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On the behavior at infinity of logarithmic potentials

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1. Statement of results.

For a (signed) measure λ in the plane R^2, we define

$$L\lambda(x)=\int \log\frac{1}{|x-y|}d\lambda(y)$$

if the integral exists at x. We note that $L\lambda(x)$ is finite for some x if and only if

$$\int \log(1+|y|)d|\lambda|(y)<\infty, \tag{1}$$

where $|\lambda|$ denotes the total variation of λ. Denote by $B(x, r)$ the open disc with center at x and radius r. For $E\subset B(0, 2)$ we set

$$C(E)=\inf \mu(R^2),$$

where the infimum is taken over all nonnegative measures μ on R^2 such that S_μ (the support of μ)$\subset B(0, 4)$ and

$$\int \log\frac{8}{|x-y|}d\mu(y)\geqq 1 \qquad \text{for every} \quad x\in E.$$

A set E in R^2 is said to be thin at infinity if

$$\sum_{j=1}^{\infty} jC(E_j')<\infty, \quad E_j'=\{x\in B(0, 2)-B(0, 1);\ 2^j x\in E\}. \tag{2}$$

It is known (cf. Brelot [1; Theorem IX, 7]) that if μ is a nonnegative measure on R^2 satisfying (1), then there exists a set $E\subset R^2$ which is thin at infinity and for which

$$\lim_{|x|\to\infty,\, x\in R^2-E}[L\mu(x)+\mu(R^2)\log|x|]=0.$$

Our first aim is to establish the following result.

THEOREM 1. *Let μ be a nonnegative measure on R^2 satisfying* (1). *Then there exists a set E in R^2 such that*

$$\lim_{|x|\to\infty,\, x\in R^2-E}(\log|x|)^{-1}L\mu(x)=-\mu(R^2),$$

$$\sum_{j=1}^{\infty} j^2C(E_j')<\infty, \tag{3}$$

where E_j' are defined as above.

COROLLARY 1. *Let μ be as in Theorem 1. Suppose there exists a set $E\subset R^2$ with the following two properties:*

(i) $\liminf_{|x|\to\infty,\, x\in E} (\log|x|)^{-1}L\mu(x)\geqq 0$;

(ii) $\sum_{j=1}^{\infty} j^2C(E_j')=\infty$.

Then $\mu=0$.

Let F be a closed set in R^2. A positive measure μ is called an equilibrium measure on F if $S_\mu\subset F$, $\mu(F)=1$ and $L\mu$ is equal to a constant on F except for a set E with logarithmic capacity zero, which means that $C(E_j')=0$ for any integer j.

COROLLARY 2. *If a closed set F in R^2 has an equilibrium measure, then F satisfies* (3).

This result is an improvement of Ninomiya [3; Theorem 4].

Next we shall prove

THEOREM 2. *Let E be a set in R^2 which is thin at infinity. Then there exist $r_0>0$ and a (signed) measure λ on R^2 satisfying* (1) *such that $\lambda(R^2)=0$, $L\lambda(x)=1$ for all $x\in E-B(0, r_0)$ and $L\lambda(x)\leqq 1$ for all $x\in R^2$.*

Finally we shall be concerned with the existence of equilibrium measures.

THEOREM 3. *Let E be a subset of $R^2-B(0, 2)$ satisfying* (3). *Then there exist a positive measure μ on R^2 and a number γ such that $L\mu(x)=\gamma$ on E and $L\mu(x)\leqq\gamma$ on R^2.*

2. Proof of Theorem 1.

Let μ be as in Theorem 1, and write

$$\begin{aligned} &L\mu(x)+\mu(R^2)\log|x| \\ &=\int_{R^2-B(x,|x|/2)}\log\frac{|x|}{|x-y|}d\mu(y) \\ &\quad+\int_{B(x,|x|/2)}\log\frac{|x|}{|x-y|}d\mu(y)=L'(x)+L''(x)\,. \end{aligned}$$

By (1) $L'(x)$ is finite for $x\neq 0$. If $y\in R^2-B(x, |x|/2)$, then

$$\left|\log\frac{|x|}{|x-y|}\right|\leqq \text{const.}\min\left(1, \frac{|y|}{|x|}\right)\log\left(2+\frac{|y|}{|x|}\right).$$

Hence Lebesgue's dominated convergence theorem implies that $\lim_{|x|\to\infty} L'(x)=0$.

