J. Math. Soc. Japan
Vol. 36, No. 3, 1984

Minimal surfaces with constant normal curvature

By Antonio Carlos ASPERTI*

(Received March 8, 1983)

1. Introduction.

Let M be a connected 2-dimensional Riemannian manifold which is isometrically immersed into $Q^n(c)$, $n \geqq 4$, where $Q^n(c)$ stands for the sphere $S^n(c)$ of radius $1/c$, the Euclidean space $\boldsymbol{R}^n$ or the hyperbolic space $H^n(c)$, according to c is positive, zero or negative. Through this paper we assume that the normal curvature tensor $R^{\perp}$ of the immersion is nowhere zero. In this case there exists an orthogonal bundle splitting $\nu=\nu^* \oplus \nu^0$ of the normal bundle ν of the immersion, where ν^0 consists of the normal directions that annihilate $R^{\perp}$ and ν^* is a 2-plane subbundle of ν. We know by [**1**] that if M is compact and oriented, then the Gaussian curvature K of M is strongly related to the normal curvature K^{ν} of the immersion and to the intrinsic curvature K^* of ν^*. The first result of this paper is an extension of Theorem 2 of [**1**] to the case when M is not necessarily compact.

THEOREM 1. *Let M be a connected, oriented 2-dimensional Riemannian manifold immersed with nowhere zero normal curvature tensor into $Q^n(c)$. Assume that the normal curvature K^{ν} is constant and that the mean curvature vector H of the immersion is parallel in the normal connection. We have*

(a) *if M is complete and $K^* \geqq 0$, then K and K^* are constant and $K=K^*/2$;*

(b) *if K^* is constant, then $K=K^*/2$.*

It should be noted that no global assumption is made in part (b). When M is complete and minimal in the unit sphere S^n with K^{ν} constant, it follows immediately from (a) that

(a′) if $K^*>0$, then $M=S^2(K^*/2)$ is one of the Veronese surfaces studied by Calabi [**2**] and do Carmo-Wallach [**3**];

(a″) if $K^*=0$, then we obtain a minimal plane in S^n. These were studied by Kenmotsu [**7**], [**8**].

As a consequence of Theorem 1 and its proof we can deduce the following result.

THEOREM 2. (a) *If $c \leqq 0$, then there is no minimal immersion of a surface M into $Q^n(c)$ with K^{ν} constant and $K^* \geqq 0$.*

* Work done under partial support by $\mathrm{CNP_q}$, Brazil.

(b) *Let $f: M \to S^n$ be an isometric minimal immersion with K^ν and K^* positive constant. Then M is locally one of the Veronese surfaces of Calabi and do Carmo-Wallach.*

When $n=4$ then $\nu^*=\nu$, $K^*=K^\nu$ and we have the following result which was firstly proved by Wong in [**13**], Theorem 4.9. See also [**6**] for a similar result.

COROLLARY 1. *Let M be a 2-dimensional submanifold of $Q^4(c)$ with K^ν constant and H parallel. Then $c>0$ and M is locally a Veronese surface $S^2(c/3)$ in $S^4(c)$.*

The proofs of the above results are presented in Section 3. In Section 4 we present the proof of the following extension of Theorem 2 of [**9**].

THEOREM 3. *Let $f: M \to Q^6(c)$ be an isometric minimal immersion of a connected surface M of constant curvature K and with nonzero constant normal curvature K^ν. Then $c>0$ and either*

(a) *$K=c/3$ and M is locally a Veronese surface in $S^4(c)$;*

(b) *$K=0$ and f is locally one of the immersions $\boldsymbol{R}^2 \to S^5(c)$ described in [**7**]; or*

(c) *$K=c/6$ and M is locally a Veronese surface in $S^6(c)$.*

As a consequence we see that the hyperbolic 2-plane cannot be minimally immersed with constant normal curvature in the 6-sphere, even locally.

I want to thank the hospitality of the people of the SUNY at Stony Brook Mathematics Department, where this work was done. I want also to thank Professor B. Lawson for bringing [**9**] to my attention and to the referee for pointing out several mistakes.

2. Preliminaries.

Let $f: M \to Q^n(c)$ be an isometric immersion of a 2-dimensional Riemannian manifold M into the space $Q^n(c)$ and denote by $\nu=\nu(f)$ the normal bundle of the immersion. We will always assume that M is connected, oriented and with complex structure J. We denote by $\nabla^\perp$ the covariant derivative of ν associated to the induced connection and by $R^\perp$ the corresponding curvature tensor, that is,

$$R^\perp(X, Y)\xi = \nabla^\perp_X \nabla^\perp_Y \xi - \nabla^\perp_Y \nabla^\perp_X \xi - \nabla^\perp_{[X,Y]}\xi ,$$

for tangent fields X, Y and normal field ξ. Now we recall the Ricci's equation

$$R^\perp(X, Y)\xi = B(X, A_\xi Y) - B(A_\xi X, Y) , \tag{2.1}$$

where B is the second fundamental form of the immersion and A_ξ is the associated symmetric endomorphism of the tangent bundle TM. We set $B_{ij}=B(e_i, e_j)$ for a tangent frame e_1, e_2. With this notation the mean curvature vector H of the immersion is given by $H=\mathrm{tr}B/2=(B_{11}+B_{22})/2$.

