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On the Stark-Shintani conjecture and cyclotomic Z_p-extensions of class fields over real quadratic fields

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§1. Introduction.

Let F be a real quadratic field embedded in the real number field $\boldsymbol{R}$. Let M be a finite abelian extension of F in which exactly one of the two infinite primes of F, corresponding to the prescribed embedding of F into $\boldsymbol{R}$, splits. Let $\mathfrak{f}$ be the conductor of M/F. Denote by $H_F(\mathfrak{f})$ the group consisting of all narrow ray classes of F defined modulo $\mathfrak{f}$. Let G be the subgroup of $H_F(\mathfrak{f})$ corresponding to M by class field theory. Take a totally positive integer ν of F satisfying $\nu+1\in\mathfrak{f}$, and denote by the same letter ν the narrow ray class modulo $\mathfrak{f}$ represented by the principal ideal (ν). For each $c\in H_F(\mathfrak{f})$, set $\zeta_F(s, c)=\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$, where $\mathfrak{a}$ runs over all integral ideals of F belonging to the ray class c. It is known that $\zeta_F(s, c)$ is holomorphic on the whole complex plane except for a simple pole at $s=1$.

The Stark-Shintani ray class invariant $X_{\mathfrak{f}}(c)$ is defined by

$$X_{\mathfrak{f}}(c)=\exp(\zeta_F'(0, c)-\zeta_F'(0, c\nu)) \qquad (c\in H_F(\mathfrak{f})) \tag{1}$$

(see Stark [7] and Shintani [5], the notation $X_{\mathfrak{f}}(c)$ is due to [5]). Obviously, $X_{\mathfrak{f}}(c\nu)=X_{\mathfrak{f}}(c)^{-1}$. In his paper [5], T. Shintani expressed this invariant $X_{\mathfrak{f}}(c)$ as a product of certain special values of the double gamma function of E. W. Barnes. In particular, he proved that $X_{\mathfrak{f}}(c)$ is a positive real number. Put $X_{\mathfrak{f}}(c, G)=\prod_{g\in G} X_{\mathfrak{f}}(cg)$. Then $X_{\mathfrak{f}}(c, G)$ depends only on $c\in H_F(\mathfrak{f})/G$. In [7] and [8], H. M. Stark presented a striking conjecture on the arithmetic nature of the invariant $X_{\mathfrak{f}}(c, G)$. Shintani found it independently and reformulated it in [5] into a more precise form.

CONJECTURE 1. For some positive rational integer m, $X_{\mathfrak{f}}(c, G)^m$ is a unit of M $(\forall c\in H_F(\mathfrak{f})/G)$. Moreover, $\{X_{\mathfrak{f}}(c, G)^m\}^{\sigma(c_0)}=X_{\mathfrak{f}}(cc_0, G)^m$ $(\forall c_0\in H_F(\mathfrak{f})/G)$, where σ is the Artin isomorphism of $H_F(\mathfrak{f})/G$ onto the Galois group $\mathrm{Gal}(M/F)$.

Shintani introduced in [5] another invariant $Y_{\mathfrak{f}}(c, G)$ to prove Conjecture 1 in some special non-trivial cases (for the definition of $Y_{\mathfrak{f}}(c, G)$, see (18) and (20) of [5]). We may state the same conjecture for $Y_{\mathfrak{f}}(c, G)$ instead of $X_{\mathfrak{f}}(c, G)$, and call it Conjecture 1′. Denote by M^+ the maximal totally real subfield of M. Then Shintani proved the following theorem in [5].

THEOREM A (Shintani). *If M^+ is abelian over the rational number field $\boldsymbol{Q}$, then Conjecture 1 and Conjecture 1′ are true.*

For a number field k, denote by $E(k)$ and $h(k)$ the group of units of k and the class number of k respectively. Then T. Arakawa proved the following relation between the relative class number of M/M^+ and the invariants $Y_{\mathfrak{f}}(c, G)$ ([1]).

THEOREM B (Arakawa). *Assume that Conjecture 1′ is true, then we have*

$$h(M)/h(M^+)=[E(M):E_{Y,m}(M)]/2^{2n-1}m^n\,,$$

where $n=[M^+:F]$ and $E_{Y,m}(M)$ is the subgroup of $E(M)$ generated by $E(M^+)$ and $Y_{\mathfrak{f}}(c, G)^m$ $(c\in H_F(\mathfrak{f})/G)$.

