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A pinching problem for symmetric spaces of rank one

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§ 0. Introduction.

A main problem in Riemannian geometry is to investigate the influences of geometrical quantities of complete Riemannian manifolds on the topology. The pioneering work for this is the well known sphere theorem due to Rauch [12] which was improved by Klingenberg [9]. Take $M=S^n$ with the constant sectional curvature equal to 1. The theorem states that if $\bar{M}$ is a complete simply connected n-manifold with the sectional curvature $K_{\bar{M}}$, $\frac{1}{4}<K_{\bar{M}}\leqq 1$, then $\bar{M}$ is homeomorphic to S^n. A stronger assumption for curvature implies that $\bar{M}$ must be diffeomorphic to S^n ([6], [14], [16]).

Cheeger [3] defines another notion of pinching. Let M, $\bar{M}$ be compact Riemannian manifolds of $\dim M=\dim \bar{M}=n$ and $m\in M$, $\bar{m}\in\bar{M}$. Let $I: M_m\to\bar{M}_{\bar{m}}$ be a linear isometry between the tangent spaces. For a geodesic γ emanating from m, let $\bar{\gamma}$ denote the geodesic emanating from $\bar{m}$ such that $\bar{\gamma}'(0)=I(\gamma'(0))$, and P_γ the parallel translation along γ. Set $I_\gamma:=P_{\bar{\gamma}}\circ I\circ P_{\gamma^{-1}}$. I_γ induces an isomorphism on tensor spaces. We denote by R_M the curvature tensor of M and by $L(\ )$ the length of curves. Now set:

$$\tilde{\rho}(M, \bar{M}) := \inf_{m,\bar{m},I}\left[\sup\{\|R_M-I_\gamma^{-1}(R_{\bar{M}})\|\ ;\ L(\gamma)\leqq 2\,\mathrm{diam}(M)\}\right].$$

Let M be a simply connected compact rank one symmetric space (henceforth SCROSS). One of his results states that there exists an $\varepsilon>0$ such that if a compact simply connected manifold $\bar{M}$ is ε-close to M with respect to $\tilde{\rho}$, then $\bar{M}$ is piecewise linearly homeomorphic to M.

The main purpose of this paper is to consider diameter or volume-pinching for SCROSSes using a somewhat weaker one than $\tilde{\rho}$, as well as to strengthen the topological conclusion to diffeomorphism. For $m\in M$, we denote by $\mathfrak{S}_m$ the compact domain in M_m bounded by the tangent cut locus of m: $\mathfrak{S}_m:=\{v\in M_m\ ;\ d(\exp_m v, m)=\|v\|\}$, and by $U\mathfrak{S}_m$ the set of all unit tangent vectors on $\mathfrak{S}_m$. We define our pinching numbers by

DEFINITION.

$$\rho_0(M, \bar{M})=\inf_{m,\bar{m},I}\left[\sup\{|\,\|d\exp_{\bar{m}} I(v)\|-\|d\exp_m(v)\|\,|\ ;\ v\in U\mathfrak{S}_m\}\right],$$

$$\rho(M, \bar{M})=\rho_0(M, \bar{M})+|\mathrm{diam}(M)-\mathrm{diam}(\bar{M})|+|\mathrm{Vol}(M)-\mathrm{Vol}(\bar{M})|,$$

where $\mathrm{diam}(M)$ and $\mathrm{Vol}(M)$ denote the diameter and the volume of M.

We denote by $\nabla R_{\bar{M}}$ the covariant derivative of the curvature tensor $R_{\bar{M}}$.

We shall prove the following:

THEOREM 3.4. *Let M be a SCROSS. For given Λ, $\Lambda_1>0$, there exists an $\varepsilon>0$ depending only on M and Λ, Λ_1 such that if a compact manifold $\bar{M}$ satisfies $|K_{\bar{M}}|\leqq\Lambda^2$, $\|\nabla R_{\bar{M}}\|\leqq\Lambda_1$, $\rho(M, \bar{M})<\varepsilon$, then $\bar{M}$ is diffeomorphic to M.*

The outline of the proofs of the theorems in this article is sketched as follows. Let m, $\bar{m}$ and I minimize the quantity in the definition of $\rho(M, \bar{M})$. Set $\Phi:=\exp_{\bar{m}}\circ I\circ\exp_m{}^{-1}$, where $\exp_m{}^{-1} : M\to\mathfrak{S}_m$ is some inverse. Although Φ is not continuous, it will be seen in Section 3 that Φ is almost distance preserving. In Section 1, we shall show that such a map can be approximated by a diffeomorphism.

Our pinching constants can be estimated explicitly. But we shall not do this in order to avoid non-essential complexity.

§ 1. ε-mappings.

Before proceeding to our pinching situation, we begin with a general consideration. For given n, Λ, Λ_1, $R>0$, we denote by $\mathfrak{M}^n(\Lambda, \Lambda_1, R)$ the following class of n dimensional complete Riemannian manifolds M (not necessarily compact):

$$|K_M|\leqq\Lambda^2, \quad \|\nabla R_M\|\leqq\Lambda_1, \quad i(M)\geqq R,$$

where $i(M)$ denotes the injectivity radius of the exponential map on M. From now on, for given Λ, R, we will set $R_0:=\frac{1}{2}\min\{\pi/\Lambda, R\}$ implicitly. Notice that if $r<R_0$, then the r-ball $B(p, r)$ around any $p\in M$ is convex and if r is taken sufficiently small, then $\exp_p|B(0, r)$ is almost isometric, where $B(0, r)$ denotes the r-ball in M_p around the origin.

DEFINITION. We say that a map $f : X\to Y$ between metric spaces X and Y is an ε-map if $|d(f(x), f(x'))-d(x, x')|<\varepsilon$ for all x, $x'\in X$.

