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# On the asymptotic behavior of incompressible viscous fluid motions past bodies

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## § 1. Introduction.

Let $\Omega$ be a domain exterior to a finite number of bodies in $E_3$ with the smooth boundary $\partial\Omega$. The motion of the incompressible viscous fluid in $\Omega$ is described by the following system of the Navier-Stokes equations for the velocity $\boldsymbol{u}=(u_1(x, t), u_2(x, t), u_3(x, t))$ of the fluid and the pressure $\boldsymbol{p}=\boldsymbol{p}(x, t)$;

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{u}-\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\operatorname{grad}\boldsymbol{p}=0 \\ \operatorname{div}\boldsymbol{u}=0 \end{cases} \qquad (x, t)\in Q_T,$$

where $\nu$ is a positive constant — "viscousity constant", $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=u_i\partial\boldsymbol{u}/\partial x_i$, $0<T\leqq\infty$ and $Q_T$ is the space time region $\Omega\times(0, T)$. Here and in what follows we use the conventional rule of tensor that repeated suffix means the summation with respect to the suffix.

We consider a flow $\boldsymbol{u}$ satisfying initial-boundary conditions;

$$(1.2)\qquad \boldsymbol{u}(x, 0)=\boldsymbol{a}(x), \qquad x\in\Omega,$$

$$(1.3)\qquad \boldsymbol{u}(x, t)=\boldsymbol{b}(x, t) \qquad x\in\partial\Omega,\ 0\leqq t<T,$$

$$(1.4)\qquad \boldsymbol{u}(x, t)\rightarrow\boldsymbol{u}_\infty \quad \text{as} \quad |x|\rightarrow\infty, \qquad 0\leqq t<T,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are given smooth and bounded functions such that $\operatorname{div}\boldsymbol{a}=0$ and $\boldsymbol{a}(x)=\boldsymbol{b}(x, 0)$ for $x\in\partial\Omega$, and $\boldsymbol{u}_\infty$ is a prescribed constant vector. We are mainly concerned with the decay rate of $|\boldsymbol{u}(x, t)-\boldsymbol{u}_\infty|$ as $|x|\rightarrow\infty$; for the existence of solutions, see [7], [11], [14], [15] and especially [9], [10], [17] and [18].

In the case that $\boldsymbol{b}(x, t)$ is independent of $t$, R. Finn [4, 5, 6] showed that if a stationary solution $\boldsymbol{u}_s$ of (1.1), (1.3), and (1.4) has finite Dirichlet norm; $\|\nabla\boldsymbol{u}_s\|_{L^2(\Omega)}<\infty$, and satisfies

$$(1.5)\qquad \boldsymbol{u}_s(x)=\boldsymbol{u}_\infty+O(|x|^{-\alpha})$$

where $\alpha$ is a constant, $\alpha>1/2$, then

(1.6) $$u_s(x)=u_\infty+O(|x|^{-1}(1+s_x)^{-1})$$

where $s_x=|x|-u_\infty\cdot x/|u_\infty|$. Later, K.I. Babenko [1] proved that (1.5) with $\alpha=1$ holds if $u_s$ has finite Dirichlet norm. Further, D. Clark [3] and K.I. Babenko-M.M. Vasil'ev [2] independently proved that (1.5) implies, for arbitrarily fixed $\varepsilon>0$,

(1.7) $$\text{rot}\, u_s(x)=(\mathrm{R}/4\pi)\nabla s_x\times f_0|x|^{-1}\exp(-\mathrm{R}s_x) + O(|x|^{-2}\exp(-(\mathrm{R}-\varepsilon)s_x)$$

where $\mathrm{R}=|u_\infty|/2\nu$ (the Reynolds number), $f_0$ is the vector of force exerted by the flow on the bodies, and the notation $\times$ is the vector product. The equalities (1.6) and (1.7) explain the existence of a paraboloidal wake region behind the bodies.

In the case of non-stationary solutions, G. Knightly [12, 13] obtained a space-time estimate of $|u(x,t)-u_\infty|$. From his result, one can deduce $|u(x,t)-u_\infty| \leqq M|x|^{-1/2}(1+s_x)^{-3/2+\varepsilon}$, where $M$ is a constant and $0<\varepsilon\leqq 1$. But it is impossible to deduce $|u(x,t)-u_\infty|\leqq M|x|^{-1}(1+s_x)^{-1}$. Moreover, his assumptions seem too strong. It remains to investigate whether every classical non-stationary solution of (1.1)-(1.4) satisfies $|u(x,t)-u_\infty|\leqq M|x|^{-1}(1+s_x)^{-1}$.

Before stating our results, we define a notation. By the equality

$$f(x,t)=g(x,t)+O(\psi(x)) \quad \text{in} \quad Q_T$$

we mean that there is a constant $M$ such that

$$|f(x,t)-g(x,t)|\leqq M|\psi(x)|, \qquad (x,t)\in Q_T.$$

We assume $0\notin\bar{\Omega}$ without loss of generality.

Let us state our results.

THEOREM 1. *Let $T<\infty$ and $u$ be a classical solution of* (1.1)-(1.4) *in $Q_T$ with $\nabla u\in L^\infty(0,T;L^2(\Omega))$. Suppose*

(1.8) $$a(x)=u_\infty+O(|x|^{-\lambda})$$

*where $\lambda>0$. Then*

(1.9) $$u(x,t)=u_\infty+O(|x|^{-\mu}) \quad \text{in} \quad Q_T$$

*where $\mu=\min(2,\lambda)$. If the total flux through $\partial\Omega$ is 0;*

(1.10) $$\iint_{\partial\Omega} u(x,t)\cdot n\,dx=0 \qquad 0\leqq t<T$$

*where $n$ is the outer normal vector on $\partial\Omega$, then we can take $\mu=\min(3,\lambda)$.*

THEOREM 2. *Let $T=\infty$. Let $u$ be a classical solution of* (1.1)-(1.4) *in $Q_\infty$ with $\nabla u\in L^\infty(0,\infty;L^2(\Omega))$. Suppose there exists a stationary solution $u_s$ of* (1.1) *and* (1.4) *with $\nabla u_s\in L^2(\Omega)$, such that*

(1.11) $$\boldsymbol{a}(x)=\boldsymbol{u}_s(x)+O(|x|^{-2}).$$

*If* $\boldsymbol{u}$ *satisfies the following conditions:* (i) *there is* $r$, $1\leqq r<3$, *such that*

(1.12) $$\boldsymbol{u}-\boldsymbol{u}_\infty\in L^\infty(0, \infty; L^r(\Omega))$$

*and* (ii)

(1.13) $$\lim_{|x|,\, t\to\infty}|\boldsymbol{u}(x, t)-\boldsymbol{u}_\infty|=0,$$

*then*

(1.14) $$\boldsymbol{u}(x, t)=\boldsymbol{u}_\infty+O(|x|^{-1}(1+s_x)^{-1}) \qquad in \quad Q_\infty.$$

REMARK 1. The assumptions (i) and (ii) seem reasonable. Indeed, the solutions constructed by Heywood [**10**] and Masuda [**17**] satisfy (1.13) and (1.12) with $r=2+\varepsilon$, $\varepsilon>0$.

REMARK 2. We shall give the decay rate of mean value of a weak solution (see § 3.3).

If $\boldsymbol{b}(x, t)$ is time-independent and $\boldsymbol{u}$ converges (as $t\to\infty$) to a stationary solution $\boldsymbol{u}_s$ of (1.1), (1.3) and (1.4), it is possible to ask whether $\boldsymbol{u}-\boldsymbol{u}_s$ decays like $|x|^{-1}(1+s_x)^{-1}t^{-\alpha}$, $\alpha>0$, as $t$, $|x|\to\infty$. Let us introduce a class $S(\alpha)$ : $\alpha>0$.

DEFINITION 1. Let $\boldsymbol{u}$, $\boldsymbol{p}$ be a classical solution of (1.1)-(1.4) in $Q_\infty$. Suppose $\boldsymbol{b}(x, t)$ is independent of $t$. Then $\boldsymbol{u}\in S(\alpha)$ if and only if there is a stationary solution $\boldsymbol{u}_s$, $\boldsymbol{p}_s$ of (1.1), (1.3) and (1.4) such that

(1.15) $$\sup_{x\in\Omega}|\boldsymbol{u}(x, t)-\boldsymbol{u}_s(x)|+\|\nabla\boldsymbol{u}(\cdot, t)-\nabla\boldsymbol{u}_s\|_{L^2(\Omega)} \leqq M(1+t)^{-\alpha}, \qquad t\geqq 0$$

(1.16) $$\iint_{\partial\Omega}\{|\boldsymbol{p}(x, t)-\boldsymbol{p}_s(x)|+|\nabla\boldsymbol{u}(x, t)-\nabla\boldsymbol{u}_s(x)|\}\,dx \leqq M(1+t)^{-\alpha}, \qquad t\geqq 0$$

where $M$ is a constant independent of $t$.

REMARK 3. J. Heywood [**9**, **10**] and K. Masuda [**17**, **18**] showed that if $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy some additional conditions, then there is a solution of (1.1)-(1.4) contained in $S(1/4)$.

COROLLARY 1. *In addition to the assumptions of Theorem* 2, *assume* $\boldsymbol{u}\in S(\alpha)$ *and*

(1.17) $$\boldsymbol{a}(x)=\boldsymbol{u}_s(x)+O(|x|^{-2-\beta})$$

*where* $\beta>0$. *Then*

(1.18) $$|\boldsymbol{u}(x, t)-\boldsymbol{u}_s(x)|\leqq M|x|^{-1}(1+s_x)^{-1}(1+t)^{-\gamma},$$

*for* $x\in\Omega$, $0\leqq t$, *where* $\gamma$ *is an arbitrary number satisfying the following conditions: if* $\beta\neq 1$, $\gamma\leqq\min(\beta/2, 1/2)$ *and* $\gamma<\alpha/3$ *and if* $\beta=1$, $\gamma<1/2$ *and* $\gamma<\alpha/3$. $M$ *is a constant depending on* $\boldsymbol{u}$ *and* $\gamma$.

The decay of vorticity can be obtained from Theorem 2 and a proposition of Babenko-Vasil'ev [2].

COROLLARY 2. *In addition to the assumptions of Theorem 2, assume*

$$\operatorname{rot}\boldsymbol{a}(x)=\operatorname{rot}\boldsymbol{u}_s(x)+O(|x|^{-\lambda}\exp(-\mu_1 s_x)) \tag{1.19}$$

*where* $0<\mu_1\leqq 1/2$, $2<\lambda$. *Then,*

$$\operatorname{rot}\boldsymbol{u}(x, t)=O(|x|^{-3/2}\exp(-\mu_2 s_x)) \qquad in \quad Q_\infty \tag{1.20}$$

*where* $\mu_2$ *is an arbitrary number* $0<\mu_2<\mu_1$.