We shall make use of the *curvature ellipse* of $f: M\to Q^n(c)$, which is, for each p in M the subset of ν_p given by

$$\varepsilon_p=\{B(X, X)\in\nu_p\ ;\ X\in TM_p \text{ and } \|X\|=1\}\ .$$

If $X=\cos\theta\cdot e_1+\sin\theta\cdot e_2$ we can see that $B(X, X)=\cos 2\theta\cdot u+\sin 2\theta\cdot v$, where $u=(B_{11}-B_{22})/2$ and $v=B_{12}$. This shows that ε_p is in fact an ellipse with center in the tip of $H(p)$. Also it is not difficult to see that $R_p^{\perp}\neq 0$ if and only if ε_p is nondegenerate, and that this happens if and only if u and v are linearly independent. From now on we assume that $R_p^{\perp}\neq 0$ for all p in M so that we can define a 2-plane subbundle of the normal bundle, namely the bundle ν^* whose fiber over p is the subspace of ν_p spanned by u and v (one can check that this ν^* is the ν^* of the Introduction). We define an orientation on ν^* as follows: a pair (ξ, η) in ν_p^* will be positively oriented if $\langle R^{\perp}(X, JX)\eta, \xi\rangle>0$ for one (and hence all) $X\neq 0$ in TM_p. This plane bundle inherits a canonical covariant derivative from that of ν, which we denote by ∇^*. Let R^* be the corresponding curvature tensor and define the *intrinsic curvature* K^* of ν^* by

$$K^*=\langle R^*(e_1, e_2)e_4, e_3\rangle\ ,$$

where (e_1, e_2) and (e_3, e_4) are positively oriented frames of TM and ν^*, respectively. The *normal curvature* of f at p is given by

$$K_p^{\nu}=\langle R^{\perp}(e_1, e_2)e_4, e_3\rangle|_p\ ,$$

where the frames are as above (hence K^{ν} is positive by definition). It can be shown that $\text{Area}(\varepsilon_p)=K_p^{\nu}\cdot\pi/2$ (see [**11**]).

At this point it is convenient to introduce some notation related to the method of moving frames. We will be based on the framework of Section 2 of [**4**], but we remark here that our sign convention is the opposite of that of [**4**]. Let $(e_1, \cdots, e_n)$ be a local frame field tangent to $Q^n(c)$ such that (e_1, e_2) spans TM. Such a frame is said to be adapted to M. Define as usual functions h_{ij}^{α} by

$$h_{ij}^{\alpha}=\langle B_{ij}, e_{\alpha}\rangle=h_{ji}^{\alpha}\ ,$$

where we are using the following convention on the range of indices:

$$1\leqq A, B, C, \cdots\leqq n\ ;\ 1\leqq i, j, k, \cdots\leqq 2\ ;\ 3\leqq\alpha, \beta, \gamma, \cdots\leqq n\ .$$

We take the normal covariant derivative of B and define a trilinear form $\tilde{B}$ from TM into ν, and functions h_{ijk}^{α} by

$$\tilde{B}(e_i, e_j, e_k)=(\nabla_{e_k}^{\perp}B)(e_i, e_j)=\sum_{\alpha}h_{ijk}^{\alpha}e_{\alpha}\ .$$

We set $\tilde{B}_{ijk}=\tilde{B}(e_i, e_j, e_k)$. A simple calculation shows that

$$\sum_k h_{ijk}^{\alpha}w_k=dh_{ij}^{\alpha}+\sum_s h_{is}^{\alpha}w_{sj}+\sum_s h_{sj}^{\alpha}w_{si}+\sum_{\beta}h_{ij}^{\beta}w_{\beta\alpha}\ , \tag{2.2}$$

where w_A, w_{AB} are 1-forms on $Q^n(c)$ defined by

$$w_A(e_B)=\delta_{AB}, \quad w_{AB}(e_C)=\langle\nabla_{e_C}e_A, e_B\rangle .$$

These are the dual and the connection forms of $Q^n(c)$ relative to the given frame, respectively. In particular