Notice that f is not necessarily continuous and any inverse map $f^{-1} : f(X)\to X$ is also an ε-map.

ε-MAPPING THEOREM 1.1. *There exists an $\varepsilon_0=\varepsilon_0(n, \Lambda, \Lambda_1, R)>0$ such that if M, $\bar{M}\in\mathfrak{M}^n(\Lambda, \Lambda_1, R)$ and $f : M\to\bar{M}$ is an ε_0-map, then f can be approximated by a diffeomorphism.*

For the proof, we use the following general result which has been proved by using a technique of center of mass and which has been applied to finiteness theorems and a differentiable sphere theorem (See [17] and also [7], [11], [18]).

GENERAL THEOREM. *For given* n, Λ, Λ_1, $R>0$, *there exist positive constants* $\varepsilon_1=\varepsilon_1(n)$, $r_1=r_1(n, \Lambda, \Lambda_1, R)$ *such that if* $M, \bar{M}\in\mathfrak{M}^n(\Lambda, \Lambda_1, R)$ *satisfy the following condition, then they are diffeomorphic: for some* $\varepsilon\leqq\varepsilon_1$, $r\leqq r_1$, *there are r-dense and r/2-discrete subsets* $\{p_i\}\subset M$, $\{q_i\}\subset\bar{M}$ *such that the correspondence* $p_i\mapsto q_i$ *is bijective and satisfies that* $1-\varepsilon\leqq d(q_i, q_j)/d(p_i, p_j)\leqq 1+\varepsilon$ *for all* p_i, p_j *with* $d(p_i, p_j)\leqq 2^{(n+13)}r$. *The diffeomorphism* $F: M\to\bar{M}$ *is taken so as to satisfy* $d(F(p_i), q_i)\leqq\delta r$, *where* δ *is a function of* n, Λ, r, ε *such that* $\delta\to 0$ *as* r, $\varepsilon\to 0$.

By definition, a subset A of a metric space X is *δ-dense* (resp. *δ-discrete*) if any $x\in X$ has the distance $d(x, A)<\delta$ (resp. if any distinct pair $a\neq a'\in A$ has the distance $d(a, a')\geqq\delta$). A system of points $\{x_i\}$ in X is said to be a *δ-maximal system* if it is maximal with respect to the property that $d(x_i, x_j)\geqq\delta$, $i\neq j$. Notice that $\{x_i\}$ is δ-maximal if and only if it is δ-dense and δ-discrete.

In [**17**], Lemma 2.1', we have proved the following lemma essentially.

LEMMA 1.2. *Let* $\alpha\leqq 2^{-(n+7)}$, $\varepsilon\leqq 2^{-(n+14)}$. *Let* $\{x_i\}_{i=1,\dots,N}$ *be an* αr*-maximal system of* $B(0, r)\subset\boldsymbol{R}^n$ *with* $x_1=0$. *If a system* $\{y_i\}_{i=1,\dots,N}$ *of points in* $B(0, r)$ *with* $y_1=0$ *satisfies*

$$1-\varepsilon\leqq\|y_i-y_j\|/\|x_i-x_j\|\leqq 1+\varepsilon \qquad \textit{for every} \quad i\neq j,$$

then there exist a linear isometry I *of* $\boldsymbol{R}^n$ *and some constant* $c(n)$ *such that* $\|I(x_i)-y_i\|\leqq c(n)\varepsilon'^{1/2}\cdot r$ *for every* i, *where* $\varepsilon'=4(3\varepsilon(1+2\alpha^{-2}))^{1/2}$.

LEMMA 1.3. *For given* n, Λ, $\varepsilon>0$, *there exist* r, $\delta>0$ *such that if complete n-manifolds* M *and* $\bar{M}$ *with* $|K_M|$, $|K_{\bar{M}}|\leqq\Lambda^2$, $i(M)$, $i(\bar{M})\geqq R$ *admit a* δ*-map* $f: M\to\bar{M}$, *then for every* $m\in M$ *the following are satisfied:*

(1) $\exp_m|B(0, r)$ *is a* $4\Lambda r^2$*-map,*

(2) $f|B(m, r)$ *admits an* εr*-approximation of the form* $\exp_{f(m)}\circ I\circ\exp_m^{-1}$,

(3) $f(M)$ *is* $2\varepsilon r$*-dense.*

PROOF. (1). By the Rauch comparison theorem (henceforth RCT), we may assume that $r<R_0$ is chosen so small that for every x, $y\in B(0, r)$

$$e^{-\Lambda r}\leqq\frac{\sin\Lambda r}{\Lambda r}\leqq\frac{d(\exp_m x, \exp_m y)}{\|x-y\|}\leqq\frac{\sinh\Lambda r}{\Lambda r}\leqq e^{\Lambda r},$$

and hence $|d(\exp_m x, \exp_m y)-\|x-y\||<4\Lambda r^2$.

(2). (1) implies that the map $g:=\exp_{f(m)}{}^{-1}\circ f\circ\exp_m|B(0, r)$ is a $(\delta+8\Lambda r^2)$-map.

ASSERTION. There exists a linear isometry $I: M_m\to\bar{M}_{f(m)}$ such that $\|g-I\|<\eta(n, \Lambda, \delta, r)r$ on $B(0, r)$, where $\eta\to 0$ as r, δ/r, $r^3/\delta\to 0$.