To prove the theorems, we shall apply the method of Babenko. § 2 contains some preliminaries. The proof of the theorems is done in § 3. The corollaries are proved in § 4. In these sections we shall use the same $c$ for various constants independent of variables $x$, $t$, given data, or parameters. We shall use the same $M$ for various constants depending on some data. We shall use $c_p$, $c_\alpha$, $c_{p,\alpha}$, etc. for various constants depending only on the parameters $p$, $\alpha$, $p$ and $\alpha$, etc. Some notations, e.g. $\xi$, $\alpha$, will be used to represent various objects when no confusion occurs.

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## § 2. Preliminaries.

The main contents of this section are estimates for the fundamental solution tensor of the linearized system of equations for (1.1), reduction of equations (1.1)-(1.4) into integral equations, and proof of a fundamental lemma of Babenko [1].

We first normalize variables. Through a suitable change of variables, we can assume $\nu=1$, $\boldsymbol{u}_\infty=(1, 0, 0)$ without loss of generality. We set $\boldsymbol{v}=\boldsymbol{u}-\boldsymbol{u}_\infty$. Then the equations (1.1)-(1.4) are transformed into

$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{v}-\Delta\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\dfrac{\partial}{\partial x_1}\boldsymbol{v}+\operatorname{grad}\boldsymbol{p}=0 \\ \operatorname{div}\boldsymbol{v}=0\,. \end{cases} \qquad (x, t)\in Q_T \tag{2.1}$$

$$\boldsymbol{v}(x, 0)=\boldsymbol{a}(x)-\boldsymbol{u}_\infty, \qquad x\in\Omega \tag{2.2}$$

$$\boldsymbol{v}(x, t)=\boldsymbol{b}(x, t)-\boldsymbol{u}_\infty, \qquad x\in\partial\Omega, \quad 0\leqq t<T \tag{2.3}$$

$$\boldsymbol{v}(x, t)\to\boldsymbol{0} \qquad \text{as} \quad |x|\to\infty, \quad 0\leqq t<T\,. \tag{2.4}$$

**2.1. Fundamental solution tensor.**

The fundamental solution tensor $E=(E_{ij}, Q_i)$ of the linearized system of equations for (2.1);

$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{v}-\Delta\boldsymbol{v}+\dfrac{\partial}{\partial x_1}\boldsymbol{v}+\operatorname{grad}\boldsymbol{p}=\boldsymbol{f} \\ \operatorname{div}\boldsymbol{v}=0 \end{cases}$$

is given by

$$(2.5)\qquad \begin{cases} E_{ij}(x, t)=\delta_{ij}\Gamma(x, t)+\dfrac{1}{4\pi}\dfrac{\partial^2}{\partial x_i\partial x_j}\displaystyle\iiint_{E_3}\Gamma(x-y, t)|y|^{-1}dy \\ Q_i(x, t)=-\dfrac{1}{4\pi}\delta(t)\dfrac{\partial}{\partial x_i}(|x|^{-1}) \end{cases}$$

where $\delta_{ij}$ is the Kronecker's delta, $\delta(\cdot)$ is the delta function, and $\Gamma$ is given by

$$\Gamma(x, t)=(4\pi t)^{-3/2}\exp(-|x-t\boldsymbol{u}_\infty|^2/4t).$$

$E$ is the fundamental solution tensor in the sense

$$(2.6)\qquad \begin{cases} \left(\dfrac{\partial}{\partial t}-\Delta+\dfrac{\partial}{\partial x_1}\right)E_{ij}(x, t)+\dfrac{\partial}{\partial x_j}Q_i(x, t)=\delta_{ij}\delta(x)\delta(t) \\ \dfrac{\partial}{\partial x_j}E_{ij}(x, t)=0. \end{cases}$$

This follows from the obvious equality

$$\left(\frac{\partial}{\partial t}-\Delta+\frac{\partial}{\partial x_1}\right)\Gamma(x, t)=\delta(x)\delta(t).$$

We shall give some estimates of $E$ for later use. We first state the estimates essentially due to V. A. Solonnikov [**20**].

LEMMA 1. (Solonnikov) *The inequalities*

$$(2.7)\qquad \iiint_{E_3}\Gamma(x-y, t)(1+|y|)^{-\lambda}dy \leq c_\lambda\begin{cases} k(x, t; -\lambda/2), & 0<\lambda<3, \\ \log(1+t)k(x, t; -3/2), & \lambda=3, \\ k(x, t; -3/2), & 3<\lambda \end{cases}$$

$$(2.8)\qquad \left|\nabla^l\iiint_{E_3}\Gamma(x-y, t)|y|^{-1}dy\right|\leq c_l k(x, t; -(1+l)/2)$$

$$(2.9)\qquad |E_{ij}(x, t)|\leq ck(x, t; -3/2)$$

$$(2.10)\qquad |\nabla E_{ij}(x, t)|\leq ck(x, t; -2)$$

*hold for* $x\neq 0$, $t\geq 0$, *where* $\nabla^l$ *stands for an arbitrary* $l$-*th order* $x$-*derivative and*

*$k(x, t; \lambda)$ for real $\lambda$ is given by*

$$k(x, t; \lambda)=(t+|x-t\boldsymbol{u}_\infty|^2)^\lambda. \tag{2.11}$$

We evaluate $k(x, t; \lambda)$ and its certain integrals. To this end, we set $\rho_x=(x_2^2+x_3^2)^{1/2}$ and

$$\Phi_{p,q}(x)=\begin{cases} |x|^{-p/2}(1+s_x)^{-p/2} & \text{for} \quad |x|>1 \\ |x|^{-q} & \text{for} \quad |x|\leqq 1 \end{cases} \tag{2.12}$$

where $p, q>0$.

LEMMA 2. *Let $\alpha>0$, $p>1$ and $q>0$. The following inequalities* (2.13)-(2.17) *hold for $x\neq 0$, $t\geqq 0$;*

$$k(x, t; -q)\leqq c^q(x_1^2/(1+t)+\rho_x^2)^{-q} \tag{2.13}$$

$$k(x, t; -q)\leqq c^q\Phi_{2q,2q}(x) \tag{2.14}$$

$$\int_0^t k(x, \tau; -3/2)d\tau\leqq ct(1+t)^{3/2}|x|^{-3} \tag{2.15}$$

$$\int_0^t k(x, \tau; -2)d\tau\leqq ct(1+t)^2|x|^{-4} \tag{2.16}$$

$$\begin{aligned} &\int_0^t k(x, \tau; -p)(1+t-\tau)^{-\alpha}d\tau \\ &\leqq c_{\alpha,p}\Phi_{2p-1,2p-2}(x) \\ &\times\begin{cases} J_\alpha(|t-x_1|)(1+t+\rho_x^2)^{-\min(1,\alpha)/2} & \text{if} \quad |x|>1, \quad x_1>\rho_x/2 \\ J_\alpha(t)(1+t+|x|)^{-\min(1,\alpha)} & \text{otherwise,} \end{cases} \end{aligned} \tag{2.17}$$

*where*

$$J_\alpha(t)=\begin{cases} 1, & \alpha\neq 1 \\ 1+\log(1+t), & \alpha=1. \end{cases}$$

PROOF. Simple calculation can show (2.13) and (2.14). (2.15) and (2.16) follow from (2.13). To show (2.17) in the case $|x|\leqq 1$ or $2x_1\leqq\rho_x$, we note

$$k(x, \tau; -p)\leqq c^p(\tau+\tau^2+|x|^2)^{-p}, \qquad |x|\leqq 1 \quad \text{or} \quad 2x_1\leqq\rho_x, \quad 0\leqq\tau.$$

Integration of the both sides multiplied by $(1+t-\tau)^{-\alpha}$ gives (2.17) in this case.

We show (2.17) in the excluded case $|x|>1$ and $2x_1>\rho_x$. We set $\eta=\rho_x^2+x_1-1/4$. Note $\eta>3/4$. We put $\xi(\tau)=(\tau-x_1+1/2)/\sqrt{\eta}$ and set $\xi_0=\xi(0)$, $\xi_1=\xi(t)$. Changing variables from $\tau$ to $\xi$, we get

$$\begin{aligned} &\int_0^t k(x, \tau; -p)(1+t-\tau)^{-\alpha}d\tau \\ &=\eta^{-p-\alpha/2+1/2}\int_{\xi_0}^{\xi_1}(1+\xi^2)^{-p}(1/\sqrt{\eta}+\xi_1-\xi)^{-\alpha}d\xi. \end{aligned} \tag{2.18}$$

Evaluation of the right hand side of (2.18) is carried out separately in cases $\xi_1<-1$, $|\xi_1|\leqq 1$ and $\xi_1>1$. In the case $\xi_1<-1$, put $\zeta=\xi/\xi_1$. Evaluating $(1+\xi^2)^{-1}$ by $|\xi|^{-2}$ and changing variables from $\xi$ to $\zeta$, we have

$$(2.19)\qquad \int_{-\infty}^{\xi_1}(1+\xi^2)^{-p}(1/\sqrt{\eta}+\xi_1-\xi)^{-\alpha}d\xi$$

$$\leqq |\xi_1|^{-2p+1-\alpha}\int_1^{\infty}\zeta^{-2p}(1/\sqrt{\eta}|\xi_1|-1+\zeta)^{-\alpha}d\zeta$$

$$\leqq c_{\alpha,p}\begin{cases}|\xi_1|^{-2p+1-\alpha}, & 0\leqq\alpha<1,\\ |\xi_1|^{-2p}\log(\sqrt{\eta}|\xi_1|+1), & \alpha=1,\\ |\xi_1|^{-2p}\eta^{(\alpha-1)/2}, & \alpha>1.\end{cases}$$

A brief observation for $|\xi_1|\leqq 1$ shows

$$(2.20)\qquad \int_{-\infty}^{\xi_1}(1+\xi^2)^{-p}(1/\sqrt{\eta}+\xi_1-\xi)^{-\alpha}d\xi$$

$$\leqq c_{\alpha,p}\begin{cases}1, & 0\leqq\alpha<1,\\ \log\sqrt{\eta}, & \alpha=1,\\ \eta^{(\alpha-1)/2}, & \alpha>1.\end{cases}$$

In the remaining case $\xi_1>1$, separate the interval of integration into two parts: $(-\infty, \xi_1/2]$ and $(\xi_1/2, \xi_1)$. We obtain

$$(2.21)\qquad \int_{-\infty}^{\xi_1}(1+\xi^2)^{-p}(1/\sqrt{\eta}+\xi_1-\xi)^{-\alpha}d\xi$$

$$\leqq \sup_{\xi\leqq\xi_1/2}(1/\sqrt{\eta}+\xi_1-\xi)^{-\alpha}\int_{-\infty}^{\xi_1}(1+\xi^2)^{-p}d\xi$$

$$+\sup_{\xi_1/2<\xi<\xi_1}(1+\xi^2)^{-p}\int_{\xi_1/2}^{\xi_1}(1/\sqrt{\eta}+\xi_1-\xi)^{-\alpha}d\xi$$

$$\leqq c_{\alpha,p}(1/\sqrt{\eta}+\xi_1)^{-\alpha}+c_{\alpha,p}\xi_1^{-2p}\begin{cases}\xi_1^{1-\alpha}, & 0\leqq\alpha<1,\\ \log(\sqrt{\eta}\xi_1+1), & \alpha=1,\\ \eta^{(\alpha-1)/2}, & 1<\alpha.\end{cases}$$