$$w_{i\alpha}=\sum_j h_{ij}^\alpha w_j$$

when restricted to M. Since we are in a space of constant curvature, it follows from the Codazzi equations that $\tilde{B}_{ijk}=\tilde{B}_{ikj}$, that is, $\tilde{B}$ is symmetric. This implies that $h_{ijk}^\alpha=h_{ikj}^\alpha$ for all α. If M is minimal in $Q^n(c)$ then from (2.2) we have

$$\begin{aligned} h_{111}^\alpha&=-h_{221}^\alpha=-h_{212}^\alpha=-h_{122}^\alpha , \\ h_{222}^\alpha&=-h_{112}^\alpha=-h_{121}^\alpha=-h_{211}^\alpha , \end{aligned} \tag{2.3}$$

for all α. Suppose that M is minimal and that the frame is chosen with (e_3, e_4) spanning ν^*. Then $h_{ij}^\gamma=0$ for $\gamma\geqq 5$ and using (2.2) and (2.3) we obtain

$$\begin{aligned} K^\nu&=K^*+\sum_{\gamma\geq 5}((w_{3\gamma}(e_2)w_{4\gamma}(e_1)-w_{3\gamma}(e_1)w_{4\gamma}(e_2)) \\ &=K^*+\frac{2}{K^\nu}\sum_{\gamma\geq 5}((h_{111}^\gamma)^2+(h_{112}^\gamma)^2) . \end{aligned} \tag{2.4}$$

Now we take the normal covariant derivative of $\tilde{B}$ and define functions h_{ijkl}^α by

$$(\nabla_{e_l}^\perp\tilde{B})(e_i, e_j, e_k)=\sum_\alpha h_{ijkl}^\alpha e_\alpha .$$

A simple calculation shows that

$$\sum_l h_{ijkl}^\alpha w_l=dh_{ijk}^\alpha+\sum_s h_{sjk}^\alpha w_{si}+\sum_s h_{isk}^\alpha w_{sj}+\sum_s h_{ijs}^\alpha w_{sk}+\sum_\beta h_{ijk}^\beta w_{\beta\alpha} . \tag{2.5}$$

If the frame is such that (e_3, e_4) spans ν^* then $h_{11}^\gamma=h_{22}^\gamma$, $h_{12}^\gamma=0$ for $\gamma\geqq 5$. In this case we apply equation (2.15) of [4] to obtain

$$h_{ij12}^\gamma=h_{ij21}^\gamma ,$$

for all $\gamma\geqq 5$. From (2.3) and (2.5) it follows that

$$\begin{aligned} h_{1212}^\gamma&=h_{1221}^\gamma=h_{2121}^\gamma=h_{2211}^\gamma , \\ h_{1112}^\gamma&=h_{1121}^\gamma=h_{1211}^\gamma=h_{2111}^\gamma , \end{aligned} \tag{2.6}$$

for all $\gamma\geqq 5$, in such a frame.

3. Surfaces with constant normal curvature.

For our purposes it is convenient to divide M into two subsets F and $A=M-F$, where F consists of the points p in M where ε_p is a circle. Obviously A is open and F is closed in M. For each p in A there exist (cf. [1]) a neighborhood U of p and smooth positively oriented frames (e_1, e_2) in $TM|U$ and (e_3, e_4) in $\nu^*|U$ such that

$$B_{11}-H=\lambda e_3=-B_{22}+H, \quad B_{12}=\mu e_4, \tag{3.1}$$

where λ and μ are the length of the semi-axes of the curvature ellipse. We may in addition assume that $\lambda>\mu$ on U. On the other hand, if the ellipse is a circle on a *neighborhood* of a point p in F, we can start with any positively oriented (e_3, e_4) and choose (e_1, e_2) in a way to obtain (3.1) again. In this case $\lambda=\mu$. The Gauss equation takes the form

$$K=c+\|H\|^2-\lambda^2-\mu^2=C-S \tag{3.2}$$

where $S=\lambda^2+\mu^2$ and $C=c+\|H\|^2$, which is a constant whenever H is parallel. Also from (2.1) and (3.1) we obtain

$$K^{\nu}=2\lambda\mu. \tag{3.3}$$

Suppose that we are in A or in $\mathrm{Int}(F)$ with a frame as in (3.1) and assume from now on that H is parallel and K^{ν} is constant. By the Codazzi equations we obtain

$$e_i(\lambda)=(\mu w_{34}-2\lambda w_{12})\circ J(e_i), \qquad e_i(\mu)=(\lambda w_{34}-2\mu w_{12})\circ J(e_i). \tag{3.4}$$

Then

$$d\lambda=(\mu w_{34}-2\lambda w_{12})\circ J, \qquad d\mu=(\lambda w_{34}-2\mu w_{12})\circ J. \tag{3.5}$$