PROOF. Set $\alpha:=\sqrt[4]{\delta/r}$ and take an αr-maximal system $\{x_i\}$ on $B(0, r)$ with $x_1=0$. Now $|\|g(x_i)-g(x_j)\|-\|x_i-x_j\||<\delta+8\Lambda r^2$. By discreteness, this implies

$$|\|g(x_i)-g(x_j)\|/\|x_i-x_j\|-1|<\alpha^3+8\Lambda r/\alpha=:\varepsilon'.$$

By Lemma 1.2, if α and ε are taken sufficiently small, then there exists a linear isometry I such that

$$\|I(x_i)-g(x_i)\| \leqq c(n)\varepsilon''^{1/2}r$$

for some constant $c(n)$, where $\varepsilon''=4(3\varepsilon'(1+2\alpha^{-2}))^{1/2}$. Notice that $\varepsilon''\to 0$ as r, δ/r, $r^3/\delta\to 0$. For any $x\in B(0, r)$, by denseness we may take an x_i such that $\|x_i-x\|<\alpha r$. Then we get

$$\begin{aligned}\|I(x)-g(x)\| &\leqq \|I(x)-I(x_i)\|+\|I(x_i)-g(x_i)\|+\|g(x_i)-g(x)\| \\ &<(2\alpha+\alpha^3+8\Lambda r+c(n)\varepsilon''^{1/2})r \\ &=: \eta(n, \Lambda, \delta, r)r .\end{aligned}$$

Q.E.D.

Now for any $p\in B(m, r)$, we have

$$\begin{aligned}d(f(p), \exp_{f(m)}\circ I\circ\exp_m^{-1}(p)) &<\|g(\exp_m^{-1}(p))-I(\exp_m^{-1}(p))\|+4\Lambda r^2 \\ &<(\eta+4\Lambda r)r .\end{aligned}$$

Hence for the proof of (2), it suffices to choose r, δ so small that $\eta+4\Lambda r<\varepsilon$.

(3). We first show that

(*) $f(B(m, r))$ is $2\varepsilon r$-dense in $B(f(m), r)$ for every $m\in M$.

Let $\{x_i\}\subset B(0, r)$ be the αr-maximal system as in the proof of (2). For any $q\in B(f(m), r)$, there is an x_i such that $\exp_m x_i\in(\exp_{f(m)}\circ I\circ\exp_m^{-1})^{-1}(B(q, 2\alpha r))$. Then (2) yields

$$\begin{aligned}d(f(\exp_m x_i), q) &\leqq d(f(\exp_m x_i), \exp_{f(m)}\circ I(x_i))+d(\exp_{f(m)}\circ I(x_i), q) \\ &<\varepsilon r+2\alpha r<2\varepsilon r .\end{aligned}$$

By induction, we assert that $f(B(m, (2k-1)r))$ is $2\varepsilon r$-dense in $B(f(m), kr)$ for $k=1, 2, \cdots$. This will complete the proof of (3). Suppose that the assertion is true for k. For any q in the ball $B(f(m), (k+1)r)$, take $q_1\in B(f(m), kr)$ with $q\in B(q_1, r)$. The induction hypothesis assures the existence of such a point $p\in B(m, (2k-1)r)$ that $d(f(p), q_1)<2\varepsilon r$. Since $d(f(p), q)<(1+2\varepsilon)r$, it is possible to take a point q_2 on the unique minimal geodesic from $f(p)$ to q such that $d(f(p), q_2)<r$, $d(q_2, q)<2\varepsilon r$. Then by (*) there exists a $p'\in B(p, r)$ with $d(f(p'), q_2)<2\varepsilon r$. Since

$$d(f(p'), q)\leqq d(f(p'), q_2)+d(q_2, q)<4\varepsilon r<r ,$$

(*) implies again the existence of such a point $p''\in B(p', r)$ that $d(f(p''), q)<2\varepsilon r$, where

$$d(m, p'')\leqq d(m, p)+d(p, p')+d(p', p'')<(2k+1)r .$$

This completes the induction argument. Q.E.D.

Proof of ε-mapping Theorem 1.1. Let $\varepsilon_1(n)$, $r_1(n, \Lambda, \Lambda_1, R)$ be the constants as in General Theorem. For an $\varepsilon_0<\frac{3}{4}r_1\varepsilon_1$, let $f: M\to\bar{M}$ be an ε_0-map. Take a $\frac{3}{4}r_1$-maximal system $\{p_i\}$ on M and set $q_i:=f(p_i)$. Now the inequality $|d(q_i, q_j)-d(p_i, p_j)|<\varepsilon_0$ implies

$$\left|\frac{d(q_i, q_j)}{d(p_i, p_j)}-1\right|<\varepsilon_0\Big/\left(\frac{3}{4}r_1\right)<\varepsilon_1 .$$

In particular, the correspondence $p_i\mapsto q_i$ is bijective and $\{q_i\}$ is $r_1/2$-discrete. It remains to prove the r_1-denseness of $\{q_i\}$. In Lemma 1.3 take ε, r so small that $2\varepsilon r<r_1/8$. Let δ be the constant given in the lemma. Then setting $\varepsilon_0<\delta$ we see that for any $q\in\bar{M}$, there exists a $p\in M$ such that $d(q, f(p))<2\varepsilon r$. Taking a p_i with $d(p, p_i)<\frac{3}{4}r_1$ yields

$$d(q, q_i)\leqq d(q, f(p))+d(f(p), q_i)<2\varepsilon r+\frac{3}{4}r_1+\varepsilon_0<r_1 .$$

Hence $\{q_i\}$ is r_1-dense and therefore General Theorem implies the existence of a diffeomorphism $F: M\to\bar{M}$ such that $d(F(p_i), q_i)<\delta r_1$, where $\delta\to 0$ as ε_0, $r_1\to 0$.

Q.E.D.

It should be mentioned that Cheeger [3] showed the following: Let M be a compact Riemannian manifold. Then for given Λ, $R>0$, there exists an $\varepsilon>0$ such that if a compact manifold $\bar{M}$ satisfies $|K_{\bar{M}}|\leqq\Lambda^2$, $i(\bar{M})\geqq R$, then every ε-map from M to $\bar{M}$ can be approximated by a piecewise linear homeomorphism. But it seems to us that the constant ε can not be estimated explicitly in terms of the given constants.