If we can take $m=1$ in Conjecture 1′, then the formula in Theorem B becomes a better one. In this direction, Stark presented the following conjecture in [7].

CONJECTURE 2. Conjecture 1 holds with $m=1$.

Let p be a prime number and let $M_\infty=\bigcup_{n\geqq 0} M_n$ be the cyclotomic $\boldsymbol{Z}_p$-extension of M $([M_n:M]=p^n)$. Arakawa pointed out in [1] that Conjecture 1 is true for each M_n if M^+ is abelian over $\boldsymbol{Q}$. But it was not known whether or not the integer m could be taken constantly in the tower of fields. In this paper, we assume that M^+ is abelian over $\boldsymbol{Q}$, and we study the integer m which makes $X_{\mathfrak{f}}(c, G)^m$ $(c\in H_F(\mathfrak{f})/G)$ into units of M. In particular, we study it in the tower of fields of cyclotomic $\boldsymbol{Z}_p$-extensions of M. Now we state one of the main results of this paper (cf. Propositions 8, 9, 10 and 13).

THEOREM 1. *Let α be an integer of F such that $\alpha>0$ and $\alpha'<0$ (α' is the conjugate of α), and let $M=F(\sqrt{\alpha})$. Let p be an odd prime and let $M_\infty=\bigcup_{n\geqq 0} M_n$ be the cyclotomic $\boldsymbol{Z}_p$-extension of M $([M_n:M]=p^n)$. If no prime divisor of p in F splits in M, then $X_{\mathfrak{f}_n}(c, G_n)$ is a unit of M_n for each $c\in H_F(\mathfrak{f}_n)/G_n$ $(\forall n\geqq 0)$, where $\mathfrak{f}_n$ is the conductor of M_n/F and G_n is the subgroup of $H_F(\mathfrak{f}_n)$ corresponding to M_n.*

Stark presented more general conjectures on special values of L-functions for totally real fields (see [7] and [9]). He also formulated in [6] conjectures on special values of non-abelian Artin L-functions (cf. Tate [10]). T. Chinburg made computations confirming the existence of expected units in tetrahedral cases over $\boldsymbol{Q}$ ([2]). He also studied how the expected units lie in the tower of fields of cyclotomic $\boldsymbol{Z}_p$-extensions of certain non-abelian fields. Let K be the fixed field of an odd, irreducible two dimensional Galois representation ρ : $\mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})\rightarrow$

$GL_2(\boldsymbol{C})$. Let p be an odd prime and let $K(n)$ be the field obtained by adjoining to K the p^{n+1}th roots of unity. Then Chinburg proved in [2] that the expected units in $K(\infty)$ $(=\bigcup_{n\geq 0} K(n))$ satisfy distribution and norm relation assuming the Stark conjecture for each $K(n)$. On the other hand, we have really constructed certain cyclotomic $\boldsymbol{Z}_p$-extensions $M_\infty=\bigcup_{n\geq 0} M_n$ such that the Stark-Shintani invariants for M_n are units of M_n for each $n\geqq 0$.

In our subsequent paper, we shall discuss some consequences of Theorem B and the results of this paper, and we shall study the image of $X_{\mathfrak{f}_n}(c, G_n)$'s in the completion of M_∞ at a prime over p by using a result of Coleman [3].

The base field M of our cyclotomic $\boldsymbol{Z}_p$-extension is abelian over a real quadratic field, but not abelian over $\boldsymbol{Q}$. Moreover, M is neither a totally real field nor a totally imaginary quadratic extension of a totally real field. It seems that few things are known about such $\boldsymbol{Z}_p$-extensions except what the general theory of $\boldsymbol{Z}_p$-extensions tells, and investigation of those Iwasawa invariants seems to be an interesting problem. For example, it is conjectured by Iwasawa that the Iwasawa invariant μ vanishes in any cyclotomic $\boldsymbol{Z}_p$-extension, and it was proved by Ferrero and Washington in the case of any abelian base field. Now we should try to prove it in our cases.

§ 2. Some results of Stark [9].