Since $d(\lambda\mu)=0$, (3.5) implies

$$Sw_{34}=2K^{\nu}w_{12}. \tag{3.6}$$

Therefore we can rewrite (3.5) as

$$d\lambda=2\frac{\mu K^{\nu}-\lambda S}{S}w_{12}\circ J, \qquad d\mu=2\frac{\lambda K^{\nu}-\mu S}{S}w_{12}\circ J, \tag{3.7}$$

and then

$$\langle \mathrm{grad}\,\lambda, \mathrm{grad}\,\mu\rangle=-2K^{\nu}\frac{S^2-(K^{\nu})^2}{S^2}\|w_{12}\|^2. \tag{3.8}$$

Differentiating (3.6) we have by the definition of w_{12} and w_{34}

$$2K=\frac{8(S^2-(K^\nu)^2)}{S^2}\|w_{12}\|^2+\frac{K^*S}{K^\nu} \quad \text{in } A, \qquad 2K=K^* \quad \text{in Int}(F). \tag{3.9}$$

This immediately implies the following

(3.10) PROPOSITION. *Let* $f: M\to Q^n(c)$ *be an isometric immersion with* K^ν *constant and* H *parallel. If* $K^*\geqq 0$ *on* M, *then* $K\geqq 0$ *on* M.

In fact, $K\geqq 0$ in A and in Int(F) by (3.9). By continuity, $K\geqq 0$ on M.

Now from (3.5) we have

$$dS=\frac{4((K^\nu)^2-S^2)}{S}w_{12}\circ J \tag{3.11}$$

which with (3.9) gives

$$\|\operatorname{grad} S\|^2=-2\Big(2+\frac{K^*}{K^\nu}\Big)S^3+4CS^2+2K^\nu(2K^\nu+K^*)S-4C(K^\nu)^2. \tag{3.12}$$

Also, by a simple calculation using (3.4) and (3.6), we have

$$\Delta\lambda=4\Big(\frac{\lambda(K^\nu)^2}{S^2}-\frac{2\mu K^\nu}{S}+\lambda\Big)\|w_{12}\|^2-\mu K^*+2\lambda K, \qquad \Delta\mu=4\Big(\frac{\mu(K^\nu)^2}{S^2}-\frac{2\lambda K^\nu}{S}+\mu\Big)\|w_{12}\|^2-\lambda K^*+2\mu K. \tag{3.13}$$

Then

$$\frac{1}{2}\Delta S=\frac{8(S^2-(K^\nu)^2)}{S}\|w_{12}\|^2+2KS-K^\nu K^*.$$

By applying (3.9) to the last equation in two different ways we get

$$\frac{1}{4}\Delta S=\frac{4(S^4-(K^\nu)^4)}{S^3}\|w_{12}\|^2+\frac{(S^2-(K^\nu)^2)}{S}K, \qquad \Delta S=-2\Big(4+\frac{K^*}{K^\nu}\Big)S^2+8CS-2K^\nu K^*. \tag{3.14}$$

(3.15) PROOF OF THEOREM 1. Suppose first that M is complete and that $K^*\geqq 0$. Then $K\geqq 0$ on M by (3.10), which jointly with (3.2) imply that $0<S\leqq C$ on M. This also implies, by (3.14), that $\Delta S\geqq 0$ on M. In summary, S is a bounded subharmonic function defined in a complete surface of nonnegative curvature. It is well known that such a function must be constant, that is, $K=C-S$ is constant. It also follows that λ and μ are constant and then $M=A$ or $M=F$. If $M=F$, from (3.9) we have $2K=K^*$. If $M=A$ we cannot have $w_{12}\neq 0$ otherwise from (3.11) we obtain $0=S^2-(K^\nu)^2=(\lambda^2-\mu^2)^2$, which is impossible in A. So $w_{12}=0$, $K=0$ and $K^*=0$ in this case. This completes the proof of part (a). To prove part (b), we follow closely Wong [**13**], p. 486. We claim that if K^ν and K^* are constant then $dS=0$, that is, S is constant. It is clear,

by (3.11), that $dS=0$ in Int(F). If we are in A, by (3.12) and (3.14) we know that $\|\mathrm{grad}S\|^2=g(S)$, $\Delta S=h(S)$, where $g(S)$ and $h(S)$ are polynomials in S with constant coefficients. If S is not constant it is known (cf. [**13**], [**9**]) that there exist local coordinates (S, T) in A such that the first fundamental form of M is given locally by

$$ds^2=\frac{1}{g(S)}\left(dS^2+\exp\left(2\int\frac{h}{g}dS\right)dT^2\right).$$

Then the Gaussian curvature K of M satisfies

$$2gK+\left(h-\frac{dg}{dS}\right)\left(2h-\frac{dg}{dS}\right)+g\left(2\frac{dh}{dS}-\frac{d^2g}{dS^2}\right)=0,$$

which is equivalent to

$$(6K^\nu-K^*)S^4+C(7K^*-12K^\nu)S^3+K^\nu(10C^2-10(K^\nu)^2-9K^\nu K^*-6(K^*)^2)S^2$$
$$+C(K^\nu)^2(12K^\nu-7K^*)S+2(K^\nu)^3(-5C^2+2(K^\nu)^2+5K^\nu K^*+3(K^*)^2)=0.$$

This is a polynomial equation in S with constant coefficients. Therefore S must be constant, which is a contradiction. This proves our claim and an argument as in part (a) shows that $K^*=2K$. So Theorem 1 is proved.