§ 2. Property *CM*.

From now on, let M, $\bar{M}$ denote compact Riemannian manifolds of dimension n unless otherwise stated. We will often assume that $m\in M$, $\bar{m}\in\bar{M}$ and $I: M_m\to\bar{M}_{\bar{m}}$ have been chosen so as to minimize the pinching number under consideration. Let $C(m)$ denote the cut locus of m. Then the interior $\mathring{\mathfrak{S}}_m$ of $\mathfrak{S}_m$ is mapped diffeomorphically onto $M-C(m)$ by the exponential map at m. Let $\exp_m^{-1}: M\to\mathfrak{S}_m$ be some extension of $(\exp_m|\mathring{\mathfrak{S}}_m)^{-1}$. Then we define the map $\Phi: M\to\bar{M}$ by $\Phi=\exp_{\bar{m}}\circ I\circ\exp_m^{-1}$. For an $\varepsilon\in\boldsymbol{R}$, we set $\mathfrak{S}_m^{\varepsilon}:=\left\{\left(1+\frac{\varepsilon}{\|v\|}\right)v\,;\ v\in\mathfrak{S}_m\right\}$. Although $\exp_m^{-1}$ is not continuous, we will try to show that in case M is a SCROSS, Φ is an ε-map if $\bar{M}$ is sufficiently close to M with respect to ρ. This is done in the next section.

LEMMA 2.1. *For given M, ε_1, $\varepsilon_2>0$, there exists a $\delta>0$ such that $\rho_0(M, \bar{M})<\delta$ implies that $|\|d\Phi\|-1|<\varepsilon_2$ on* $\exp_m(\mathfrak{S}_m^{-\varepsilon_1})$.

PROOF. We may find an $\varepsilon>0$ such that $\|d\exp_m\|>\varepsilon$ on $\mathfrak{S}_m^{-\varepsilon_1}$. Set $\delta:=\varepsilon\cdot\varepsilon_2$. Then for any unit vector v on $\mathfrak{S}_m^{-\varepsilon_1}$, we have

$$\left|\frac{\|d\exp_{\bar{m}}\circ I(v)\|}{\|d\exp_m(v)\|}-1\right|<\frac{\rho(M, \bar{M})}{\varepsilon}<\varepsilon_2 .$$

For any unit vector u on $\exp_m(\mathfrak{S}_m^{-\varepsilon_1})$, setting $w:=d\exp_m^{-1}(u)$, we may rewrite $\|d\Phi\|$ as:

$$\|d\Phi(u)\|=\left\|d\exp_{\bar{m}}\circ I\left(\frac{w}{\|w\|}\right)\right\|\Big/\left\|d\exp_m\left(\frac{w}{\|w\|}\right)\right\| .$$

This implies the required estimate. Q.E.D.

In order to obtain some more information about Φ, we require the model space M to have the following property.

DEFINITION. The pair (M, m) is said to have *property CM* if for any $p, q\in M-C(m)$ and for any $\varepsilon>0$, there exists a curve h_ε from p to q which does not intersect $C(m)$ such that $L(h_\varepsilon)<d(p, q)+\varepsilon$. If (M, m) has property CM for all $m\in M$, then M is said to have property CM.

We show that if (M, m) has property CM, then every geodesic emanating from m minimizes up to its first conjugate point, in particular M must be simply connected. Otherwise, there is a geodesic emanating from m along which the cut point, say p, of m is not conjugate to m. Then some ball $B(p, r)$ is the diffeomorphic image by $\exp_m$, and is divided into two connected components by $C(m)$. For $q, q'\in B(p, r/2)$ in the distinct components, let h be a curve from q to q' which does not meet $C(m)$. Then it turns out that

$$L(h)>2r>d(q, q')+r .$$

Hence (M, m) does not have property CM.

From now on, all geodesics will be assumed to be parametrized by arc length, and the diameter of the model space M will be denoted by D for simplicity.

LEMMA 2.2. *Let M have property CM. Then for a given ε there exists a $\delta>0$ such that $\rho_0(M, \bar{M})<\delta$ implies that $d(\Phi(p), \Phi(q))<d(p, q)+\varepsilon$ for all $p, q\in M$.*

PROOF. There are the unique minimal geodesics γ, σ from m to p, q which are compatible with the choice of $\exp_m^{-1}$. Set $p':=\gamma(d(m, p)-\varepsilon)$, $q':=\sigma(d(m, q)-\varepsilon)$. By the compactness of $\exp_m(\mathfrak{S}_m^{-\varepsilon})$, we may find an $\varepsilon'>0$ such that for all $p'', q''\in\exp_m(\mathfrak{S}_m^{-\varepsilon})$, there exists a curve h_ε from p'' to q'' such that

$$L(h_\varepsilon)<d(p'', q'')+\varepsilon , \quad d(h_\varepsilon, C(m))>\varepsilon'.$$

By Lemma 2.1, we may choose a $\delta>0$ such that $\rho_0(M, \bar{M})<\delta$ implies

$$\begin{aligned} d(\Phi(p'), \Phi(q'))&\leqq L(\Phi\circ h_\varepsilon)<L(h_\varepsilon)\left(1+\frac{\varepsilon}{D+1}\right)\\ &<d(p', q')+3\varepsilon , \end{aligned}$$

and therefore,

$$d(\Phi(p),\ \Phi(q))\leqq d(\Phi(p'),\ \Phi(q'))+2\varepsilon$$
$$<d(p',\ q')+5\varepsilon$$
$$<d(p,\ q)+7\varepsilon\,. \qquad \text{Q.E.D.}$$

For completeness, we give the proof of the following lemma which was proved in [3] where the assumption $d(p, q)<R_0$ was not assumed. But it seems to us that the proof of the part is incomplete.