On the other hand we get

$$(2.22)\qquad \eta(1+\xi_1^2)=t+(t-x_1)^2+\rho_x^2\geqq c(1+t+\rho_x^2)$$

since $x_1>c>0$. The estimate (2.17) follows from (2.18)-(2.22) and the following inequality:

$$\eta^{-p+1/2}\leqq c\Phi_{2p-1,2p-2}(x), \qquad x\neq 0.$$

LEMMA 3. *Let* $p>3/2$, $1\leqq q<3/2$, $1\leqq r<5/4$. *Then*

$$\left\{\int_0^t \iiint_{E_3} k(x, \tau ; -2r) dx d\tau\right\}^{1/r} \leqq c_r t^{-2+5/2r} \tag{2.23}$$

$$\int_0^t \left\{\iiint_{E_3} k(x, \tau ; -2q) dx\right\}^{1/q} d\tau \leqq c_q t^{-1+3/2q} \tag{2.24}$$

$$\int_0^\infty \left\{\iiint_{\rho_x > R} k(x, \tau ; -2p) dx\right\}^{1/p} d\tau \leqq c_p R^{-2+3/p} \tag{2.25}$$

$$\int_0^\infty \left\{\iiint_{|x_1| > R} k(x, \tau ; -2p) dx\right\}^{1/p} d\tau \leqq c_p R^{-1+3/2p} \tag{2.26}$$

$$\int_0^\infty \left\{\iiint_{\rho_x < 1} k(x, \tau ; -2q) dx\right\}^{1/q} d\tau \leqq c_q \tag{2.27}$$

*hold for* $t \geqq 0$, *where* $R > 1$.

PROOF. Putting $z = x - \tau \boldsymbol{u}_\infty$, the integrations with respect to $x$ in (2.23) and (2.24) can be reduced to integrations with respect to $|z|$. Then, changing variables from $|z|$ to $|z|/\sqrt{\tau}$, straightfoward calculation gives (2.23) and (2.24). (2.25) and (2.27) can be shown analogously. Let us show (2.26). Integrating $k(x, \tau ; -2p)$ with respect to $\rho_x$, we get

$$\begin{aligned} \iiint_{|x_1| > R} k(x, \tau ; -2p) dx &= \frac{\pi}{2p-1} \left\{\int_{-\infty}^{-R} + \int_R^\infty (\tau + (x_1 - \tau)^2)^{-2p+1} dx_1\right\} \\ &\leqq \frac{2\pi}{2p-1} \tau^{-2p+3/2} \int_{(R-\tau)/\sqrt{\tau}}^\infty (1+\xi^2)^{-2p+1} d\xi \end{aligned} \tag{2.28}$$

where $\xi = (x_1 - \tau)/\sqrt{\tau}$. For $\tau \leqq R/2$ we have

$$\int_{(R-\tau)/\sqrt{\tau}}^\infty (1+\xi^2)^{-2p+1} d\xi \leqq \frac{1}{4p-3} ((R-\tau)/\sqrt{\tau})^{-4p+3},$$

and for $\tau > R/2$,

$$\int_{(R-\tau)/\sqrt{\tau}}^\infty (1+\xi^2)^{-2p+1} d\xi \leqq c_p.$$

Thus we get

$$\begin{aligned} \int_0^\infty \left\{\iiint_{|x_1| > R} k(x, \tau ; -2p) dx\right\}^{1/p} d\tau &= \int_0^{R/2} \{\quad\} d\tau + \int_{R/2}^\infty \{\quad\} d\tau \\ &\leqq c_p R^{-3+3/p} + c_p R^{-1+3/2p}. \end{aligned} \tag{2.29}$$

Thus (2.26) has been proved.

We give some estimates for $\Gamma(x, t)$, which will be used in the proof of Corollary 2.

LEMMA 4. *Let $l$ be a non-negative integer and $0<\mu<1/2$. Then*

$$\int_0^\infty |\nabla^l \Gamma(x, t)|\, dt \tag{2.30}$$
$$\leqq c_{l,\mu}|x|^{-(1+l/2)}(1+|x|^{-l/2})\exp(-\mu s_x)$$

*holds for $x\neq 0$.*

PROOF. Let $l=l_1+l_2+l_3$, where $l_1$, $l_2$ and $l_3$ are non-negative integers. The derivative $(\partial/\partial x_1)^{l_1}(\partial/\partial x_2)^{l_2}(\partial/\partial x_3)^{l_3}\Gamma(x, t)$ of $\Gamma(x, t)$ is a linear combination of

$$t^{-l-3/2+j_1+j_2+j_3}(x_1-t)^{l_1-2j_1}x_2^{l_2-2j_2}$$
$$\times x_3^{l_3-2j_3}\exp\{-(|x_1-t|^2+\rho_x^2)/4t\},$$

where $0\leqq j_i\leqq l_i/2$, $i=1, 2, 3$. We can show

$$\int_0^\infty t^{-p}|x_1-t|^q \exp\left(-\frac{|x_1-t|^2+\rho_x^2}{4t}\right)dt \tag{2.31}$$
$$\leqq c_{p,q}|x|^{-p+1/2}(1+|x|^{-p+q/2+3/2})(s_x+\sqrt{|x|})^q$$
$$\times \exp(-s_x/2)$$

for $x\neq 0$, where $0\leqq q<p$. To show this, put $\zeta=(|x_1-t|^2+\rho_x^2)/4t-s_x/2$ and change the variable of integration of the left hand side of (2.31), obtaining

$$\int_0^\infty t^{-p}|x_1-t|^q \exp\left(-\frac{|x_1-t|^2+\rho_x^2}{4t}\right)dt=I^++I^- \tag{2.32}$$

where

$$I^\pm=\int_0^\infty \{|x|+2\zeta\pm 2\sqrt{\zeta^2+\zeta|x|}\}^{-p}|s_x+2\zeta\pm\sqrt{\zeta^2+\zeta|x|}|^q$$
$$\times\left(\pm 2+\frac{2\zeta+|x|}{\sqrt{\zeta^2+\zeta|x|}}\right)\exp(-\zeta-s_x/2)d\zeta.$$

Calculation yields

$$I^+\leqq c_{p,q}|x|^{-p+1/2}(s_x+\sqrt{|x|})^q\exp(-s_x/2)$$
$$I^-\leqq c_{p,q}|x|^{-p+1/2}(s_x+\sqrt{|x|})^q(1+|x|^{-p+q/2+3/2})$$
$$\times\exp(-s_x/2)$$

for $x\neq 0$. Then the inequality (2.31) immediately follows from these inequalities. By (2.31) we can prove this lemma. Indeed, setting $p=l-(j_1+j_2+j_3)+3/2$, $q=l_1-2j_1$, we obtain (2.30) since $|x_i|\leqq\sqrt{2|x|s_x}$, $i=2, 3$.

LEMMA 5. *Let $\lambda>2$ and $0<\mu'\leqq 1/2$. Then,*

$$\iiint_{E_3} \Gamma(x-y, t)(1+|y|)^{-\lambda} \exp(-\mu' s_y) dy \leqq c_{\lambda,\mu'}(1+|x|)^{-3/2} \exp(-\mu' s_x) \tag{2.33}$$

*holds for* $x \in E_3$, $0 \leqq t$.

PROOF. We put $\xi=x/\sqrt{t}$, $\eta=y/\sqrt{t}$, $r=|\eta|$, and $\cos\theta=\cos(\xi, \eta)$. In the left hand side of (2.33), we evaluate $\exp(-|x-y-t\boldsymbol{u}_\infty|^2/4t)$ by $\exp(-\mu' \times |x-y-t\boldsymbol{u}_\infty|^2/2t)$ and change the variables of integration from $y$ to polar coordinates $r$, $\theta$, $\psi$. Since the integrand does not depend on $\psi$, we have

$$\begin{aligned} \iiint_{E_3} \Gamma(x-y, t)(1+|y|)^{-\lambda} \exp(-\mu' s_y) dy &\leqq (4\pi)^{-3/2} \exp(-\mu' t/2+\mu' x_1) \\ &\quad \times \int_0^\infty (1+\sqrt{t}\, r)^{-\lambda} r^2 \exp\left\{-\frac{\mu'}{2}(|\xi|^2+r^2+2\sqrt{t}\, r)\right\} dr \\ &\quad \times 2\pi \int_0^\pi \exp(\mu' |\xi| r \cos\theta) \sin\theta \, d\theta \\ &\leqq c_{\mu'} \exp(-\mu' s_x)(I_1+I_2) \end{aligned} \tag{2.34}$$

where

$$I_1=\int_0^{1/|\xi|} (1+\sqrt{t}\, r)^{-\lambda} r^2 \exp\left\{-\frac{\mu'}{2}(r+\sqrt{t}-|\xi|)^2\right\} dr,$$

$$I_2=\frac{1}{|\xi|}\int_{1/|\xi|}^\infty (1+\sqrt{t}\, r)^{-\lambda} r \exp\left\{-\frac{\mu'}{2}(r+\sqrt{t}-|\xi|)^2\right\} dr.$$

We give a list of estimates of $I_j$, $j=1, 2$. For $t \geqq 2|x|$,

$$\begin{aligned} I_1 &\leqq \int_0^{1/|\xi|} r^2 \exp\left\{-\frac{\mu'}{2}(r+\sqrt{t}/2)^2\right\} dr \\ &\leqq \frac{1}{3}\left(\frac{\sqrt{t}}{|x|}\right)^3 \exp(-\mu' t/8) \\ &\leqq c_{\mu'} |x|^{-3} \end{aligned}$$

and

$$\begin{aligned} I_2 &\leqq \frac{1}{|\xi|}\int_{1/|\xi|}^\infty r \exp\left\{-\frac{\mu'}{2}(r+\sqrt{t}/2)^2\right\} dr \\ &\leqq c_{\mu'} \frac{1}{|x|} \exp\left(-\frac{\mu'}{8}|x|\right). \end{aligned}$$

For $0 \leqq t < 2|x|$,

$$I_1 \leqq \int_0^{1/|\xi|} r^2 dr \leqq c|x|^{-3/2}.$$

For $|x|/2 \leqq t < 2|x|$,

$$I_2 \leqq \frac{1}{|\xi|}\int_0^\infty (1+\sqrt{t}r)^{-\lambda} r\,dr$$
$$\leqq c_\lambda |x|^{-3/2}.$$

For $0 \leqq t < |x|/2$,

$$I_2 \leqq \frac{1}{|\xi|}\Big\{\Big(\int_0^{|\xi|/6} + \int_{3|\xi|}^\infty\Big) r \exp\Big\{-\frac{\mu'}{2}(r/2+|\xi|/4)^2\Big\} dr$$
$$+ \int_{|\xi|/6}^{3|\xi|} r(1+\sqrt{t}r)^{-\lambda} \exp\Big\{-\frac{\mu'}{2}(r+\sqrt{t}-|\xi|)^2\Big\} dr\Big\}$$
$$\leqq \frac{1}{|\xi|}\Big\{c_{\mu'} \exp\Big(-\frac{\mu'}{32}|\xi|^2\Big) + 3c_{\mu'}|\xi|(1+\sqrt{t}|\xi|/6)^{-\lambda}\Big\}$$
$$\leqq c_{\mu'} \frac{1}{\sqrt{|x|}} \exp\Big(-\frac{\mu'}{16}|x|\Big) + c_{\lambda,\mu'}(1+|x|)^{-\lambda}.$$

Since the left hand side of (2.33) is clearly bounded for $x \in E_3$, $t \geqq 0$, the above estimates together with (2.34) assert (2.33).