(3.16) PROOF OF THEOREM 2. To prove part (a), we observe that if M is minimal in $Q^n(c)$ with $c\leqq 0$, then $K<0$. Therefore we cannot have K^ν constant and $K^*\geqq 0$, by (3.10). For the second part, we note that in view of Theorem 1-(b), f is now a minimal immersion of a surface with constant positive Gaussian curvature $K=K^*/2$. By a theorem of Wallach [**12**], f can be extended to a minimal immersion of the whole 2-sphere $S^2(K)$ into S^n. This completes the proof of Theorem 2.

(3.17) PROOF OF COROLLARY 1. We have $H\equiv 0$, otherwise from Theorem 4 of [**14**], M is contained in a 3-dimensional umbilic submanifold of $Q^4(c)$, which is impossible because $R^\perp$ never vanishes. The corollary then follows from part (b) of Theorem 2.

(3.18) REMARKS. (1) If M is minimal in $Q^n(c)$ with $K^\nu\equiv 0$, then either M is totally geodesic in $Q^n(c)$ and $K=c$ is constant, or the first normal space N_1 of the immersion has constant dimension 1. In the later case, using a theorem on reduction of codimension of [**5**], we can say that M is minimal in a totally geodesic 3-dimensional submanifold of $Q^n(c)$. By Lemma 1 of [**9**] the only constant curved minimal surfaces in $Q^3(c)$ with $\dim N_1=1$ are locally Clifford surfaces, for which $K=0$ and $c>0$.

(2) The arguments used in this section can be easily adapted to prove the following. Let $f : M\to Q^n(c)$ be an immersion under the same hypothesis of Theorem 1 but without assuming that K^ν is constant. We have

(a) if M is complete and K is a nonnegative constant, then K^{ν} and K^{*} are constant and $K^{*}=2K$;

(b) if K and K^{*} are constant, then K^{ν} is also constant and $K^{*}=2K$. When we assume further that M is minimal in S^{n} and K is a positive constant, we can use again the result of [12] to see that we do not need completeness in (a) (or the constancy of K^{*} in (b)) to get the same conclusions. This fact leads us to conjecture that Theorem 1-(a) still holds without any global assumption on M, at least in the case $K^{*}>0$.

4. Minimal surfaces with constant Gaussian and normal curvatures.

Through this section we assume that M is minimal in $Q^{n}(c)$ with K constant and $K^{\nu}>0$. If K^{ν} is also constant, equations (3.2) and (3.3) imply that λ and μ are constant. Choosing an adapted frame as in (3.1), it follows from (2.2) and (3.5) that

$$h_{ijk}^{3}=h_{ijk}^{4}=0\,. \tag{4.1}$$

Sometimes we will have to rotate (e_1, e_2) and (e_3, e_4) but we still want (4.1) to hold in the new frame. This will cause no problem when $\lambda=\mu$. In case that $\lambda>\mu$ we have

(4.2) LEMMA. *Let M be a minimal surface in $Q^{n}(c)$ with K and K^{ν} constant and let $(e_1, \cdots, e_n)$ be any local adapted frame field such that (e_3, e_4) spans ν^{*}. Then $h_{ijk}^{3}=h_{ijk}^{4}=0$ in such a frame, provided that $\lambda>\mu$.*

PROOF. Since $h_{ij}^{\gamma}=0$ for $\gamma\geqq 5$,

$$c-K=(h_{11}^{3})^2+(h_{12}^{3})^2+(h_{11}^{4})^2+(h_{12}^{4})^2$$

and

$$K^{\nu}/2=h_{11}^{3}\cdot h_{12}^{4}-h_{12}^{3}\cdot h_{11}^{4}$$

are constant functions on M. Differentiating them and using (2.2) and (2.3), gives $L\cdot(h_{111}^{3}, h_{112}^{3}, h_{111}^{4}, h_{112}^{4})=0$, where L is a certain 4×4 matrix whose entries are $\pm h_{ij}^{\alpha}$, $1\leqq i, j\leqq 2$, $3\leqq\alpha\leqq 4$. The determinant of L is $(\lambda^2-\mu^2)^2$ and this proves the lemma. Q.E.D.

