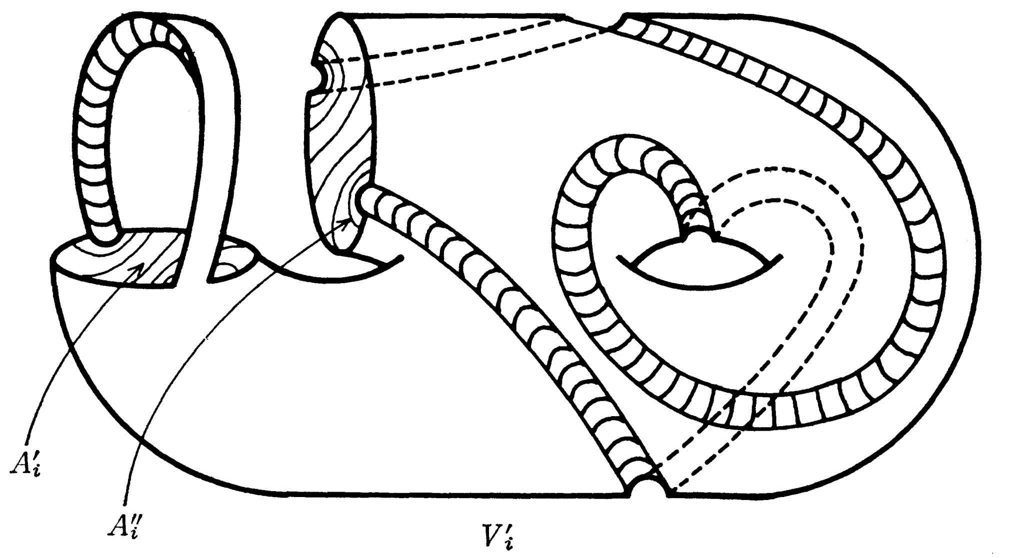


Figure 7.

Now, we will show that M can be given by the construction (*).

Let $c_i=c_{i1}\cup c_{i2}$ $(i=1, 2)$ be two disjoint trivial arcs properly embedded in a 3-cell B_i, $U_i=\mathrm{cl}(B_i-N(c_{i1}\cup c_{i2}\,; B_i))$, $A^{ij}=U_i\cap N(c_{ij}\,; B_i)$ $(j=1, 2)$, where $N(c\,; B_i)$ denotes the regular neighborhood of a polyhedron c in B_i. A^{ij} is an annulus on

∂U_i. Let m'_i be a simple closed curve obtained by pushing the core of A^{i2} into U_i. If we perform a proper Dehn surgery on U_i along m'_i then we get such a genus two handlebody U'_i that there exists a homeomorphism from U'_i to V'_i which takes the image of A^{i1} to A'_i and the image of A^{i2} to A''_i. Hence, the attaching homeomorphism $\partial V_1 \to \partial V_2$ of the Heegaard splitting $(V_1, V_2; F)$ induces a homeomorphism $\mathrm{cl}(\partial U_1-(A^{11}\cup A^{12}))\to \mathrm{cl}(\partial U_2-(A^{21}\cup A^{22}))$. If we paste U_1 and U_2 by this homeomorphism then we get a link space $M(L)$ where $L=k_1\cup k_2$ is a two bridge link (see [**6**] 115p). Let m_i $(i=1, 2)$ be the image of m'_i in $M(L)$. Then m_i is isotopic to the meridians of L. Since A^{i1} and A^{i2} are identified in M, we may suppose that m_i is isotopic to the meridian of k_i. Then by tracing the above procedure conversely we see that M can be given by the construction (*). This completes the proof of the first part of Theorem 2.

Let M be an element of θ_1 and not an element of θ_2. By the construction of M, there exists a non-separating torus T in M. Then by the loop theorem [**2**] there exists a non-separating 2-sphere in M. By Lemma 3.8 of [**2**], Haken's theorem ([**3**]) and Lemma 4.1 $M=S^2\times S^1\# M'$, where M' has a Heegaard splitting of genus one. Hence, the second part of Theorem 2 follows.