**2.2. Reduction to integral equations.**

Let $\boldsymbol{v}$, $\boldsymbol{p}$ be a classical solution of (2.1)-(2.4) in $Q_T$. Let us compute

$$\int_0^t \iiint_D \Big[\Big\{\Big(\frac{\partial}{\partial \tau} + \Delta_y + \frac{\partial}{\partial y_1}\Big) E_{ij}(x-y, t-\tau)$$
$$+ \frac{\partial}{\partial y_j} Q_i(x-y, t-\tau)\Big\} v_j(y, \tau)$$
$$+ \Big\{\Big(\frac{\partial}{\partial \tau} - \Delta_y + \frac{\partial}{\partial y_1}\Big) v_j(y, \tau) + \frac{\partial}{\partial y_j} \boldsymbol{p}(y, \tau)\Big\}$$
$$\times E_{ij}(x-y, t-\tau)\Big] dy d\tau$$

where $D$ is an arbitrary bounded subdomain of $\Omega$. Using (2.6) and the Green formula in the hydrodynamics, we obtain

$$\boldsymbol{v}(x, t) = \sum_{k=1}^{5} L_D^{(k)}[\boldsymbol{v}](x, t) \tag{2.35}$$

for $x \in D$ and $0 \leqq t < T$. Here, the $i$-th component $L_D^{(k)}[\boldsymbol{v}]_i$ of $L_D^{(k)}[\boldsymbol{v}]$ is given by

$$L_D^{(1)}[\boldsymbol{v}]_i(x, t) = \iiint_D \Gamma(x-y, t) v_i(y, 0) dy, \tag{2.36}$$

$$L_D^{(2)}[\boldsymbol{v}]_i(x, t) = -\frac{1}{4\pi}\iint_{\partial D}\Big\{\iiint_{E_3} \frac{\partial}{\partial y_i}\Gamma(x-y-z, t) \times |z|^{-1} dz\Big\} \boldsymbol{v}(y, 0)\cdot \boldsymbol{n}\, dy, \tag{2.37}$$

$$(2.38)\qquad L_D^{(3)}[\boldsymbol{v}]_i(x, t)=-\frac{1}{4\pi}\iint_{\partial D}\frac{\partial}{\partial y_i}(|x-y|)^{-1}\boldsymbol{v}(y, t)\cdot\boldsymbol{n}\,dy,$$

$$(2.39)\qquad L_D^{(4)}[\boldsymbol{v}]_i(x, t)$$

$$=\int_0^t\iint_{\partial D}\Big[\Big\{E_{ij}\Big(-\delta_{jk}\boldsymbol{p}+\frac{\partial v_j}{\partial y_k}+\frac{\partial v_k}{\partial y_j}\Big)-v_j\Big(\frac{\partial E_{ij}}{\partial y_k}+\frac{\partial E_{ik}}{\partial y_j}\Big)\Big\}n_k$$

$$-E_{ij}v_jn_1-E_{ij}v_jv_kn_k\Big]dyd\tau,$$

$$(2.40)\qquad L_D^{(5)}[\boldsymbol{v}]_i(x, t)=\int_0^t\iiint_D\frac{\partial}{\partial y_k}E_{ij}v_jv_kdyd\tau$$

where $E_{ij}=E_{ij}(x-y, t-\tau)$, $\boldsymbol{v}=\boldsymbol{v}(y, \tau)$, $\boldsymbol{p}=\boldsymbol{p}(y, \tau)$, and $\boldsymbol{n}$ is the outer normal vector on $\partial D$. We also define $L_D^{(k)}[\boldsymbol{f}]$ by (2.36)-(2.40) for any vector valued function $\boldsymbol{f}$ and for any domain $D$. Especially if $D=\Omega$, we use the notation $L^{(k)}[\boldsymbol{v}]$ for $L_\Omega^{(k)}[\boldsymbol{v}]$.

The formula (2.35) is shown in [**12**]. To establish this formula for a weak solution and for $D=\Omega$, we need

DEFINITION 2. A vector valued function $\boldsymbol{u}$ associated with a scalar function $\boldsymbol{p}$ is a weak solution in $Q_T$ of (1.1)-(1.4) if and only if $\boldsymbol{u}$, $\boldsymbol{p}$ satisfy (1.1) a.e. in $Q_T$, $\nabla\boldsymbol{u}\in L^2(Q_T)$, $\boldsymbol{u}(x, t)\to\boldsymbol{a}(x)$ as $t\to 0$ in the sense of distribution, and $\boldsymbol{u}(x, t)=\boldsymbol{b}(x, t)$ a.e. in $\Omega\times(0, T)$.

We say $\boldsymbol{v}$ is a weak solution of (2.1)-(2.4) if and only if $\boldsymbol{v}+\boldsymbol{u}_\infty$ is a weak solution of (1.1)-(1.4). By the properties of our weak solutions studied by Golovkin and Ladyzenskaya [**8**] and Solonnikov [**20**],

$$(2.41)\qquad \int_0^T\iiint_{|x-y|<1,\, y\in\Omega}\Big\{\Big|\frac{\partial\boldsymbol{u}}{\partial t}\Big|+|\nabla\boldsymbol{p}|+|\Delta\boldsymbol{u}|\Big\}^{5/4}dyd\tau\leqq M,\quad x\in\Omega,$$

where $\boldsymbol{p}$ can be chosen so that

$$(2.42)\qquad \iint_{\partial\Omega}\{|\boldsymbol{p}(y, \tau)|+|\nabla\boldsymbol{u}(y, \tau)|\}dy$$

$$\leqq M\sup_{0\leq t<T}\|\nabla\boldsymbol{u}(\cdot, t)\|_{L^2(\Omega)}$$

and $M$ is a constant depending on $\boldsymbol{a}$, $\boldsymbol{b}$ and $\|\nabla\boldsymbol{u}\|_{L^2(Q_T)}$.

PROPOSITION 1. *Let* $\boldsymbol{v}$ *be a weak solution of* (2.1)-(2.4) *in* $Q_T$. *Then,*

$$(2.43)\qquad \boldsymbol{v}(x, t)=\sum_{k=1}^{5}L^{(k)}[\boldsymbol{v}](x, t),\qquad a.\ e.\ (x, t)\in Q_T.$$

PROOF. By (2.41) and the theorem of trace, the right-hand side of (2.35) has meaning, and (2.35) holds a.e. in $D\times(0, T)$. Let $D=D_R\equiv\Omega\cap\{x\ ;\ |x|<R\}$ in (2.35) and integrate the both sides with respect to $R$ in $R_0\leqq R<R_0+1$. Then let $R_0$ go to infinity. We obtain

(2.44) $$v_i(x, t)=\sum_{k=1}^{5} L^{(k)}[\boldsymbol{v}]_i(x, t)$$

$$-\lim_{R_0\to\infty}\int_0^t\iiint_{R_0\leqq|y|\leqq R_0+1} E_{ij}(x-y, t-\tau)\boldsymbol{p}(y, \tau)\frac{y_j}{|y|}dyd\tau,$$

$$a.\ e.\quad (x, t)\in Q_T$$

by Lemma 1 and by the fact $\boldsymbol{v}\in L^2(0, T; L^6(\Omega))$, which is assured by the assumption $\nabla\boldsymbol{v}\in L^2(Q_T)$ and by the imbedding theorem of Sobolev. To prove the proposition, it remains to show the last term of (2.44) vanishes. We set

$$P_i^\infty(x, t)=\lim_{R_0\to\infty}\int_0^t\iiint_{R_0\leqq|y|\leqq R_0+1} E_{ij}(x-y, t-\tau)\boldsymbol{p}(y, \tau)\frac{y_j}{|y|}dyd\tau.$$

We can choose $\boldsymbol{p}$ such that

(2.45) $$\iint_{\partial\Omega}\boldsymbol{p}(y, t)dy=0, \qquad 0\leqq t<T.$$

Then by (2.41) we obtain

$$\int_0^T\iiint_{|x-y|<1,\ y\in\Omega}|\boldsymbol{p}(y, t)|\,dy\,dt\leqq M(1+|x|)$$

for $x\in\Omega$, where $M$ is a constant depending on $\boldsymbol{a}$, $\boldsymbol{b}$ and $\|\nabla\boldsymbol{v}\|_{L^2(Q_T)}$. Then, by Lemma 1, we have $|P_i^\infty(x, t)|\leqq M$, $P_i^\infty(x, t)-P_i^\infty(z, t)=0$, and $|P_i^\infty(x, t)-P_i^\infty(x, \tau)|\leqq M|t-\tau|$ for $x, z\in\Omega$, $0\leqq t, \tau<T$. Since $P_i^\infty(x, t)\to 0$ as $|x|\to\infty$ a.e. in $0\leqq t<T$, we obtain $P_i^\infty(x, t)=0$ in $Q_T$. This completes the proof.

### 2.3. On $L^{(5)}$.

Here we prove

PROPOSITION 2. *$L^{(5)}$ is a bounded operator from $L^s(0, T; L^r(\Omega))$ to $L^{s'}(0, T; L^{r'}(\Omega))$, where $s$, $s'$, $r$ and $r'$ are arbitrary numbers satisfying either $1<s, s', r, r'<\infty$, $2/s'+3/r'\geqq 2(2/s+3/r)-1$ and $0\leqq 2/r-1/r'<1/3$, or $s=s'=\infty$ and $0\leqq 2/r-1/r'<1/3$.*

To prove this proposition we use the following theorem of Mihlin-Lizorkin.