Next we discuss the behavior at infinity of L''. For this purpose we take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j=\infty$ and

$$\sum_{j=1}^{\infty} a_j j \mu(B_j) < \infty, \quad B_j = B(0, 2^{j+2}) - B(0, 2^{j-1}).$$

Consider

$$E_j = \{x \in B(0, 2^{j+1}) - B(0, 2^j) \,;\, L''(x) \geqq a_j^{-1} \log|x|\}$$

and set $E = \bigcup_{j=1}^{\infty} E_j$. If $x \in B(0, 2^{j+1}) - B(0, 2^j)$, then $B(x, |x|/2) \subset B_j$, and hence

$$0 \leqq (\log|x|)^{-1} L''(x) \leqq (j \log 2)^{-1} \int_{B_j} \log \frac{2^{j+3}}{|x-y|} d\mu(y).$$

Therefore, $C(E_j') \leqq a_j (j \log 2)^{-1} \mu(B_j)$, from which we see that E satisfies (3). Moreover it follows that

$$\lim_{|x| \to \infty,\, x \in R^2 - E} (\log|x|)^{-1} L''(x) = 0.$$

Thus the proof of Theorem 1 is complete.

3. Proof of Theorem 2.

For $E \subset R^2$ denote by E^* the inversion of E, i.e., $E^* = \{x^* = x/|x|^2 \,;\, x \in E\}$. First note that E is thin at infinity if and only if E^* is thin at 0, i.e., E^* satisfies

$$(2)' \qquad \sum_{j=1}^{\infty} jC(E_{-j}^{*\prime}) < \infty, \quad E_{-j}^{*\prime} = \{x \in B(0, 2) - B(0, 1) \,;\, 2^{-j} x \in E^*\},$$

which is equivalent to

$$(2)'' \qquad \sum_{j=1}^{\infty} jC(E_{-j}^{*}) < \infty, \quad E_{-j}^{*} = \{x \in E^* \,;\, x \in B(0, 2^{-j+1}) - B(0, 2^{-j})\}.$$

Now assume that E is thin at infinity, so that E^* is thin at 0. Since $C(\cdot)$ is an outer capacity, there exists an open set G^* in R^2 such that $E^* \subset G^*$ and G^* is thin at 0. In view of [2; No. 12, Chap. IV], we can find r, $0<r<1$, and a positive measure μ^* on R^2 such that $S_{\mu^*} \subset \overline{G^* \cap B(0, r)}$, $L\mu^*(0) < 1$, $L\mu^*(x^*) = 1$ for all $x^* \in G^* \cap B(0, r)$ and $L\mu^*(x^*) \leqq 1$ for all $x^* \in R^2$. Define μ by setting $\mu(A) = \mu^*(A^*)$ for a Borel set $A \subset R^2$, and note that

$$L\mu(x) = L\mu^*(x^*) + \mu^*(R^2) \log|x^*| - L\mu^*(0), \qquad x^* = x/|x|^2.$$

Set $a = 1 - L\mu^*(0)$ and $\lambda = a^{-1}(\mu - \mu(R^2)\delta_0)$, where δ_0 denotes the dirac measure at 0. Then

$$L\lambda(x) = a^{-1}\{L\mu^*(x^*) - L\mu^*(0)\},$$

which is equal to 1 for $x \in E - \overline{B(0, r^{-1})}$ and is not greater than 1 for all $x \in R^2$. Thus λ satisfies all the conditions in our theorem, and we conclude the proof.

4. Proof of Theorem 3.

Before the proof of Theorem 3 we prepare the following lemma.