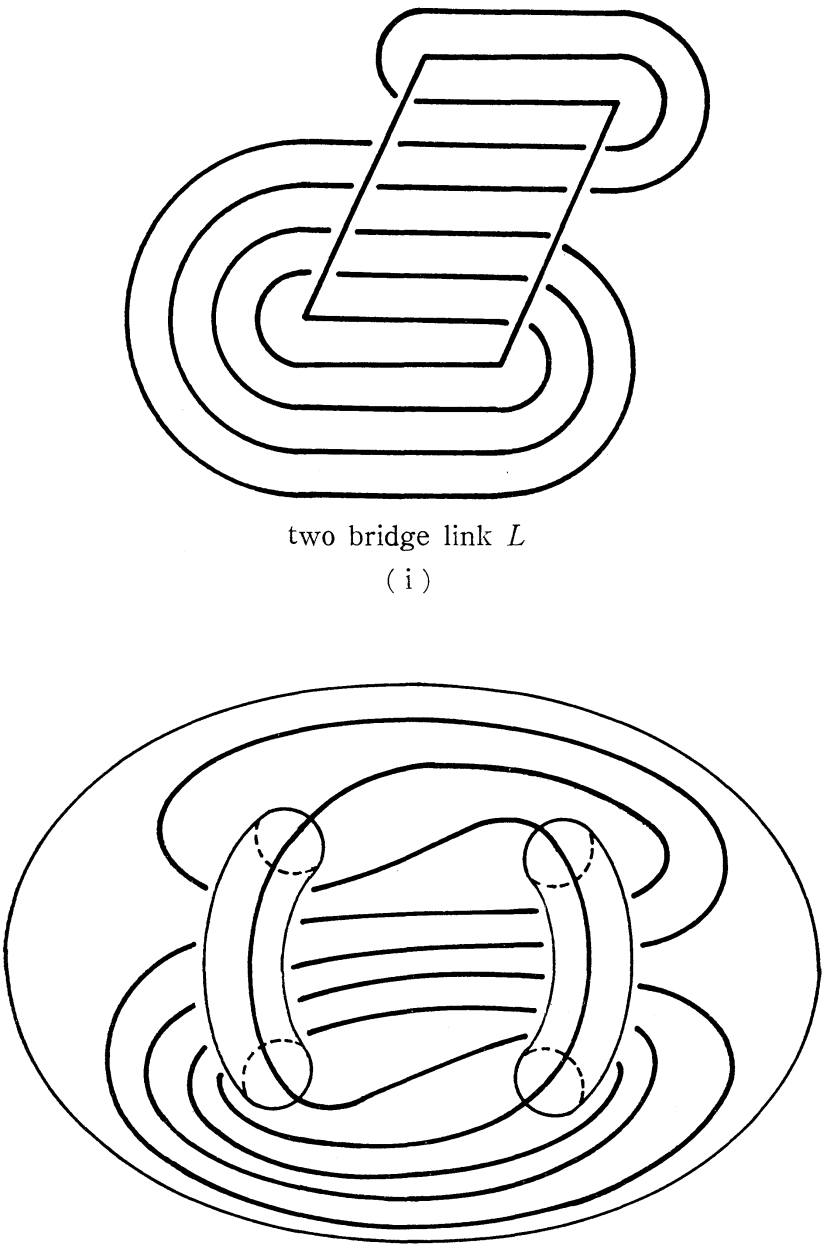
REMARK. By Theorem 2 and the fact that a torus bundle over S^1 contains a non-separating, incompressible torus, we have the following relations of inclusion.

$$\left\{\begin{array}{l}\text{closed orientable 3-manifolds}\\ \text{with Heegaard splittings of}\\ \text{genus two}\end{array}\right\}\supset\theta_2\supset\theta_1\supset\left\{\begin{array}{l}\text{torus bundles with}\\ \text{Heegaard splittings}\\ \text{of genus two}\end{array}\right\}$$

It is known that there exists infinitely many topologically distinct torus bundles with Heegaard splittings of genus two [**5**].

We note that there is an element of θ_1 which is not a torus bundle. We claim that if M is a torus bundle and T is a non-separating, incompressible torus in M then T cuts M into $T^2\times I$, where T^2, I denote the two dimensional torus, the unit interval $[0, 1]$, respectively. Since M is a torus bundle, there exists a non-separating, incompressible torus T' which cuts M into a $T^2\times I$, say M'. Since T and T' are incompressible and M is irreducible, by the standard arguments of the three dimensional topology we may suppose that $T\cap T'=\emptyset$ or T and T' intersect transversely, each component of $T\cap T'$ is an essential loop in T' (hence T) and each component of $T\cap M'$ is not boundary parallel in M'. If $T\cap T'=\emptyset$ then $T\subset M'$ and by [**1**] T and T' are parallel in M. Hence T cuts M into $T^2\times I$. If $T\cap T'\neq\emptyset$ then the image of T in M' is a system of parallel annuli $A_1, \cdots, A_r$ $(r\geqq 1)$. On the other hand, the closures of components of $T'-(T\cap T')$ are annuli $A'_1, \cdots, A'_r$. By cutting M' along $A_1\cup\cdots\cup A_r$ we get r solid tori. If we paste these solid tori along A'_i $(1\leqq i\leqq r)$ then we get $T^2\times I$. Hence M cut along T is $T^2\times I$ and we establish the claim.