THEOREM [**16**, theorem 8]. *Let $\Phi:\boldsymbol{R}^n\to\boldsymbol{C}$. Suppose $\Phi$, the derivative $\partial^n\Phi/\partial\xi_1\cdots\partial\xi_n$ and all lower derivatives are continuous for $\xi_j\neq 0$, $j=1, \cdots, n$. Let $\mathfrak{F}[f]$ be the Fourier transform of $f$. Then the following transformation*

$$f\longrightarrow(2\pi)^{-n/2}\int\cdots\int_{\boldsymbol{R}^n}\Phi(\xi)\mathfrak{F}[f](\xi)e^{\sqrt{-1}x\cdot\xi}d\xi$$

*is a bounded operator $L^p(\boldsymbol{R}^n)\to L^q(\boldsymbol{R}^n)$ if*

$$\left|\xi_1^{\alpha_1+\sigma}\cdots\xi_n^{\alpha_n+\sigma}\frac{\partial^{|\alpha|}\Phi}{\partial^\alpha\xi}\right|\leqq M, \qquad \xi\neq 0$$

*where $\sigma=1/p-1/q$, $\alpha$ is a multi-index, $\alpha_j$ take the values 0 or 1 and $M$ is a* *constant.*

PROOF OF PROPOSITION 2. The Fourier transform $\mathcal{F}(\nabla E(\cdot, t))$ of $\nabla E(\cdot, t)$ is given by

$$\mathcal{F}\Big(\frac{\partial}{\partial x_k}E_{ij}(\cdot, t)\Big)(\xi)$$
$$=(2\pi)^{-3/2}\sqrt{-1}\,\xi_k\Big(\delta_{ij}-\frac{\xi_i\xi_j}{|\xi|^2}\Big)\exp(-t|\xi|^2-\sqrt{-1}\,t\xi_1).$$

The $\alpha$-th derivative for any multi-index $\alpha$ is estimated by

$$|\xi^\alpha\partial_\xi^\alpha\mathcal{F}(\nabla E(\cdot, t))|$$
$$\leqq|\xi|\,\{p_\alpha(t|\xi|^2)+q_\alpha(t|\xi|)\}\exp(-t|\xi|^2)$$

where $p_\alpha(\zeta)$ and $q_\alpha(\zeta)$ are polynomials in $\zeta$ of order $|\alpha|$. Hence,

$$|\xi_1\xi_2\xi_3|^{1/\sigma}|\xi^\alpha\partial_\xi^\alpha\mathcal{F}(\nabla E(\cdot, t))|$$
$$\leqq c_{\sigma,\alpha}t^{-1/2-3/2\sigma}(1+t^{|\alpha|/2})$$

for $t>0$, where $\sigma>0$. By virtue of the theorem of Mihlin-Lizorkin, we obtain

$$\|L^{(5)}[f](\cdot, t)\|_{L^{p'}(\Omega)} \tag{2.46}$$
$$\leqq\int_0^t\Big\|\iiint_\Omega|\nabla E(x-y, t-\tau)|\,|f(y, \tau)|^2dy\Big\|_{L^{p'}(\Omega)}d\tau$$
$$\leqq c_{p,p'}(1+t^{3/2})\int_0^t(t-\tau)^{-1/2-3/2\sigma}\|f(\cdot, \tau)\|^2_{L^{2p}(\Omega)}d\tau$$

where $1<p\leqq p'<\infty$ and $1/\sigma=1/p-1/p'$. Hence by virtue of the same theorem, we get

$$\|L^{(5)}[f]\|_{L^{q'}(0,T;\,L^{p'}(\Omega))}$$
$$\leqq c_{p,p',q,q'}(1+T^{3/2})\|f\|^2_{L^{2q}(0,T;\,L^{2p}(\Omega))}$$

where $1<q\leqq q'<\infty$, $1/q'\geqq1/q-(1/2-3/2\sigma)$, $\sigma>3$. Setting $2p=r$, $2q=s$, $p'=r'$ and $q'=s'$, we have proposition 2 in the case $1<s, s', r, r'<\infty$, $2/s'+3/r'\geqq 2(2/s+3/r)-1$, and $0\leqq2/r-1/r'<1/3$. The conclusion of Proposition 2 also follows from (2.46) in the other case.

### 2.4. A fundamental lemma.

Here we prove a fundamental lemma due to Babenko [1].

LEMMA 6. (K.I. Babenko) *Let $R_0$ be a fixed positive number and $\phi(R)$ be a non-negative function defined for $R\geqq R_0$. Suppose*

$$\limsup_{R\to\infty}\phi(R)<(2^{\alpha+1}c_2)^{-1/(\beta-1)} \tag{2.47}$$

(2.48) $$\psi(R) \leqq c_1 R^{-\alpha} + c_2\{\psi(R/2)\}^{\beta}, \qquad R > 2R_0,$$

*where $c_1$, $c_2$, $\alpha$, $\beta$ are constants, $\alpha>0$, $\beta>1$. Then for any $\varepsilon>0$ there exists a number $R_\varepsilon$ such that*

(2.49) $$\psi(R) \leqq (c_1+\varepsilon) R^{-\alpha}, \qquad R \geqq R_\varepsilon.$$

PROOF. We set $c_3=(2^{\alpha+1}c_2)^{-1/(\beta-1)}$ and $R_1=\max\{(2c_1/c_3)^{1/\alpha},\ 1+\sup\{R: \psi(R) \geqq c_3\}\}$. By (2.47), $R_1$ is finite. For $R \geqq R_1$, there is a non-negative integer $m$ and a number $R_2$, $R_1 \leqq R_2 < 2R_1$ such that $R=2^m R_2$. Using (2.48), we can show $\psi(2^m R_2) \leqq 2^{-m\alpha} c_3$ for all $m$ by induction on $m$. Hence

$$\psi(R) \leqq c_3(2R_1)^{\alpha} R^{-\alpha}, \qquad R \geqq R_1.$$

We set $R_\varepsilon=\max\{2R_1,\ (\varepsilon^{-1}c_2 c_3^{\beta}(4R_1)^{\alpha\beta})^{1/(\alpha\beta-\alpha)}\}$. Then, we obtain by (2.48)

$$\begin{aligned}\psi(R) &\leqq c_1 R^{-\alpha} + c_2\{c_3(2R_1)^{\alpha}(R/2)^{-\alpha}\}^{\beta} \\ &\leqq (c_1+\varepsilon) R^{-\alpha}\end{aligned}$$

for $R \geqq R_\varepsilon$. Thus we have proved the lemma.

## § 3. Proofs of theorems.

We shall show Theorem 1 follows from Lemma 6 by setting $\psi(R) = \sup_{|x| \geqq R,\, 0 \leqq t < T} |\boldsymbol{v}(x, t)|$. To prove Theorem 2, we shall use the theorem of Finn as well as Lemma 6. In this section, every equality and inequality shall hold for $|x| \geqq 2\,\mathrm{diam}(\Omega^c)$, $0 \leqq t < T$.

### 3.1. Proof of Theorem 1.

We first show under the assumptions of Theorem 1

(3.1) $$|L^{(k)}[\boldsymbol{v}](x, t)| \leqq M|x|^{-\mu}, \qquad 1 \leqq k \leqq 4,$$

where $\mu$ is given in the statement of Theorem 1. Indeed, one sees by (2.36), (1.8), (2.7) and (2.13),

(3.2) $$|L^{(1)}[\boldsymbol{v}](x, t)| \leqq c_\lambda M |x|^{-\min(3,\lambda)}$$

where $M$ is a constant depending on $\boldsymbol{a}$ and $T$. To evaluate $L^{(2)}[\boldsymbol{v}]$, we set $K_i(x, t) = \dfrac{1}{4\pi}\dfrac{\partial}{\partial x_i}\iiint_{E_3} \Gamma(x-y, t)|y|^{-1}dy$, $i=1, 2, 3$. It follows from the mean-value theorem that

$$K_i(x-y, t) = K_i(x, t) - y\cdot\nabla_x K_i(x-\theta_i y, t)$$

where $0<\theta_i<1$. Hence the identity (2.37) and the estimates (2.8) and (2.13) yield

$$|L^{(2)}[\boldsymbol{v}](x,t)| \leqq c(1+t)|x|^{-2}\left|\iint_{\partial\Omega}\boldsymbol{v}(y,0)\cdot\boldsymbol{n}\,dy\right|$$
$$+c(1+t)^{3/2}|x|^{-3}\iint_{\partial\Omega}|y||\boldsymbol{v}(y,0)|\,dy.$$

We notice that by the divergence theorem

$$\iint_{\partial\Omega}\boldsymbol{v}(y,t)\cdot\boldsymbol{n}\,dy=\iint_{\partial\Omega}(\boldsymbol{v}(y,t)+\boldsymbol{u}_\infty)\cdot\boldsymbol{n}\,dy$$
$$=\iint_{\partial\Omega}\boldsymbol{u}(y,t)\cdot\boldsymbol{n}\,dy.$$

Thus we have

(3.3) $$|L^{(2)}[\boldsymbol{v}](x,t)| \leqq M(1+t)^{3/2}|x|^{-\mu},$$

where $M$ is a constant depending on $\boldsymbol{a}$ and $\Omega$. Similarly we get

(3.4) $$|L^{(3)}[\boldsymbol{v}](x,t)| \leqq M|x|^{-\mu},$$

by (2.38), where $M$ is a constant depending on $\boldsymbol{b}$ and $\Omega$. Let us estimate $L^{(4)}[\boldsymbol{v}]$. To do so, we can assume (2.42). Then the identity (2.39) and the estimates (2.9), (2.10), (2.15) and (2.16) assure

(3.5) $$|L^{(4)}[\boldsymbol{v}](x,t)| \leqq Mt(1+t)^2|x|^{-3}$$

where $M$ is a constant depending on $\boldsymbol{a}$, $\boldsymbol{b}$, $\Omega$ and $\sup_{0\leqq t<T}\|\nabla\boldsymbol{v}(\cdot,t)\|_{L^2(\Omega)}$. The inequalities (3.2) to (3.5) prove (3.1).

Let us evaluate $L^{(5)}[\boldsymbol{v}]$. Let us fix $x$ and set $D^{(1)}(R)=\{y\in\Omega : |x-y|>R\}$, $D^{(2)}(R)=\Omega-D^{(1)}(R)$ for $R>0$. It is clear that

$$\sum_{k=1}^{2} L^{(5)}_{D^{(k)}(R)}[\boldsymbol{v}]=L^{(5)}[\boldsymbol{v}].$$

Applying Hölder's inequality and using the estimates (2.10) and (2.13), we get

(3.6) $$|L^{(5)}_{D^{(1)}(R)}[\boldsymbol{v}](x,t)|$$
$$\leqq c_p t(1+t)^2 R^{-4+3/p}\sup_{0\leqq\tau\leqq t}\|\boldsymbol{v}(\cdot,\tau)\|^2_{L^{2p'}(\Omega)}$$

where $p\geqq 1$, $1/p+1/p'=1$. The estimate (2.24) with $q=1$ yields

(3.7) $$|L^{(5)}_{D^{(2)}(R)}[\boldsymbol{v}](x,t)| \leqq c\sqrt{t}\sup_{\substack{|x-y|\leqq R,\,0\leqq\tau\leqq t\\ y\in\Omega}}|\boldsymbol{v}(y,\tau)|^2.$$

To choose a suitable value of $p$ in (3.6), we need to prove

PROPOSITION 3. *Under the assumptions of Theorem* 1, $\boldsymbol{v}\in L^\infty(0,T:L^r(\Omega))$, *where $r$ satisfies either $r\geqq 6$ or $r>3/\mu$.*

PROOF. The case $r=6$ follows from the Sobolev imbedding theorem and the case $r=\infty$ from the assumption. Thus for $r\geqq 6$ Proposition 3 holds. For $r<6$, we note

$$|v(x, t)| \leqq M|x|^{-\mu} + |L^{(5)}[v](x, t)|,$$

which follows from Proposition 1 and (3.1). Proposition 2 with $s=s'=\infty$, $r=6$, $r' \geqq 3$ assures $L^{(5)}[v] \in L^\infty(0, T; L^{r'}(\Omega))$, $r' \geqq 3$. Hence $v \in L^\infty(0, T; L^r(\Omega))$ for $r > 3/\mu$ and $r \geqq 3$. But then we have $L^{(5)}[v] \in L^\infty(0, T; L^{r/2}(\Omega))$ for $r > 3/\mu$ and $r \geqq 3$, owing to Proposition 2. Repeating the similar arguments proves Proposition 3.