LEMMA. *If G is an open subset of $B(0, 1)$ satisfying*

$$(3)' \qquad \sum_{j=1}^{\infty} j^2 C(G'_{-j}) < \infty, \quad G'_{-j} = \{x \in B(0, 2) - B(0, 1)\ ;\ 2^{-j} x \in G\},$$

then there exists a positive measure ν such that $S_\nu \subset B(0, 1)$, $L\nu$ is bounded on $B(0, 2) - B(0, r)$ for any $r > 0$, $\langle \nu, \nu \rangle \equiv \int L\nu(x) d\nu(x) < \infty$ and

$$\lim_{x \to 0,\, x \in G} \left(\log \frac{1}{|x|} \right)^{-1} L\nu(x) = \infty .$$

PROOF. By [2; Theorem 2.6′], for each positive integer j there exists a positive measure ν_j such that $S_{\nu_j} \subset B(0, 2^{-j+2}) - B(0, 2^{-j-1})$, $\nu_j(R^2) < C(G'_{-j}) + \varepsilon_j$,

$$\int \log \frac{2^{-j+3}}{|x-y|} d\nu_j(y) = 1 \qquad \text{for all} \quad x \in G_{-j}$$

and

$$\int \log \frac{2^{-j+3}}{|x-y|} d\nu_j(y) \leqq 1 \qquad \text{for all} \quad x \in R^2,$$

where $\{\varepsilon_j\}$ is a sequence of positive numbers such that $\sum_{j=1}^{\infty} j^2 (C(G'_{-j}) + \varepsilon_j) < \infty$. Take a sequence $\{a_j\}$ of positive numbers which increases to ∞ and satisfies

$$\sum_{j=1}^{\infty} (a_{j+3} j)^2 (C(G'_{-j}) + \varepsilon_j) < \infty .$$

Define $\nu = \sum_{j=2}^{\infty} a_j j \nu_j$. Then $\nu(R^2) = \sum_{j=2}^{\infty} a_j j \nu_j(R^2) \leqq \sum_{j=2}^{\infty} a_j j (C(G'_{-j}) + \varepsilon_j) < \infty$. If $x \in B(0, 2^{-k+1}) - B(0, 2^{-k})$, then

$$L\nu(x) \leqq \text{const.} \left\{ \sum_{j=2}^{k-3} a_j j^2 (C(G'_{-j}) + \varepsilon_j) + \sum_{j=k-2}^{k+2} a_j j (1 + j \nu_j(R^2) \log 2) \right.$$
$$\left. + \sum_{j=k+3}^{\infty} a_j j k (C(G'_{-j}) + \varepsilon_j) \right\} \leqq \text{const.}\ a_{k+2} k .$$

Consequently we derive

$$\langle \nu, \nu \rangle = \sum_{k=2}^{\infty} a_k k \int L\nu(x) d\nu_k(x) \leqq \text{const.} \sum_{k=2}^{\infty} (a_k k)(a_{k+3} k)(C(G'_{-k}) + \varepsilon_k) < \infty .$$

On the other hand we have for $x \in G_{-k}$,

$$L\nu(x) + \nu(R^2) \log 2 \geqq a_k k \{L\nu_k(x) + \nu_k(R^2) \log 2\}$$
$$\geqq a_k k \{1 + (k-2) \nu_k(R^2) \log 2\} .$$

This implies that $\lim_{x\to 0,\, x\in G}\left(\log\frac{1}{|x|}\right)^{-1}L\nu(x)=\infty$. Thus the lemma is established.

We are now ready to prove Theorem 3.

PROOF OF THEOREM 3. Let E be a subset of $R^2-B(0, 2)$ which satisfies (3). Then the inversion E^* of E satisfies (3)′ with $G=E^*$. Hence there exists an open subset G of $B(0, 1)$ such that $E^*\subset G$ and G satisfies (3)′. Let ν be a positive measure as in the Lemma.