If L is a non-trivial two bridge link, which is not the Hopf link, then the image of $\partial M(L)$ in M, say T, is a non-separating, incompressible torus and M cut along T is not a $T^2 \times I$. Hence M is not a torus bundle.



two bridge link L

(i)

genus two surface normally embedded in S^3

(ii)

Figure 8.

Let $L=k_1\cup\cdots\cup k_n$ be an n component link. L is called *strongly invertible* if there is an orientation preserving involution g of S^3 which satisfies

(i) $\mathrm{Fix}(g)$, the fixed point set of g, is a circle,

(ii) $g(k_i)=k_i$ $(1\leqq i\leqq n)$ and

(iii) $g|_{k_i}$ reverses the orientation of k_i for each i.

Then we have

LEMMA 4.3. *Every two bridge link is strongly invertible.*

PROOF. Let $L=k_1\cup k_2$ be a two bridge link and let $(V_1, V_2; F)$ be a genus two Heegaard splitting of S^3. Then by the definition of the two bridge link ([**6**]) we may suppose that $L\subset F$ and that L does not separate F (see Figure 8).

In [7] it is shown that if $(V_1', V_2'; F')$ is the genus two Heegaard splitting of a 3-manifold M' and α_1, α_2 are pairwise disjoint simple closed curves such that $\alpha_1\cup\alpha_2$ does not separate F then there is an orientation preserving involution h of M' such that $h(V_i)=V_i$ $(i=1, 2)$, $\mathrm{Fix}(h)$ is a 1-manifold, $h(\alpha_j)=\alpha_j$ and $h|_{\alpha_j}$ reverses the orientation of α_j $(j=1, 2)$. Hence there is an orientation preserving involution g of S^3 such that $\mathrm{Fix}(g)$ is a 1-manifold, $g(k_i)=k_i$ $(i=1, 2)$ and $g|_{k_i}$ reverses the orientation of k_i. By the Smith theory $\mathrm{Fix}(g)$ is a circle. Hence L is strongly invertible.

PROOF OF COROLLARY. By Theorem 2 an orientable, closed 3-manifold M which has a Heegaard splitting of genus two and contains a non-separating, incompressible torus is given by the construction (*). By Lemma 4.3 there is an involution h on $M(L)$ whose fixed point set K is a 1-manifold which intersects

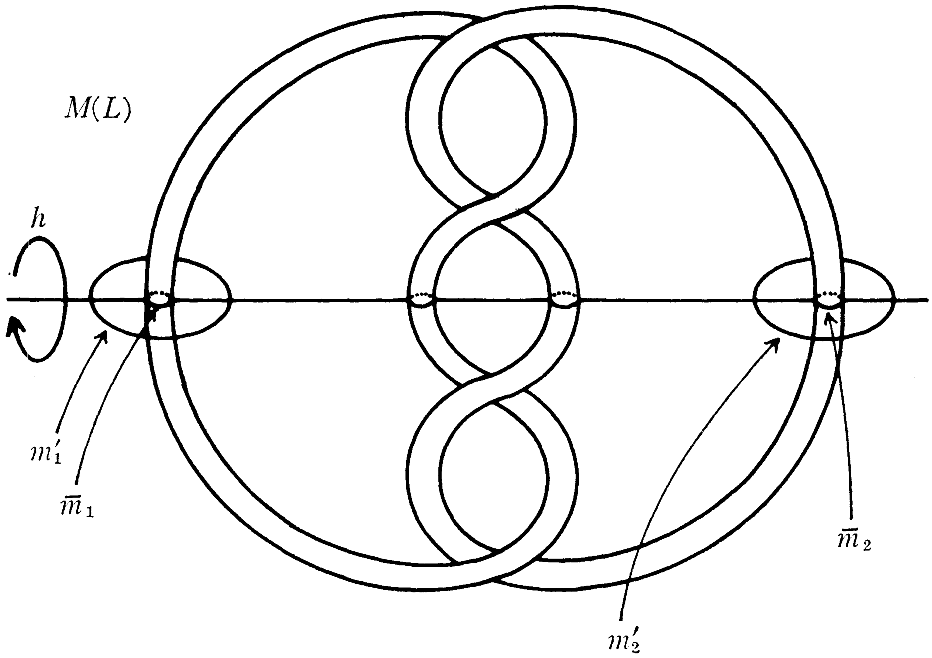


Figure 9.

each boundary component of $M(L)$ in four points. Moreover, we may suppose that each of m_i' and $\bar{m}_i$ $(i=1, 2)$ intersects K in two points and is invariant under h (see Figure 9). Then h induces an involution h' on M, and the quotient space of M under h' is $S^2 \times S^1$. This completes the proof of Corollary.

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