We choose $p = \max(1, \mu)$ in (3.6) so that $2p' > 3/\mu$. Then Proposition 3, combined with the estimates (3.6) and (3.7), gives

$$\begin{aligned} |L^{(5)}[v](x, t)| &\leqq Mt(1+t)^2 R^{-\max(1,\mu)} \\ &\quad + c\sqrt{t} \sup_{|x-y| \leq R,\, 0 \leq \tau \leq t} |v(y, \tau)|^2. \end{aligned} \tag{3.8}$$

We choose $R = |x|/2$ in (3.8). Then

$$\begin{aligned} |v(x, t)| &\leqq M(1+t)^3 |x|^{-\mu} \\ &\quad + c\sqrt{t} \sup_{|x-y| \leq |x|/2,\, 0 \leq \tau \leq t} |v(y, \tau)|^2. \end{aligned} \tag{3.9}$$

Hence if we set $\psi(R) = \sup_{|x| \geq R,\, 0 \leq t < T} |v(x, t)|$, then

$$\psi(R) \leqq M(1+T)^3 R^{-\mu} + c\sqrt{T}\{\psi(R/2)\}^2. \tag{3.10}$$

Thus Theorem 1 immediately follows from Lemma 6 if (2.47) holds. Let us verify (2.47). We have by (2.10), (2.23), (2.40) and Hölder's inequality

$$\begin{aligned} &|L^{(5)}_{D^{(2)}(R)}[v](x, t)| \\ &\quad \leqq c_r t^{-2+5/2r} \|v\|^2_{L^{2r'}(0, T;\, L^{2r'}(D^{(2)}(R)))} \end{aligned} \tag{3.11}$$

where $1 < r < 5/4$ and $1/r + 1/r' = 1$. We choose $R = |x|/2$ in (3.11), obtaining by Proposition 3

$$\|v\|_{L^{2r'}(0, T;\, L^{2r'}(D^{(2)}(|x|/2)))} \to 0 \qquad \text{as} \quad |x| \to \infty,$$

since $r' > 5$. Hence we have (2.47) by (3.1), (3.6) and (3.11). This completes the proof of Theorem 1.

**3.2. Proof of Theorem 2.**

Here we shall make use of a trivial modification of a theorem of Finn [**5**, Theorem 5.1, Corollary 5.1]:

THEOREM. (R. Finn) *Let $f$ and $g$ be functions in $Q_\infty$ satisfying*

$$\begin{aligned} &\sup_{0 \leq t} |f(x, t) - g(x, t)| \\ &\quad \leqq c \iiint_\Omega \Phi_{3,2}(x-y) \sup_{0 \leq t} |f(y, t)|^2 dy, \qquad x \in \Omega \end{aligned} \tag{3.12}$$

$$\sup_{0\leq t}|g(x,t)|\leqq c|x|^{-1}(1+s_x)^{-1}, \qquad x\in\Omega \tag{3.13}$$

$$\sup_{0\leq t}|f(x,t)|\leqq c|x|^{-\alpha}, \qquad x\in\Omega \tag{3.14}$$

*where* $\Phi_{3,2}$ *is given by* (2.12) *and* $\alpha>1/2$. *Then, there holds*

$$\begin{aligned} f(x,t)=&g(x,t)\\ &+O(|x|^{-3/2}\log(1+|x|)(1+s_x)^{-3/2}) \qquad \text{in} \quad Q_\infty. \end{aligned} \tag{3.15}$$

We set $f=\boldsymbol{v}$, $g=\sum_{k=1}^{4}L^{(k)}[\boldsymbol{v}]$. Then Theorem 2 is an immediate consequence of this theorem if the conditions (3.12) to (3.14) hold.

First we show (3.12). Since $f-g=L^{(5)}[\boldsymbol{v}]$, (3.12) follows from (2.40), (2.10) and (2.17) with $\alpha=0$, $p=2$.

Next we verify (3.13). The terms $L^{(2)}[\boldsymbol{v}]$ and $L^{(3)}[\boldsymbol{v}]$ are estimated by (2.37), (2.38), (2.8) and (2.14), leading us to

$$\sum_{k=2}^{3}|L^{(k)}[\boldsymbol{v}](x,t)|\leqq M|x|^{-1}(1+s_x)^{-1} \tag{3.16}$$

where $M$ is a constant depending on $\boldsymbol{a}$ and $\boldsymbol{b}$. We evaluate $L^{(4)}[\boldsymbol{v}]$ by (2.39), (2.9), (2.10), (2.42) and (2.17) with $\alpha=0$, $p=3/2$ or 2, concluding

$$|L^{(4)}[\boldsymbol{v}](x,t)|\leqq M|x|^{-1}(1+s_x)^{-1} \tag{3.17}$$

where $M$ is a constant depending on $\boldsymbol{a}$, $\boldsymbol{b}$ and $\sup_{0\leq t}\|\nabla\boldsymbol{v}(\cdot,t)\|_{L^2(\Omega)}$. We evaluate $L^{(1)}[\boldsymbol{v}]$ by decomposing it into $L^{(1)}[\boldsymbol{v}-\boldsymbol{v}_s]$ and $L^{(1)}[\boldsymbol{v}_s]$. First we evaluate $L^{(1)}[\boldsymbol{v}-\boldsymbol{v}_s]$. Since $\boldsymbol{v}-\boldsymbol{v}_s=\boldsymbol{u}-\boldsymbol{u}_s$, we have

$$|\boldsymbol{v}(x,0)-\boldsymbol{v}_s(x)|\leqq M|x|^{-2}$$

by the assumption (1.11), where $M$ is a constant depending on $\boldsymbol{a}$. Hence we get by (2.36), (2.7) and (2.14)

$$|L^{(1)}[\boldsymbol{v}-\boldsymbol{v}_s](x,t)|\leqq M|x|^{-1}(1+s_x)^{-1}. \tag{3.18}$$

We evaluate $L^{(1)}[\boldsymbol{v}_s]$ by applying Proposition 1 to $\boldsymbol{v}_s$, i.e. by

$$L^{(1)}[\boldsymbol{v}_s]=\boldsymbol{v}_s-\sum_{k=2}^{5}L^{(k)}[\boldsymbol{v}_s].$$

By (1.6), (3.16) and (3.17) for $\boldsymbol{v}=\boldsymbol{v}_s$, we obtain

$$|\boldsymbol{v}_s(x)|+\sum_{k=2}^{4}|L^{(k)}[\boldsymbol{v}_s](x,t)|\leqq M|x|^{-1}(1+s_x)^{-1},$$

where $M$ depends on $\boldsymbol{v}_s$. By the theorem of Finn [**5**, theorem 5.1] and by (1.6) we get

$$|L^{(5)}[\boldsymbol{v}_s](x,t)|\leqq M|x|^{-1}(1+s_x)^{-1},$$

where $M$ depends on $\boldsymbol{v}_s$. Thus we have

$$|L^{(1)}[\boldsymbol{v}_s](x, t)| \leqq M|x|^{-1}(1+s_x)^{-1}. \tag{3.19}$$

Combining (3.16), (3.17), (3.18) and (3.19), we get (3.13).

It remains to show (3.14). Let us set $\psi(R)=\sup_{|x|\geqq R, 0\leqq t}|\boldsymbol{v}(x, t)|$. Then the condition (3.14) follows from Lemma 6 if the assumptions (2.47) and (2.48) of the lemma are valid. The inequality (2.47) follows from Theorem 1 and the assumption (1.13) of Theorem 2. Let us prove (2.48) for some $\alpha>1/2$. Since $|g(x, t)|\leqq M|x|^{-1}$, we need to estimate $L^{(5)}[\boldsymbol{v}]$ $(=f-g)$. To this end, we fix $x$ in $\Omega$ and $R\geqq 1$ and define

$$\begin{aligned}
D^{(3)} &= \{y\in\Omega : \rho_{x-y}>R\} \\
D^{(4)} &= \{y\in\Omega : |x_1-y_1|>R,\ \rho_{x-y}\leqq R\} \\
D^{(5)} &= \{y\in\Omega : |x_1-y_1|\leqq R,\ 1<\rho_{x-y}\leqq R\} \\
D^{(6)} &= \{y\in\Omega : |x_1-y_1|\leqq R,\ \rho_{x-y}\leqq 1\}\,.
\end{aligned}$$

It is clear that

$$L^{(5)}[\boldsymbol{v}]=\sum_{k=3}^{6} L^{(5)}_{D^{(k)}}[\boldsymbol{v}]\,.$$

Using Hölder's inequality and the estimates (2.25) and (2.26) we get

$$\begin{aligned}
\sum_{k=3}^{4} &|L^{(5)}_{D^{(k)}}[\boldsymbol{v}](x, t)| \\
&\leqq c_{p_1} R^{-1+3/2p_1}\sup_{0\leqq\tau\leqq t}\|\boldsymbol{v}(\cdot, \tau)\|^2_{L^{2p_1'}(\Omega)}
\end{aligned} \tag{3.20}$$

where $1/p_1+1/p_1'=1$, $2p_1'=r$. Note that we can use (2.25) since $p_1>3$ by the assumption (1.12) of Theorem 2. We assume $2<r<3$ without loss of generality. Then by Hölder's inequality and by (2.25) with $R=1$, we get

$$\begin{aligned}
|L^{(5)}_{D^{(5)}}&[\boldsymbol{v}](x, t)| \\
&\leqq c_{p_2'}\sup_{0\leqq\tau\leqq t}\|\boldsymbol{v}(\cdot, \tau)\|^2_{L^{2p_2'}(\Omega)} \\
&\leqq c_{p_2'}\sup_{0\leqq\tau\leqq t}\|\boldsymbol{v}(\cdot, \tau)\|^{1-\varepsilon}_{L^r(\Omega)}\cdot\sup_{y\in D^{(5)}, 0\leqq\tau\leqq t}|\boldsymbol{v}(y, \tau)|^{1+\varepsilon}
\end{aligned} \tag{3.21}$$

where $0<\varepsilon<1-r/3$, $1/p_2+1/p_2'=1$ and $p_2'=r(1-\varepsilon)^{-1}$. We note we can use (2.25) since $p_2>3/2$. By (2.27) we get

$$|L^{(5)}_{D^{(6)}}[\boldsymbol{v}](x, t)|\leqq c\sup_{y\in D^{(6)}, 0\leqq\tau\leqq t}|\boldsymbol{v}(x, t)|^2\,. \tag{3.22}$$