For any unit vector $X=\cos\theta\cdot e_1+\sin\theta\cdot e_2$ tangent to M, let us denote by $\tilde{B}(\theta)$ the normal vector $\tilde{B}(X, X, X)$. Then

$$\begin{aligned}\tilde{B}(\theta)&=(\nabla_X^{\perp}B)(X, X)=\cos 3\theta\cdot\tilde{B}_{111}+\sin 3\theta\cdot\tilde{B}_{112}\\&=A\cdot(\cos 3\theta, \sin 3\theta)\,,\end{aligned}$$

where $A: TM\to\nu$ is the operator given in the bases (e_1, e_2) and $(e_3, \cdots, e_n)$ by the $(n-2)\times 2$ matrix

$$A=\begin{pmatrix} h_{111}^{3} & h_{112}^{3}\\ \vdots & \vdots\\ h_{111}^{n} & h_{112}^{n}\end{pmatrix}.$$

It follows that the image of S_p^1 under $\tilde{B}$ is an ellipse $\tilde{\varepsilon}_p$ in ν_p with area given by

$$\mathrm{Area}(\tilde{\varepsilon}_p)=\Big(\sum_{\alpha,\beta}(h_{111}^{\alpha}h_{112}^{\beta}-h_{112}^{\alpha}h_{111}^{\beta})^2\Big)^{1/2}\cdot\pi/2\,.$$

We list below some properties of $\tilde{B}(\theta)$ and $\tilde{\varepsilon}_p$.

(4.3) LEMMA. (i) $\tilde{B}(\theta+(2k+1)\pi/3)=-\tilde{B}(\theta)$, $\tilde{B}(\theta+2k\pi/3)=\tilde{B}(\theta)$, *for all* $k\in\boldsymbol{Z}$.

(ii) *The line tangent to* $\tilde{\varepsilon}_p$ *by the point* $\tilde{B}(\theta+\pi/6)$ *is parallel to the vector* $\tilde{B}(\theta)$.

(iii) *If* K *and* K^ν *are constant, then* $\tilde{\varepsilon}_p$ *is contained in the normal space* $\nu_p^{*\perp}$ *of* ν_p^* *in* ν_p. *Moreover, if* $\tilde{\varepsilon}_p$ *is nondegenerate we can choose the adapted frame in a way that* (e_3, e_4) *spans* ν^* *and that*

$$\tilde{B}_{111}=\tilde{\lambda}e_5\,,\qquad \tilde{B}_{112}=\tilde{\mu}e_6\,,$$

where $\tilde{\lambda}\geqq\tilde{\mu}$ *are the length of the semi-axes of* $\tilde{\varepsilon}_p$.

PROOF. The verification of (i) is routine. To verify (ii), define a curve $c(\theta)=A\cdot(\cos 3\theta, \sin 3\theta)$ and observe that $(dc/d\theta)(\theta+\pi/6)=-3\tilde{B}(\theta)$. To verify the first half of (iii), it is sufficient to note that $\tilde{B}_{ijk}=\sum_{\gamma\geqq 5}h_{ijk}^{\gamma}e_\gamma$ for a frame as in Lemma (4.2). For the second half of (iii), it is clear that we can choose the frame such that e_5 and e_6 give the directions of the semi-axes of $\tilde{\varepsilon}_p$. We can also rotate (e_1, e_2) so that the frame satisfies $\tilde{B}_{111}=\tilde{\lambda}e_\gamma$, $\gamma=5$ or 6. Then $\tilde{B}_{112}=\tilde{B}(\pi/2)$ is normal to $\tilde{B}_{111}$ by (ii). To conclude the proof we only have to change (if necessary) e_5 by e_6 or $-e_6$. Q.E.D.

Assume that p is a point where $\tilde{\varepsilon}_p$ is nondegenerate, for a minimal immersion with K and K^ν constant. We believe that the following is now clear: if $\tilde{\varepsilon}_p$ is not a circle, or if $\tilde{\varepsilon}$ is a circle on a neighborhood of p, then we can always choose a local adapted frame field around p such that (e_3, e_4) spans ν^*, (e_5, e_6) spans the bundle generated by $\tilde{\varepsilon}$ and that

$$\tilde{B}_{111}=\tilde{\lambda}e_5\,,\qquad \tilde{B}_{112}=\tilde{\mu}e_6\,.$$

In such a frame we have $h_{111}^5=\tilde{\lambda}$, $h_{112}^6=\tilde{\mu}$ and $h_{112}^5=h_{111}^6=h_{ijk}^\gamma=0$ for $\gamma\neq 5, 6$. We are in a position to state the following proposition, whose proof is similar to that of Theorem 1-(b).