LEMMA 2.3. *For given* n, Λ, R, ε_1, *there exists a* $\delta>0$ *such that for a given integer* N *there exists some* $\eta>0$ *such that the following is true: Let* M *be a complete n-manifold such that* $|K_M|\leqq\Lambda^2$, $i(M)\geqq R$ *and let a subset* $C\subset M$ *admits a cover by balls* $\{B(p_i, r_i)\}_{i=1,\cdots,N}$ *such that* $\sum_1^N r_i^{n-1}<\delta$. *Then if* $p, q\in M$, $d(p, C)$, $d(q, C)>\varepsilon_1$, $d(p, q)<R_0$, *then there exists a curve* h *from* p *to* q *such that*

(1) $L(h)<d(p, q)+\varepsilon_1$,

(2) $d(h, C)>\eta$.

PROOF. We denote by d_E the Euclidean distance on $B(p, R_0)$ with respect to a normal coordinate system at p. RCT implies

$$e^{-\Lambda s}\leqq\frac{d(x,\ y)}{d_E(x,\ y)}\leqq e^{\Lambda s}$$

for all x, $y\in B(p, s)$.

Case 1). $d(p, q)<\varepsilon_1$. By the triangle inequality, the unique minimal geodesic γ from p to q satisfies $d(\gamma, C)>\varepsilon_1/2$. Hence it suffices to set $h:=\gamma$, $\eta=\varepsilon_1/2$.

Case 2). $R_0>d(p, q)\geqq\varepsilon_1$. Set $s:=d(p, q)$. We denote by d_S the distance on the sphere $S=S(p, s)$ of radius s around p induced from d_E. Clearly

$$1\leqq\frac{d_S(x,\ y)}{d_E(x,\ y)}\leqq\frac{\pi}{2}\,.$$

Let $\phi: B(p, s)-B(p, \varepsilon_1/2)\rightarrow S$ be the radial projection from p. Then RCT implies the existence of $\Omega(\Lambda, s, \varepsilon_1)>0$ such that

$$(*) \qquad d_S(\phi(x),\ \phi(y))<\Omega d(x,\ y)\,.$$

Let A_t denote the Euclidean volume of a t-ball in S and $\mathfrak{A}$ denote the Euclidean volume measure on S. Set $\varepsilon:=\varepsilon_1/2$, $\delta:=\frac{1}{2}A_\varepsilon/\Omega^{n-1}$. From the given balls $\{B(p_i, r_i)\}_{i=1,\cdots,N}$, we choose such balls that intersect $C\cap B(p, s)$, say $B(p_1, r_1)$, $\cdots$, $B(p_k, r_k)$. Set $B:=(B(p_1, r_1)\cup\cdots\cup B(p_k, r_k))\cap B(p, s)$. Then $(*)$ yields

$$\mathfrak{A}(\phi(B))\leqq\sum_{i=1}^{k}\mathfrak{A}(\phi(B(p_i,\ r_i)\cap B(p,\ s)))$$
$$<\sum A_{(\Omega r_i)}<\sum(\Omega r_i)^{n-1}<\frac{1}{2}A_\varepsilon\,.$$

Then we assert that there exists an $\eta_1(n, N, \varepsilon, s)>0$ independent of p_i, r_i such that there exists a $q'\in S\cap B(q, \varepsilon)$ with

$$d(q', \phi(C\cap B(p, s)))>d(q', \phi(B))>\eta_1 .$$

In order to see this, define a compact subset L of R^N by

$$L:=\left\{(t_1, \cdots, t_N)\,;\ 0\leqq t_i,\ \sum_1^N t_i^{n-1}\leqq \frac{1}{2}A_\varepsilon\right\}.$$

Consider the following function $f: \underbrace{S\times\cdots\times S}_{N}\times L\to[0, \pi s]$,

$$f(q_1, \cdots, q_N, t_1, \cdots, t_N):=\sup\{d(x, \bigcup_1^N B(q_i, t_i))\,;\ x\in B^S(q, \varepsilon)\},$$

where B^S denotes a ball in S. Notice that $f>0$. Since $\phi(B(p_i, r_i)\cap B(p, s))\subset B^S(\phi(p_i), \Omega r_i)$, it suffices to observe that f is continuous and hence takes the positive minimum $\eta_1(n, N, \varepsilon, s)$.

Now let γ be the minimal geodesic from p to q'. Then we have

$$d(\gamma, C)>\min\{\varepsilon, d(\gamma, C\cap B(p, s))\}\geqq\min\{\varepsilon, \eta_1/\Omega\}=\eta_1/\Omega=:\eta .$$

Since $\varepsilon_1\leqq s<R_0$, we may choose η independent of s. Let σ be the minimal geodesic from q' to q. Then the required h is given as the broken geodesic $\gamma\cup\sigma$. Q.E.D.

The previous lemma will be very useful in case the $n-1$ dimensional Hausdorff measure $H^{n-1}(C(m))$ of $C(m)$ is equal to zero.

COROLLARY 2.4. *If $H^{n-1}(C(m))=0$, then (M, m) has property CM.*

PROOF. For given $p, q\in M-C(m)$, and $\varepsilon>0$, we take a minimal geodesic from p to q and choose $p_0=p, p_1, \cdots, p_k=q$ on the geodesic so that $d(p_i, p_{i+1})<R_0$, $k<[D/R_0]+1$. Take $p_i'\in B(p_i, \varepsilon R_0/6D)-C(m)$, $i=1, \cdots, k-1$. For each pair (p_i', p_{i+1}'), by Lemma 2.3 there exists a curve h_i from p_i' to p_{i+1}' such that

$$L(h_i)<d(p_i', p_{i+1}')+\varepsilon R_0/4D ,\quad h_i\cap C(m)=\emptyset .$$

Set $h:=h_0\cup h_1\cup\cdots\cup h_{k-1}$. Then we have

$$\begin{aligned} L(h)&<\sum d(p_i', p_{i+1}')+\varepsilon/3 \\ &<\sum (d(p_i, p_{i+1})+\varepsilon R_0/3D)+\varepsilon/3 \\ &<d(p, q)+\varepsilon . \end{aligned}$$

Q.E.D.