Let $R=|x|/2\sqrt{2}$. Then the estimates (3.20) to (3.22) gives

$$\begin{aligned}
|L^{(5)}&[\boldsymbol{v}](x, t)| \\
&\leqq M|x|^{-1+3/2p_1}+M\sup_{|x-y|\leqq|x|/2, 0\leqq\tau\leqq t}|\boldsymbol{v}(y, \tau)|^{1+\varepsilon}
\end{aligned}$$

where $M$ is a constant depending on $\boldsymbol{v}$, $r$, and $\varepsilon$. Hence it follows from (3.13) with $g=\sum_{k=1}^{4} L^{(k)}[\boldsymbol{v}]$ that

$$|\boldsymbol{v}(x, t)| \leqq M|x|^{-1+3/2p_1} + M \sup_{|x-y| \leqq |x|/2,\, 0 \leqq \tau \leqq t} |\boldsymbol{v}(y, \tau)|^{1+\varepsilon}.$$

Taking the supremum of the both sides of this inequality over the domain $\{x \in \Omega : |x|>R\} \times (0, \infty)$, we obtain (2.48) for $\alpha=1-3/2p_1>1/2$. This completes the proof of Theorem 2.

**3.3. A remark on the decay of weak solutions.**

Here we prove

PROPOSITION 4. *Let $\boldsymbol{v}$ be a weak solution of* (2.1)-(2.4) *in $Q_T$. Suppose $\boldsymbol{v} \in L^s(0, T; L^r(\Omega))$ with $r>3$ and $3/r+2/s=1$. Then under the assumption* (1.8), *there holds*

$$\left\{\int_0^T \left\{\iiint_{|x-y|<1,\, y \in \Omega} |\boldsymbol{v}(y, \tau)|^r dy\right\}^{s/r} d\tau\right\}^{1/s} \leqq M|x|^{-\mu} \tag{3.23}$$

*where $M$ is a constant depending on $\boldsymbol{v}$, $T$, and $\mu$, and $\mu$ is the constant given in the statement of Theorem* 1.

REMARK 4. If $3/r+2/s<1$, then $\boldsymbol{v}$ is a classical solution (Serrin [**19**]) and Proposition 4 follows from Theorem 1.

PROOF. We define

$$\bar{f}_{r,s}(x, t)=\left\{\int_0^t \left\{\iiint_{|x-y|<1,\, y \in \Omega} |f(y, \tau)|^r dy\right\}^{s/r} d\tau\right\}^{1/s}$$

where $f$ is a function on $Q_T$. Proposition 1 yields

$$\bar{\boldsymbol{v}}_{r,s}(x, t) \leqq \sum_{k=1}^{5} \overline{L^{(k)}[\boldsymbol{v}]}_{r,s}(x, t), \qquad (x, t) \in Q_T.$$

It is clear that

$$\sum_{k=1}^{4} \overline{L^{(k)}[\boldsymbol{v}]}_{r,s}(x, t) \leqq M|x|^{-\mu}, \qquad (x, t) \in Q_T,$$

where $M$ is a constant depending on $\boldsymbol{v}$, $T$ and $\mu$. To evaluate $\overline{L^{(5)}[\boldsymbol{v}]}_{r,s}$, let $x_0$, $|x_0|>3$, be arbitrarily fixed in $\Omega$. Let us express a ball centred at $x$ and of a radius $R$ by $B(x, R)$. Let $\{B(y^{(j)}, 1)\}_{j=1}^{N}$ be a finite covering of $B(x_0, |x_0|/2)$. We assume $y^{(j)} \in B(x_0, |x_0|/2)$ and the multiplicity of the covering is not more than 8, without loss of generality. Let $\{\lambda_j\}_{j=1}^{N}$ be a partition of unity subordinate to this covering. We define

$$I_j(x, t)=\int_0^t\iiint_\Omega \lambda_j(y)|\nabla E(x-y, t-\tau)||\boldsymbol{v}(y, \tau)|^2 dy d\tau,$$

$j=1, 2, \cdots, N$. Then, by Proposition 2 and the estimates (2.10) and (2.13), the following inequality holds:

$$\bar{I}_{jr,s}(x_0, t)\leqq c\min\{1, t(1+t)^2|y^{(j)}-x_0|^{-4}\}$$
$$\times\{\bar{\boldsymbol{v}}_{r,s}(y^{(j)}, t)\}^2.$$

On the other hand, if $|x-x_0|<1$ and $|y-x|<|x_0|/2-1$, then $y\in B(x_0, |x_0|/2)$ and hence there holds by Hölder's inequality and by (2.16)

$$\int_0^t\iiint_\Omega\Big(1-\sum_{j=1}^N\lambda_j(y)\Big)|\nabla E(x-y, t-\tau)||\boldsymbol{v}(y, \tau)|^2 dy d\tau$$
$$\leqq Mt(1+t)^2(|x_0|/2-1)^{-\mu}$$
$$\leqq Mt(1+t)^2|x_0|^{-\mu},$$

where $M$ is a constant depending on $\boldsymbol{v}$ and $\mu$. Hence we have

$$\overline{L^{(5)}[\boldsymbol{v}]}_{r,s}(x_0, t)\leqq M|x_0|^{-\mu}+\sum_{j=1}^N\bar{I}_{jr,s}(x_0, t)$$
$$\leqq M|x_0|^{-\mu}+M\max_{1\leq j\leq N}\{\bar{\boldsymbol{v}}_{r,s}(y^{(j)}, t)\}^2$$

where $M$ is a constant depending on $\boldsymbol{v}$, $\mu$ and $T$. Finally we get

$$\bar{\boldsymbol{v}}_{r,s}(x_0, t)$$
$$\leqq M|x_0|^{-\mu}+M\{\sup_{|y-x_0|\leq|x_0|/2,\, 0\leq\tau\leq t}\bar{\boldsymbol{v}}_{r,s}(y, \tau)\}^2,$$

which implies, as a consequence of Lemma 6, Proposition 4, if one sets $\phi(R)=\sup_{|x_0|\geq R,\, 0\leq t\leq T}\bar{\boldsymbol{v}}_{r,s}(x_0, t)$.

## §4. Proofs of corollaries.

### 4.1. Proof of Corollary 1.

We set $\boldsymbol{v}_s=\boldsymbol{u}_s-\boldsymbol{u}_\infty$ and $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{v}_s$. Let us define

$$L_i^{(5)}[f, g](x, t)=\int_0^t\iiint_\Omega\frac{\partial}{\partial y_k}E_{ij}(x-y, t-\tau)\times\{f_j(y, \tau)g_k(y, \tau)+f_k(y, \tau)g_j(y, \tau)\}dy d\tau, \tag{4.1}$$

$i=1, 2, 3$, where $f$ and $g$ are vector valued functions defined in $Q_\infty$. Then Proposition 1 yields

$$\boldsymbol{w}=\sum_{k=1}^5 L^{(k)}[\boldsymbol{w}]+L^{(5)}[\boldsymbol{w}, \boldsymbol{v}_s]. \tag{4.2}$$

We evaluate each term of the right hand side of (4.2). Let $\sigma^*$ be $\min(\sigma, 1)$ if

$\sigma\neq 1$ and represent any number less than 1 if $\sigma=1$. The assumption (1.17), on account of the estimates (2.7) and (2.14), implies

$$(4.3)\qquad |L^{(1)}[\boldsymbol{w}](x, t)|\leqq M(1+t)^{-\beta^*/2}\Phi_{2,0}(x), \qquad (x, t)\in Q_\infty$$

where $M$ is a constant depending on $\boldsymbol{a}$ and $\beta^*$. Since $\boldsymbol{w}|_{\partial\Omega}=0$, $L^{(k)}[\boldsymbol{w}]$, $k=2, 3$, also vanish. Owing to Lemma 2, the assumption (1.16) implies

$$(4.4)\qquad |L^{(4)}[\boldsymbol{w}](x, t)|\leqq c_\alpha M(1+t)^{-\alpha^*/2}\Phi_{2,0}(x), \qquad (x, t)\in Q_\infty,$$

where $M$ is a constant depending on $\boldsymbol{v}$. Let us evaluate $L^{(5)}[\boldsymbol{w}, \boldsymbol{v}_s]$. By the assumption (1.15) and Theorem 2, we get

$$(4.5)\qquad \begin{aligned}|\boldsymbol{w}(x, t)|&\leqq \sup_{x\in\Omega}|\boldsymbol{w}(x, t)|^{1/2}\sup_{0\leq t}|\boldsymbol{w}(x, t)|^{1/2}\\ &\leqq M(1+t)^{-\alpha/2}\Phi_{1,0}(x), \qquad (x, t)\in Q_\infty,\end{aligned}$$

where $M$ is a constant depending on $\boldsymbol{v}$. The inequalities (1.6) and (4.5) imply

$$|\boldsymbol{w}(x, t)\boldsymbol{v}_s(x)|\leqq M(1+t)^{-\alpha/2}\Phi_{3,0}(x), \qquad (x, t)\in Q_\infty,$$

where $M$ is a constant depending on $\boldsymbol{v}$ and $\boldsymbol{v}_s$. We note that Finn's theorem 5.1 ([5]) assures

$$\iiint_\Omega \Phi_{3,2}(x-y)\Phi_{3,0}(y)dy\leqq c\Phi_{2,0}(x), \qquad x\in\Omega.$$

Hence the following inequality follows from Lemma 2 and (2.10);

$$\begin{aligned}&|L^{(5)}[\boldsymbol{w}, \boldsymbol{v}_s](x, t)|\\ &\leqq\iiint_\Omega\int_0^t M(1+\tau)^{-\alpha/2}\Phi_{3,0}(y)k(x-y, t-\tau; -2)d\tau dy\\ &\leqq c_\alpha M\sqrt{1+t}^{-(\alpha/2)^*}\iiint\Phi_{3,0}(y)\Phi_{3,2}(x-y)dy\\ &\leqq c_\alpha M\sqrt{1+t}^{-(\alpha/2)^*}\Phi_{2,0}(x), \qquad (x, t)\in Q_\infty.\end{aligned}$$

Similarly we get

$$|L^{(5)}[\boldsymbol{w}](x, t)|\leqq cM\sqrt{1+t}^{-(\alpha/2)^*}\Phi_{2,0}(x), \qquad (x, t)\in Q_\infty,$$

where $M$ is a constant depending on $\boldsymbol{v}$. Thus we have

$$(4.6)\qquad |\boldsymbol{w}(x, t)|\leqq M(1+t)^{-\gamma_1}\Phi_{2,0}(x), \qquad (x, t)\in Q_\infty,$$

where $\gamma_1=\min(\beta^*/2, (\alpha/2)^*/2)$ and $M$ is a constant depending on $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{v}$, $\boldsymbol{v}_s$, $\Omega$ and $\gamma_1$. By (4.6) and (1.15) we get

$$|\boldsymbol{w}(x, t)|\leqq M(1+t)^{-\alpha/2-\gamma_1/2}\Phi_{1,0}(x), \qquad (x, t)\in Q_\infty.$$

Substituting the estimate (4.5) by this and repeating the argument, we get instead of (4.6)

$$|\boldsymbol{w}(x, t)|\leqq M(1+t)^{-\gamma_2}\Phi_{2,0}(x), \qquad (x, t)\in Q_\infty,$$

where $\gamma_2=\min(\beta^*/2, (\alpha/2+\gamma_1/2)^*/2)$ and $M$ is a constant depending on $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{v}$, $\boldsymbol{v}_s$, $\Omega$ and $\gamma_2$. A finite number of iterations complete the proof.