(4.4) PROPOSITION. *Let* $f: M\to Q^6(c)$ *be a minimal immersion of a surface with constant Gaussian and normal curvatures. Assume that there exists a point* p *in* M *such that* $\tilde{\varepsilon}_p$ *is nondegenerate. Then* $c>0$, $K>0$ *and* M *is locally a Veronese surface* $S^2(c/6)$ *in* $S^6(c)$.

PROOF. Let $(e_1, \cdots, e_6)$ be an adapted frame field around p as above. From the minimality of M and from (2.6) we have

$$
\begin{aligned}
e_1(\tilde{\lambda}) &= -e_1\langle \tilde{B}_{221}, e_5\rangle = -\langle \nabla^{\perp}_{e_1}\tilde{B}_{221}, e_5\rangle \\
&= -\langle (\nabla^{\perp}_{e_1}\tilde{B})(e_2, e_2, e_1), e_5\rangle = -\langle (\nabla^{\perp}_{e_2}\tilde{B})(e_1, e_2, e_1), e_5\rangle \\
&= -\langle \nabla^{\perp}_{e_2}\tilde{B}_{112} - 2\tilde{B}(\nabla_{e_2}e_1, e_2, e_1) - \tilde{B}(e_1, e_1, \nabla_{e_2}e_2), e_5\rangle \\
&= (\tilde{\mu}w_{56} - 3\tilde{\lambda}w_{12})\circ J(e_1)\,.
\end{aligned}
$$

Analogously we determine $e_2(\tilde{\lambda})$ and $e_i(\tilde{\mu})$, $i=1, 2$. The conclusion is

$$
d\tilde{\lambda} = (\tilde{\mu}w_{56} - 3\tilde{\lambda}w_{12})\circ J \qquad d\tilde{\mu} = (\tilde{\lambda}w_{56} - 3\tilde{\mu}w_{12})\circ J\,.
$$

Since K and K^{ν} are constant, K^* is obviously constant. Using (2.4) we see that $\tilde{S} = \tilde{\lambda}^2 + \tilde{\mu}^2$ is also constant and then $d\tilde{S} = 0$ gives

$$
2\tilde{\lambda}\tilde{\mu}w_{56} = 3\tilde{S}w_{12}\,. \tag{4.5}
$$

So we can write

$$
\begin{gathered}
d\tilde{\lambda} = -\frac{3(\tilde{\lambda}^2 - \tilde{\mu}^2)}{2\tilde{\lambda}}w_{12}\circ J\,, \qquad d\tilde{\mu} = \frac{3(\tilde{\lambda}^2 - \tilde{\mu}^2)}{2\tilde{\mu}}w_{12}\circ J\,, \\
d(\tilde{\lambda}\tilde{\mu}) = \frac{3(\tilde{\lambda}^2 - \tilde{\mu}^2)^2}{2\tilde{\lambda}\tilde{\mu}}w_{12}\circ J\,.
\end{gathered} \tag{4.6}
$$

Let us call $\tilde{\lambda}\tilde{\mu} = X$ for simplicity. Then (4.6) gives

$$
\begin{gathered}
\langle \operatorname{grad}\tilde{\lambda}, \operatorname{grad}\tilde{\mu}\rangle = -\frac{9(\tilde{\lambda}^2 - \tilde{\mu}^2)^2}{4X}\|w_{12}\|^2\,, \\
\|\operatorname{grad}X\|^2 = \frac{9(\tilde{\lambda}^2 - \tilde{\mu}^2)^4}{4X^2}\|w_{12}\|^2\,.
\end{gathered} \tag{4.7}
$$

Now a long but simple calculation using (4.6) and (4.7) shows that

$$
\Delta X = -\frac{9(\tilde{\lambda}^2 - \tilde{\mu}^2)^2(\tilde{S}^2 + 4X^2)}{4X^3}\|w_{12}\|^2 - \frac{3(\tilde{\lambda}^2 - \tilde{\mu}^2)^2}{2X}K\,. \tag{4.8}
$$

On the other hand, by differentiating (4.5) and using (2.2) and (2.3) we obtain

$$
\frac{9\tilde{S}(\tilde{\lambda}^2 - \tilde{\mu}^2)^2}{2X^2}\|w_{12}\|^2 = \frac{8SX^2}{(K^{\nu})^2} - 3\tilde{S}K\,. \tag{4.9}
$$

Bringing (4.9) into (4.7) and (4.8) gives

$$
\begin{gathered}
\|\operatorname{grad}X\|^2 = -\frac{16S}{(K^{\nu})^2\tilde{S}}X^4 + \left(6K + \frac{4S\tilde{S}}{(K^{\nu})^2}\right)X^2 - \frac{3\tilde{S}K}{2}\,, \\
\Delta X = -\frac{16S}{(K^{\nu})^2\tilde{S}}X^3 + \left(12K - \frac{4S\tilde{S}}{(K^{\nu})^2}\right)X\,.
\end{gathered} \tag{4.10}
$$