It is well known that the cut locus $C(m)$ in a SCROSS is a submanifold of codimension $\geqq 2$, in particular $H^{n-1}(C(m))=0$. More generally it is also known in [3] that the following classes of manifolds satisfy $H^{n-1}(C(m))=0$:

1) simply connected symmetric spaces of the compact type,

2) simply connected manifolds with the property that all geodesics emanating from m have the first conjugate points of order $\geqq 2$.

§ 3. Pinching theorems.

In our pinching situation, we would like to show that Φ is an ε-map if $\bar{M}$ is sufficiently close to M. For this, it will be needed that the tangent cut locus of $\bar{m}$ is close to that of m. For this reason, we adopt a SCROSS as the model space M. A crucial property which M possesses is that $D=\mathrm{diam}(M)=i(M)$.

We prepare an estimate for Jacobi fields.

LEMMA 3.1. *Let M be a manifold with $|K_M|\leqq \Lambda^2$ and $J(t)$ a Jacobi field along a geodesic γ in M such that $J(0)=0$, $\|J'(0)\|=1$. Then for $0<a<b$, there exists an $\Omega(\Lambda, a, b)>0$ such that*

$$\left|\left(\frac{\|J(t)\|}{t}\right)'\right|\leqq \Omega \qquad on \quad [a, b].$$

PROOF. The estimates from the theory of ordinary differential equations as in [3], § 2 implies the existence of $\Omega_1(\Lambda, b)>0$ such that $\|J(t)\|\leqq \Omega_1 t$ on $[0, b]$. Then the Jacobi equation: $J''=R_M(\gamma', J)\gamma'$ implies that on $[0, b]$

$$\begin{aligned}\|J'(t)\| &\leqq \|J'(0)\|+\int_0^t \|J''(t)\|dt \\ &\leqq 1+\int_0^t \|R_M\|\,\|J(t)\|dt \leqq 1+\Omega_1\Lambda^2 t^2.\end{aligned}$$

Therefore we have

$$\begin{aligned}\left|\left(\frac{\|J(t)\|}{t}\right)'\right| &= \left|\frac{1}{t}\left\langle\frac{J(t)}{\|J(t)\|}, J'(t)\right\rangle - t^{-2}\|J(t)\|\right| \\ &< \Omega_1\Lambda^2 b+\frac{1}{a}(1+\Omega_1)=:\Omega.\end{aligned}$$

Q.E.D.

LEMMA 3.2. *Let M be a SCROSS. Then for given Λ, $\varepsilon>0$, there exists a $\delta>0$ such that $|K_{\bar{M}}|\leqq \Lambda^2$, $\rho(M, \bar{M})<\delta$ implies that $d(\bar{m}, C(\bar{m}))\geqq D-\varepsilon$.*

PROOF. We may find a $\delta_1>0$ such that $\rho_0(M, \bar{M})<\delta_1$ implies that $d\exp_{\bar{m}}$ is non-singular on $I(\mathfrak{S}_{\bar{m}}^{-\varepsilon})$. Now suppose that $d(\bar{m}, C(\bar{m}))=: l<D-\varepsilon$. Then there exists a geodesic loop $\sigma : [0, 2l]\to M$ at $\bar{m}$ such that $\sigma(l)=: x\in C(\bar{m})$. We observe the influence of the existence of σ on the total volume of $\bar{M}$. For any $y\in C(\bar{m})$, let $\sigma_1 : [0, l_1]\to \bar{M}$ be a minimal geodesic from $\bar{m}$ to y. Set $\theta :=\sphericalangle(\dot{\sigma}(0), \dot{\sigma}_1(0))$, $l_2 :=d(x, y)$. Then the Toponogov comparison theorem implies

$$\cosh\Lambda l_2\leqq \cosh\Lambda l\cosh\Lambda l_1-\sinh\Lambda l\sinh\Lambda l_1\cos\theta\,.$$

If τ denotes a minimal geodesic from x to y, then we have immediately

$$\sphericalangle(\dot{\tau}(0),\ \dot{\sigma}(l))\leqq\frac{\pi}{2},\quad \text{or}\quad \sphericalangle(\dot{\tau}(0),\ -\dot{\sigma}(l))\leqq\frac{\pi}{2}.$$

Hence the Toponogov comparison theorem implies again

$$\cosh\Lambda l_1\leqq\cosh\Lambda l\cosh\Lambda l_2\,.$$

The above inequalities yield

$$\cos\theta\leqq\coth\Lambda l_1\tanh\Lambda l\,.$$

Now if $l_1\geqq D-\varepsilon/2$, then

$$\cos\theta<\coth\Lambda(D-\varepsilon/2)\tanh\Lambda(D-\varepsilon)<1\,.$$

Define $\theta_0=\theta_0(\Lambda,\ \varepsilon)$ by $\cos\theta_0=\coth\Lambda(D-\varepsilon/2)\tanh\Lambda(D-\varepsilon)$ and set

$$C:=\{v\in M_m\ ;\ \sphericalangle(v,\ I^{-1}(\dot{\sigma}(0)))\leqq\theta_0,\ D-\varepsilon/2\leqq\|v\|\leqq D\}\,.$$

We have just verified that $I(C)$ does not meet $\mathfrak{S}_{\bar{m}}$. Set $v(\varepsilon):=\mathrm{Vol}(\exp_m(C))$. Then we may find an $\varepsilon_1>0$ such that

$$\mathrm{Vol}(\exp_m(\mathfrak{S}_m^{\varepsilon_1}-\mathfrak{S}_m^{-\varepsilon_1}))<\frac{1}{4}v(\varepsilon)\,.$$