Let $U_1(G)$ be the totality of positive measures μ such that $S_\mu\subset G$, $\mu(G)=1$ and $\langle\mu, \mu\rangle<\infty$. Set

$$V(\mu)=\langle\mu, \mu\rangle-2L\mu(0),$$

and consider

$$\gamma=\inf\{V(\mu);\ \mu\in U_1(G)\}.$$

Take r, $0<r<1$, such that $\log|x|^{-1}<L\nu(x)$ for every $x\in G\cap B(0, r)$. Then we obtain for $\mu\in U_1(G)$,

$$\begin{aligned}\int_{B(0,r)}\log\frac{1}{|x|}d\mu(x)&\leqq\int_{B(0,r)}L\nu(x)d\mu(x)\\ &\leqq\iint\log\frac{2}{|x-y|}d\nu(y)d\mu(x)\\ &\leqq 2^{-1}(\langle\nu, \nu\rangle+\langle\mu, \mu\rangle)+\nu(R^2)\log 2,\end{aligned}$$

which implies that $V(\mu)\geqq-\langle\nu, \nu\rangle-2\log\frac{1}{r}-2\nu(R^2)\log 2>-\infty$. Take a sequence $\{\mu_j\}$ of positive measures in $U_1(G)$ such that $\lim_{j\to\infty}V(\mu_j)=\gamma$. We may assume that $\{\mu_j\}$ converges vaguely to a positive measure μ_0. For $0<r<1$, define $\Lambda(r)=\inf\left\{\left(\log\frac{1}{|x|}\right)^{-1}L\nu(x);\ x\in G\cap B(0, r)\right\}$. Let ϕ_r be a function in $C_0(B(0, r))$ such that $0\leqq\phi_r\leqq 1$ on R^2 and $\phi_r=1$ on $B(0, r/2)$. Then we have

$$L\mu_j(0)\leqq\Lambda(r)^{-1}\int_{B(0,r)}L\nu(x)d\mu_j(x)+\int(1-\phi_r(x))\log\frac{1}{|x|}d\mu_j(x).$$

It follows that $\{\langle\mu_j, \mu_j\rangle\}$ is bounded and $\limsup_{j\to\infty}L\mu_j(0)\leqq L\mu_0(0)$. Since $\liminf_{j\to\infty}L\mu_j(0)\geqq L\mu_0(0)$, $\lim_{j\to\infty}L\mu_j(0)=L\mu_0(0)$. Note here that $L\mu_0(0)$ is finite. On the other hand we see that $\lim_{j\to\infty}\langle\mu_j, \mu_j\rangle\geqq\langle\mu_0, \mu_0\rangle$. Since $U_1(G)$ is convex, $V((\mu_j+\mu_k)/2)\geqq\gamma$ for any positive integers j and k. Letting first $j\to\infty$ and next $k\to\infty$, we establish $V(\mu_0)\geqq\gamma$. Hence $V(\mu_0)=\gamma$ and $\lim_{j\to\infty}\langle\mu_j, \mu_j\rangle=\langle\mu_0, \mu_0\rangle$. The last equality also implies that $\lim_{j\to\infty}\langle\mu_j-\mu_0, \mu_j-\mu_0\rangle=0$.

If $\mu\in U_1(G)$ and $0<t<1$, then

$$\gamma \leqq \liminf_{j\to\infty} V((1-t)\mu_j + t\mu)$$
$$= V(\mu_0) + 2t\left\{\langle \mu_0, \mu-\mu_0\rangle - \int \log\frac{1}{|x|}d(\mu-\mu_0)\right\} + t^2\langle \mu-\mu_0, \mu-\mu_0\rangle,$$

which yields

$$\int L\mu_0(x)d\mu(x) - L\mu(0) \geqq \int L\mu_0(x)d\mu_0(x) - L\mu_0(0).$$

For $x^0 \in G$, by taking as μ the unit uniform surface measure on the circle $\partial B(x^0, r)$ and letting $r \downarrow 0$, we obtain

$$(4) \qquad L\mu_0(x^0) \geqq L\mu_0(0) + \gamma + \log\frac{1}{|x^0|}.$$

Let $x^0 \in S_{\mu_0}$, and suppose

$$L\mu_0(x^0) > L\mu_0(0) + \gamma + \log\frac{1}{|x^0|}.$$

Since $L\mu_0$ is lower semicontinuous, there exists $r > 0$ such that

$$(5) \qquad L\mu_0(x) > L\mu_0(0) + \gamma + \log\frac{1}{|x|} \quad \text{for every} \quad x \in B(x^0, r).$$