**4.2. Proof of Corollary 2.**

Corollary 2 is a consequence of a result of Babenko-Vasil'ev [**2**], which states

PROPOSITION 5. (Babenko-Vasil'ev) *Let $\phi(x)$ be a nonnegative function in $\Omega$ satisfying*

$$\phi(x)\leqq M|x|^{-3/2}(1+s_x)^{-1}, \qquad x\in\Omega \tag{4.7}$$

$$\phi(x)\leqq M\iiint_\Omega \phi(y)|y|^{-1}(1+s_y)^{-1}G_\mu(x-y)dy + M|x|^{-3/2}\exp(-\mu s_x), \qquad x\in\Omega \tag{4.8}$$

*where $0<\mu<1/2$, $M$ is a constant and $G_\mu$ is given by*

$$G_\mu(x)=|x|^{-3/2}(1+|x|^{-1/2})\exp(-\mu s_x). \tag{4.9}$$

*Then $\phi(x)$ satisfies*

$$\phi(x)\leqq M_\varepsilon|x|^{-3/2}\exp\{-(\mu-\varepsilon)s_x\}, \qquad x\in\Omega \tag{4.10}$$

*where $0<\varepsilon<\mu$ and $M_\varepsilon$ is a constant depending on $\varepsilon$ and $M$.*

We derive a representation formula for $\operatorname{rot}\boldsymbol{u}$ and prove the inequalities (4.7) and (4.8) for $\phi(x)=|\operatorname{rot}\boldsymbol{u}(x)|$. We start from the obvious formula

$$\boldsymbol{\omega}=\sum_{k=1}^{5}\operatorname{rot}L^{(k)}[\boldsymbol{v}]$$

where $\boldsymbol{\omega}=\operatorname{rot}\boldsymbol{u}$. The terms $\operatorname{rot}L^{(k)}[\boldsymbol{v}]$, $k=2, 3$ clearly vanish. Note

$$\operatorname{rot}_x(E^*f)(x, y, t, \tau)=\operatorname{rot}_x\Gamma(x-y, t-\tau)f(y, \tau)$$

for a vector valued function $f$, where the $i$-th component of $E^*f$ is given by

$$(E^*f)_i(x, y, t, \tau)=E_{ij}(x-y, t-\tau)f_j(y, \tau).$$

Hence the linear part of $\operatorname{rot}L^{(4)}[\boldsymbol{v}]$ with respect to $(\boldsymbol{v}, \boldsymbol{p})$ can be rewritten as $\operatorname{rot}L^{(6)}[\boldsymbol{v}]$, where

$$\begin{aligned} L^{(6)}[\boldsymbol{v}]_i(x, t) &= \int_0^t\iint_{\partial\Omega}\Gamma\Big\{\Big(-\delta_{ij}\boldsymbol{p}+\frac{\partial v_i}{\partial y_j}+\frac{\partial v_j}{\partial y_i}\Big)n_j-v_i n_1\Big\}dyd\tau \\ &\quad -\int_0^t\iint_{\partial\Omega}\{v_i(\boldsymbol{n}\cdot\nabla)\Gamma+n_i(\boldsymbol{v}\cdot\nabla)\Gamma\}dyd\tau, \end{aligned} \tag{4.11}$$

where $\Gamma=\Gamma(x-y, t-\tau)$, $\boldsymbol{v}=\boldsymbol{v}(y, \tau)$, $\boldsymbol{p}=\boldsymbol{p}(y, \tau)$. The sum of the nonlinear part of $\mathrm{rot}L^{(4)}[\boldsymbol{v}]$ and $\mathrm{rot}L^{(5)}[\boldsymbol{v}]$ equals the left hand side of the following equality

$$-\mathrm{rot}\int_0^t\iiint_\Omega \Gamma(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\,dyd\tau=\sum_{k=7}^{8} L^{(k)}[\boldsymbol{v}]$$

where

$$L^{(7)}[\boldsymbol{v}](x, t)=\frac{1}{2}\int_0^t\iint_{\partial\Omega}\Gamma\,\boldsymbol{n}\times\mathrm{grad}(|\boldsymbol{v}|^2)dyd\tau \tag{4.12}$$

$$L^{(8)}[\boldsymbol{v}]_i(x, t)=\int_0^t\iiint_\Omega \{((\boldsymbol{v}\cdot\nabla)\Gamma)\boldsymbol{\omega}-((\boldsymbol{\omega}\cdot\nabla)\Gamma)\boldsymbol{v}\}\,dyd\tau. \tag{4.13}$$

Thus the representation formula is obtained:

$$\boldsymbol{\omega}=\mathrm{rot}L^{(1)}(\boldsymbol{v})+\mathrm{rot}L^{(6)}[\boldsymbol{v}]+\sum_{k=7}^{8} L^{(k)}[\boldsymbol{v}]. \tag{4.14}$$

Let us prove (4.7) and (4.8) for $\phi(x)=\sup_{0\leq t}|\boldsymbol{\omega}(x, t)|$. The inequality (4.7) is a consequence of Theorem 2 and of a theorem of Finn [**5**, theorem 5.3]. We show (4.8) by (4.14). Theorem 2 and Lemma 4 implies

$$\begin{aligned}|L^{(8)}[\boldsymbol{v}](x, t)| \\ \leqq M\iiint_\Omega \phi(y)|y|^{-1}(1+s_y)^{-1}G_\mu(x-y)dy, \qquad (x, t)\in Q_\infty,\end{aligned} \tag{4.15}$$

where $0<\mu<1/2$ and $M$ is a constant depending on $\boldsymbol{v}$ and $\mu$. By Lemma 4 we also get

$$|\mathrm{rot}\,L^{(6)}[\boldsymbol{v}](x, t)|\leqq M|x|^{-3/2}\exp(-\mu s_x), \qquad (x, t)\in Q_\infty, \tag{4.16}$$

where $0<\mu<1/2$ and $M$ is a constant depending on $\boldsymbol{v}$ and $\mu$. To evaluate $L^{(7)}[\boldsymbol{v}]$, we note

$$\begin{aligned}\iint_{\partial\Omega}\boldsymbol{n}\times\mathrm{grad}(|\boldsymbol{v}|^2)dy&=\iiint_\Omega \mathrm{rot}(\mathrm{grad}(|\boldsymbol{v}|^2))dy \\ &=0.\end{aligned}$$

Let us expand $\Gamma(x-y, t-\tau)$ into the series of powers in $y$ and apply Lemma 4, concluding

$$|L^{(7)}[\boldsymbol{v}](x, t)|\leqq M|x|^{-3/2}\exp(-\mu s_x), \qquad (x, t)\in Q_\infty, \tag{4.17}$$

where $0<\mu<1/2$ and $M$ is a constant depending on $\boldsymbol{v}$ and $\mu$. The evaluation for $\mathrm{rot}L^{(1)}[\boldsymbol{v}]$ remains. To do this, let us decompose $\mathrm{rot}L^{(1)}[\boldsymbol{v}]$;

$$\begin{aligned}\mathrm{rot}L^{(1)}[\boldsymbol{v}]=\mathrm{rot}L^{(1)}[\boldsymbol{v}_s]+L^{(1)}[\mathrm{rot}(\boldsymbol{a}-\boldsymbol{v}_s)] \\ -\iint_{\partial\Omega}\Gamma(x-y, t)\,\boldsymbol{n}\times(\boldsymbol{a}-\boldsymbol{v}_s)dy.\end{aligned}$$

By the assumption (1.19) and by Lemma 5 we have

(4.18) $\quad |L^{(1)}[\mathrm{rot}(\boldsymbol{a}-\boldsymbol{v}_s)](x, t)| \leqq M|x|^{-3/2}\exp(-\mu_1 s_x), \qquad (x, t)\in Q_\infty,$

where $M$ is a constant depending on $\boldsymbol{a}-\boldsymbol{v}_s$. We also obtain

(4.19)
$$\left|\iint_{\partial\Omega}\Gamma(x-y, t)\boldsymbol{n}\times(\boldsymbol{a}-\boldsymbol{v}_s)dy\right|$$
$$\leqq M\sup_{0\leqq t}\Gamma(x, t)\leqq M|x|^{-3/2}\exp(-s_x/2),$$

for large $|x|$, where $M$ is a constant depending on $\boldsymbol{a}-\boldsymbol{v}_s$. An application of the formula (4.14) to $\boldsymbol{v}=\boldsymbol{v}_s$ gives us

$$\mathrm{rot}\, L^{(1)}[\boldsymbol{v}_s]=\mathrm{rot}\,\boldsymbol{v}_s-\mathrm{rot}\, L^{(6)}[\boldsymbol{v}_s]-\sum_{k=7}^{8} L^{(k)}[\boldsymbol{v}_s].$$

Then, the inequality (1.7) and the inequalities (4.15) to (4.17) for $\boldsymbol{v}=\boldsymbol{v}_s$ yield

(4.20) $\quad |\mathrm{rot}\, L^{(1)}[\boldsymbol{v}_s](x, t)| \leqq M|x|^{-3/2}\exp(-\mu s_x), \qquad (x, t)\in Q_\infty,$

where $0<\mu<1/2$ and $M$ is a constant depending on $\boldsymbol{v}_s$ and $\mu$. Thus the inequalities (4.18) to (4.20) give

(4.21) $\quad |\mathrm{rot}\, L^{(1)}[\boldsymbol{v}](x, t)| \leqq M|x|^{-3/2}\exp(-\mu_1 s_x), \qquad (x, t)\in Q_\infty,$

which, combined with the estimates (4.15) to (4.17), gives (4.8) with $0<\mu\leqq\mu_1$, $\mu\neq 1/2$. This completes the proof.

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