Necessarily $\varepsilon_1<\varepsilon$. By Lemma 3.1 we may choose a $\delta_2>0$ such that $\rho_0(M,\ \overline{M})<\delta_2$ implies

$$\mathrm{Vol}(\exp_{\bar{m}}\circ I(\mathfrak{S}_m^{\varepsilon_1}-\mathfrak{S}_m^{-\varepsilon_1}))<\frac{1}{3}v(\varepsilon)\,.$$

Taking δ_2 smaller if necessary, we may assume that

$$\mathrm{Vol}(\exp_{\bar{m}}\circ I(\mathfrak{S}_m^{-\varepsilon_1}-C))<\mathrm{Vol}(\exp_m(\mathfrak{S}_m^{-\varepsilon_1}-C))+\frac{1}{3}v(\varepsilon)\,.$$

Now if $\mathrm{diam}(\overline{M})<D+\varepsilon_1$, then $\mathfrak{S}_{\bar{m}}\subset I(\mathfrak{S}_m^{\varepsilon_1})$. Hence we get

$$\begin{aligned}\mathrm{Vol}(\overline{M})&<\mathrm{Vol}(\exp_{\bar{m}}\circ I(\mathfrak{S}_m^{-\varepsilon_1}-C)\cup\exp_{\bar{m}}\circ I(\mathfrak{S}_m^{\varepsilon_1}-\mathfrak{S}_m^{-\varepsilon_1}))\\&<\mathrm{Vol}(\exp_{\bar{m}}\circ I(\mathfrak{S}_m^{-\varepsilon_1}-C))+\mathrm{Vol}(\exp_{\bar{m}}\circ I(\mathfrak{S}_m^{\varepsilon_1}-\mathfrak{S}_m^{-\varepsilon_1}))\\&<\mathrm{Vol}(\exp_m(\mathfrak{S}_m^{-\varepsilon_1}-C))+\frac{2}{3}v(\varepsilon)\\&<\mathrm{Vol}(M)-\frac{1}{3}v(\varepsilon)\,.\end{aligned}$$

Therefore the required δ is obtained as:

$$\delta=\min\left\{\delta_1,\ \delta_2,\ \varepsilon_1,\ \frac{1}{3}v(\varepsilon)\right\}.$$

Q. E. D.

LEMMA 3.3. *Let M be a SCROSS. Then for given Λ, $\varepsilon>0$, there exists a $\delta>0$ such that $|K_{\overline{M}}|\leqq\Lambda^2$ and $\rho(M,\ \overline{M})<\delta$ implies that Φ is an ε-map.*

PROOF. Since we may assume that $\operatorname{diam}(\bar{M})<2D$, $\operatorname{Vol}(\bar{M})>\frac{1}{2}\operatorname{Vol}(M)$, Cheeger's injectivity radius estimate [4] shows that there exists an R independent of $\bar{M}$ such that $i(\bar{M})\geqq R$. For n, Λ, R and $\varepsilon_1:=\frac{\varepsilon}{30([D/R_0]+1)}$, let δ_1 be as in Lemma 2.3. Since $H^{n-1}(C(m))=0$, there is a covering $\{B(p_i, r_i)\}_{i=1,\cdots,N}$ of $C(m)$ such that $\sum_1^N (2r_i)^{n-1}<\delta_1$. Let η be the constant given in Lemma 2.3. Set $r:=\min\{r_i\,;\ 1\leqq i\leqq N\}$. By Lemma 2.3, we may find a $\delta>0$ such that $\rho(M, \bar{M})<\delta$ implies that $d(\Phi(p), \Phi(q))<d(p, q)+r/2$ for all $p, q\in M$. On the other hand, by Lemma 3.2 we may assume that

$$\partial\mathfrak{S}_{\bar{m}}\subset B(0, D+r/4)-B(0, D-r/4)\subset\bar{M}_{\bar{m}}\,.$$

Hence we can conclude that the balls $\{B(\Phi(p_i), 2r_i)\}_{i=1,\cdots,N}$ cover $C(\bar{m})$. Now for any $p, q\in M$, take the points $q_0=\Phi(p)$, $q_1, \cdots, q_k=\Phi(q)$ on a minimal geodesic from $\Phi(p)$ to $\Phi(q)$ such that $d(q_i, q_{i+1})<R_0$, $k\leqq[D/R_0]+1$. Let γ_0, γ_k be the minimal geodesics from m to p, q which are compatible with the choice of $\exp_m^{-1}$. Set $\bar{\gamma}_0:=\Phi\circ\gamma_0$, $\bar{\gamma}_k:=\Phi\circ\gamma_k$. Let $\bar{\gamma}_i:[0, l_i]\to\bar{M}$ be a minimal geodesic from $\bar{m}$ to q_i, $1\leqq i\leqq k-1$. Set $q_i':=\bar{\gamma}_i(l_i-2\varepsilon_1)$. Choosing δ smaller if necessary, we may assume that $d(q_i', C(\bar{m}))>\varepsilon_1$. Hence by Lemma 2.3 there exists a curve h_i from q_i' to q_{i+1}' such that

$$L(h_i)<d(q_i', q_{i+1}')+\varepsilon_1\,,\quad d(h_i, C(\bar{m}))>\eta\,.$$

We may assume that $d(\bar{m}, C(\bar{m}))\geqq D-\eta$, and that by Lemma 2.1, for $\varepsilon_2:=\frac{\varepsilon}{2(D+R_0)}$, $L(\Phi^{-1}\circ h_i)<(1+\varepsilon_2)L(h_i)$. It follows that

$$\begin{aligned} d(p, q)&\leqq 4\varepsilon_1+\sum_0^{k-1} L(\Phi^{-1}\circ h_i)\\ &<4\varepsilon_1+\sum(1+\varepsilon_2)L(h_i)\\ &<4\varepsilon_1+\sum(1+\varepsilon_2)(d(q_i', q_{i+1}')+\varepsilon_1)\\ &<4\varepsilon_1+\sum(1+\varepsilon_2)(d(q_i, q_{i+1})+5\varepsilon_1)\\ &<d(\Phi(p), \Phi(q))+\varepsilon\,. \end{aligned}$$

Together with Lemma 2.2, this completes the proof. Q.E.D.