Let ϕ be a function in $C_0(B(x^0, r))$ such that $0 \leqq \phi \leqq 1$ on R^2 and $\phi = 1$ on $B(x^0, r/2)$, and set $\sigma_j = (\phi\mu_j)(R^2)\mu_j - \phi\mu_j$. Then $\mu_j + t\sigma_j \in U_1(G)$ for any positive integer j and any t, $0 < t < 1$. Hence $V(\mu_j + t\sigma_j) \geqq \gamma$ for above j and t, from which it follows that

$$(\phi\mu_0)(R^2)\left\{\int L\mu_0(x)d\mu_0(x) - L\mu_0(0)\right\} \geqq \int \left\{L\mu_0(x) - \log\frac{1}{|x|}\right\}\phi(x)d\mu_0(x).$$

By (5) we derive

$$V(\mu_0) > \gamma,$$

which is a contradiction. Thus we proved that

$$(6) \qquad L\mu_0(x) \leqq L\mu_0(0) + \gamma + \log\frac{1}{|x|} \quad \text{on} \quad S_{\mu_0}.$$

Define μ_0^* by setting $\mu_0^*(A^*) = \mu_0(A)$ for a Borel set A in R^2, where $A^* = \{x/|x|^2;\ x \in A\}$. Then by (4) $L\mu_0^* \geqq \gamma$ on G^*, and by (6) $L\mu_0^* \leqq \gamma$ on $S_{\mu_0^*}$. Thus μ_0^* satisfies all the conditions in our theorem, and hence we conclude the proof.

5. Further results.

Let E be a set in R^2 whose exterior is not empty. Suppose $B(x^0, 2r_0) \subset R^2 - E$, where $r_0 > 0$. If E is thin at infinity, then the inversion of E with respect to $\partial B(x^0, r_0)$ is thin at x^0. Note here that a set A is thin at x^0 if and only if $\{x - x^0;\ x \in A\}$ is thin at 0. Moreover if E satisfies (3), then

$$\sum_{j=1}^{\infty} j^2 C(E'_j) < \infty,$$

where $E'_j = \{x \in B(x^0, 2r_0) - B(x^0, r_0)\ ;\ x^0 + 2^j(x - x^0) \in E\}$ and $C(E'_j)$ are defined to be the quantities $C(\{x - x^0\ ;\ x \in E'_j\})$. Thus, applying the routine methods as in the proof of Theorem 3, we obtain the following results.

THEOREM 2′. *Let F be a closed set in R^2 and $x^0 \in R^2 - F$. If F is thin at infinity, then there exist $M > 0$ and a (signed) measure λ satisfying* (1) *such that $S_\lambda \subset F \cup \{x^0\}$, $\lambda(R^2) = 0$, $L\lambda = 1$ on $F - B(0, M)$ except for a set with logarithmic capacity zero and $L\lambda \leqq 1$ on R^2.*

THEOREM 3′. *Let E be a subset of R^2 whose exterior is not empty. If E satisfies* (3), *then there exist a positive measure μ and a number γ such that $L\mu = \gamma$ on E and $L\mu \leqq \gamma$ on R^2.*

In view of Corollary 2 to Theorem 1, we can establish

THEOREM 4. *Let F be a closed set in R^2. Then F has an equilibrium measure if and only if F satisfies* (3).

This result gives a negative answer to the question of Ninomiya [**3**; p. 216]. Combining Theorem 4 with [**3**; Theorem 5], we derive the following result.

COROLLARY. *Let F be a closed set in R^2. Then the following statements are equivalent:*

(i) *F has an equilibrium measure.*

(ii) *F is of logarithmic capacity finite in the sense of Ninomiya* [**3**].

(iii) *F satisfies* (3).

References

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