As in the proof of part (b) of Theorem 1, it follows from (4.10) that X must be constant. Then $\tilde{\lambda}$ and $\tilde{\mu}$ are constant around p and we conclude that they are constant all over M. With this we differentiate (4.5) to obtain

$$\frac{8SX^2}{(K^\nu)^2}=3\tilde{S}K.$$

Therefore $K>0$ and then $c>0$. The proposition is now a consequence of Theorem 2 and of the fact that, according to Theorem 5.6 of Calabi [**2**], the curvature of a full minimal sphere $S^2(K)$ in $S^{2k}(c)$ must satisfy $K=\dfrac{2c}{k(k+1)}$. Q.E.D.

(4.11) PROOF OF THEOREM 3. Choose an adapted frame $(e_1, \cdots, e_6)$ in $Q^6(c)$ such that (e_3, e_4) spans ν^* and $h^\alpha_{ijk}=0$ for $\alpha=3, 4$. By (2.4) we know that $h=\sum_{\gamma\geq 5}(h^\gamma_{ijk})^2$ is constant. If $h=0$, then $2K=K^*=K^\nu>0$ and $c>0$. Also by a lemma of Ōtsuki [**10**], p. 96, M is contained in a 4-dimensional totally geodesic submanifold $Q^4(c)$ of $Q^6(c)$. Hence M must be locally a Veronese surface $S^2(c/3)$ in $S^4(c)$, thus giving (a). Now $h\neq 0$ means that $\tilde{\varepsilon}$ is never a point. If $\tilde{\varepsilon}_p$ is nondegenerate for some p in M, then Proposition (4.4) gives (c). The only possibility left is when $\tilde{\varepsilon}$ is a line segment of constant length $2\tilde{\lambda}$. In this case we choose the frame so that $\tilde{B}_{111}=\tilde{\lambda}e_5$ and of course $\tilde{B}_{112}=0$. Then $0=d\tilde{\lambda}=-3\tilde{\lambda}w_{12}\circ J$ and this immediately implies that $w_{12}=0$, $K=0$ and $c>0$. Again by the above lemma of Ōtsuki, we see that M is contained in a totally geodesic $Q^5(c)$ of $Q^6(c)$. This gives (b) and completes the proof of the Theorem.

Bibliography

[1] A. C. Asperti, D. Ferus and L. Rodriquez, Surfaces with non-zero normal curvature tensor, Atti Accad. Naz. Lincei, **73** (1982), 109-115.

[2] E. Calabi, Minimal immersions of surfaces in Euclidean spheres, J. Diff. Geom., **1** (1967), 111-125.

[3] M. do Carmo and N. Wallach, Representations of compact groups and minimal immersions of spheres into spheres, J. Diff. Geom., **4** (1970), 91-104.

[4] S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of a constant length, Functional Analysis and Related Fields, Springer-Verlag, 1970, 59-75.

[5] A. G. Colares and M. do Carmo, On minimal immersions with parallel normal curvature tensor, Lecture Notes in Math., **597**, Springer-Verlag, 1977, 104-113.

[6] T. Itoh, Minimal surfaces in 4-dimensional Riemannian manifolds of constant curvature, Kōdai Math. Sem. Rep., **22** (1971), 451-458.

[7] K. Kenmotsu, On a parametrization of minimal immersions of $\boldsymbol{R}^2$ into S^5, Tôhoku Math. J., **27** (1975), 83-90.

[8] K. Kenmotsu, On minimal immersions of $\boldsymbol{R}^2$ into S^n, J. Math. Soc. Japan, **28** (1976), 182-191.

[9] K. Kenmotsu, Minimal surfaces with constant curvature in 4-dimensional space forms, Preprint, 1981.

[10] T. Ōtsuki, Minimal submanifolds with m-index 2 and generalized Veronese surfaces, J. Math. Soc. Japan, **24** (1972), 89-122.

[11] L. Rodriquez and I. V. Guadalupe, Normal curvature of surfaces in space forms, Pacific J. Math., **106** (1983), 95-103.

[12] N. Wallach, Extensions of locally defined minimal immersions into spheres, Arch. Math., **21** (1970), 210-213.

[13] Y. C. Wong, Contributions to the theory of surfaces in a 4-space of constant curvature, Trans. Amer. Math. Soc., **59** (1946), 467-507.

[14] S. T. Yau, Submanifolds with constant mean curvature I, Amer. J. Math., **96** (1974), 346-366.

Antonio Carlos ASPERTI
Instituto de Matematica e Estatistica
Universidade de Sao Paulo
Caixa Postal 20570 (Agencia Iguatemi)
05508 - Sao Paulo - SP - Brazil