THEOREM 3.4. *Let M be a SCROSS. Then given Λ, $\Lambda_1>0$, there exists an $\varepsilon>0$ such that if $|K_{\bar{M}}|\leqq\Lambda^2$, $\|\nabla R_{\bar{M}}\|\leqq\Lambda_1$ and $\rho(M, \bar{M})<\varepsilon$ implies that $\bar{M}$ is diffeomorphic to M.*

PROOF. This is an immediate consequence of ε-mapping Theorem 1.1 and Lemma 3.3.

REMARK. It should be noted that ρ_0 is weaker than $\tilde{\rho}$ in the sense that

$\rho_0 \leqq c\tilde{\rho}$ for some constant c, but the converse inequality does not hold. Furthermore in the case where M is a SCROSS, Klingenberg's injectivity radius estimate [9] implies that if $\tilde{\rho}(M, \bar{M}) \to 0$, then $\mathrm{Vol}(\bar{M}) \to \mathrm{Vol}(M)$, $\mathrm{diam}(\bar{M}) \to \mathrm{diam}(M)$ and hence $\rho(M, \bar{M}) \to 0$.

We denote by $\mathfrak{M}_C^n$ the class of compact n-manifolds M such that $H^{n-1}(C(m)) = 0$ for all $m \in M$. It contains simply connected symmetric spaces of the compact type, and simply connected manifolds with the property that all geodesics have the first conjugate points of order $\geqq 2$, as stated before.

THEOREM 3.5. *Let* $M \in \mathfrak{M}_C^n$. *If* $\bar{M}$ *satisfies that* $\rho_0(M, \bar{M}) = 0$, $\mathrm{Vol}(M) = \mathrm{Vol}(\bar{M})$, *then* $\bar{M}$ *is isometric to* M.

PROOF. Notice that $\Phi | M - C(m)$ is a local isometry and that $\mathfrak{S}_{\bar{m}} \subset I(\mathfrak{S}_m)$. But the assumption $\mathrm{Vol}(M) = \mathrm{Vol}(\bar{M})$ implies that $\mathfrak{S}_{\bar{m}} = I(\mathfrak{S}_m)$. Hence by Lemma 3.3, $\Phi : M \to \bar{M}$ is a "0-map", that is, an isometry. Q.E.D.

Here we consider more general model spaces than SCROSSes. In this case, of course, a more strict pinching will be needed.

LEMMA 3.6. *Let* $M \in \mathfrak{M}_C^n$. *Then for a given* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\tilde{\rho}(M, \bar{M}) + |\mathrm{Vol}(M) - \mathrm{Vol}(\bar{M})| < \delta$ *implies that* Φ *is an* ε*-map.*

PROOF. Let $x \in \mathfrak{S}_m$, and $v_1, \cdots, v_{n-1}$ an orthonormal basis for the orthogonal complement to the radial line at x. Notice that

$$\text{Jacobian of } (d\exp_m)_{|x} = \| d\exp_m(v_1) \wedge \cdots \wedge d\exp_m(v_{n-1}) \| ,$$

$$\text{Jacobian of } (d\exp_{\bar{m}})_{|I(x)} = \| d\exp_{\bar{m}} I(v_1) \wedge \cdots \wedge d\exp_{\bar{m}} I(v_{n-1}) \| .$$

Now for a given $\varepsilon_1 > 0$, by compactness, there exist some $A > B > 0$, $\pi > \alpha > \beta > 0$ such that

$$A \geqq \| d\exp_m(v_i) \| \geqq B , \quad \alpha \geqq \sphericalangle(d\exp_m(v_i),\ d\exp_m(v_j)) \geqq \beta \qquad \text{on } \mathfrak{S}_m^{-\varepsilon_1} .$$

On the other hand, by Lemma 3.2 in [3], for a given ε there exists a $\delta > 0$ such that $\tilde{\rho}(M, \bar{M}) < \delta$ implies that $\mathfrak{S}_{\bar{m}} \subset I(\mathfrak{S}_m^{\varepsilon})$. We may assume that $|K_{\bar{M}}| \leqq 2 \max |K_M|$. Therefore we may find a $\delta > 0$ in the same way as in Lemma 3.2 such that $\tilde{\rho}(M, \bar{M}) + |\mathrm{Vol}(M) - \mathrm{Vol}(\bar{M})| < \delta$ implies that $\partial \mathfrak{S}_{\bar{m}} \subset I(\mathfrak{S}_m^{\varepsilon} - \mathfrak{S}_m^{-\varepsilon})$. Thus the proof completes in the same way as in Lemma 3.2. Q.E.D.

Together with ε-mapping Theorem, we have just proved the following

THEOREM 3.7. *Let* $M \in \mathfrak{M}_C^n$. *Then for a given* $\Lambda_1 > 0$, *there exists an* $\varepsilon > 0$ *such that* $\|\nabla R_{\bar{M}}\| \leqq \Lambda_1$ *and* $\tilde{\rho}(M, \bar{M}) + |\mathrm{Vol}(M) - \mathrm{Vol}(\bar{M})| < \varepsilon$ *implies that* $\bar{M}$ *is diffeomorphic to* M.

REMARK. If the assumption for $\|\nabla R\|$ in General Theorem can be removed, then the parameter Λ_1 will be negligible throughout this paper.

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