In this section, we summarize some results of Stark [9] for later applications. Let k be an imaginary quadratic field embedded in the complex number field $\boldsymbol{C}$. Let $\mathfrak{c}$ be an integral ideal of k with $\mathfrak{c}\neq(1)$. Denote by $H_k(\mathfrak{c})$ the group consisting of all ray classes of k defined modulo $\mathfrak{c}$. Let H be a subgroup of $H_k(\mathfrak{c})$, and let K be the class field over k corresponding to H. Denote by σ_k the Artin isomorphism of $H_k(\mathfrak{c})/H$ onto $\mathrm{Gal}(K/k)$. Denote by $w(K)$ the number of roots of unity contained in K.

LEMMA 2 (Stark). *For each $c\in H_k(\mathfrak{c})/H$, there exists an algebraic integer $E_\mathfrak{c}(c, H)$ of K with the following three properties:*

i) *For any character χ of the group $H_k(\mathfrak{c})$ with $\chi(H)=1$,*

$$L_k'(0, \chi)=-(1/w(K))\sum_{c\in H_k(\mathfrak{c})/H}\chi(c)\log|E_\mathfrak{c}(c, H)|^2 .$$

ii) *$E_\mathfrak{c}(1, H)^{\sigma_k(c)}=E_\mathfrak{c}(c, H)$, for any $c\in H_k(\mathfrak{c})/H$.*

iii) *If $\mathfrak{q}$ is a prime ideal of k belonging to the class c with $(\mathfrak{q}, \mathfrak{c}w(K))=1$, then $E_\mathfrak{c}(c, H)/E_\mathfrak{c}(1, H)^{N(\mathfrak{q})}$ is a $w(K)$th power of a number of K.*

Shintani introduced in [5] a certain ray class invariant $Z_\mathfrak{c}(c)$ for each $c\in H_k(\mathfrak{c})$ (for the definition of $Z_\mathfrak{c}(c)$, see (4) and (6) of [5]). We now clarify the relation between $E_\mathfrak{c}(c, H)$ and $Z_\mathfrak{c}(c)$. For each $c\in H_k(\mathfrak{c})$, set $\zeta_k(s, c)=\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$, where $\mathfrak{a}$ runs over all integral ideals of k belonging to the ray class c. It follows from Proposition 1 of [5] that

(2) $$\omega(\mathfrak{c})\zeta_k'(0, c) = -\log Z_{\mathfrak{c}}(c),$$

where $\omega(\mathfrak{c})$ is the cardinality of the group of units of k which are congruent to 1 modulo $\mathfrak{c}$. Put $Z_{\mathfrak{c}}(c, H) = \prod_{h\in H} Z_{\mathfrak{c}}(ch)$. Then $Z_{\mathfrak{c}}(c, H)$ depends only on $c\in H_k(\mathfrak{c})/H$. In view of (2), we obtain

$$L_k'(0, \chi) = -(1/\omega(\mathfrak{c}))\sum_{c\in H_k(\mathfrak{c})/H} \chi(c)\log Z_{\mathfrak{c}}(c, H)$$

for any character χ of the group $H_k(\mathfrak{c})$ with $\chi(H)=1$. Comparing the above equality with that of Lemma 2, we obtain

(3) $$Z_{\mathfrak{c}}(c, H) = |E_{\mathfrak{c}}(c, H)|^{2\omega(\mathfrak{c})/w(K)} \qquad (\mathfrak{c}\neq(1)).$$

Following Shintani [5], we are going to introduce another invariant $W_{\mathfrak{c}}(c, H)$ $(c\in H_k(\mathfrak{c})/H)$ which is closely related to the invariant $Z_{\mathfrak{c}}(c, H)$ (cf. (8) and (9) of [5]). Denote by $\mathfrak{P}(\mathfrak{c})$ the set of prime divisors of $\mathfrak{c}$. For each subset S of $\mathfrak{P}(\mathfrak{c})$, denote by $\mathfrak{c}(S)$ the intersection of all divisors of $\mathfrak{c}$ which are prime to any $\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S$. In other words, if $\mathfrak{c}=\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})}\mathfrak{p}^{a(\mathfrak{p})}$, then $\mathfrak{c}(S)=\prod_{\mathfrak{p}\in S}\mathfrak{p}^{a(\mathfrak{p})}$. Denote by $\tilde{c}$ (resp. $\tilde{H}$) the image of c (resp. H) under the natural homomorphism of $H_k(\mathfrak{c})$ onto $H_k(\mathfrak{c}(S))$. Further, put

(4) $$n(S, H) = \omega(\mathfrak{c}(S))[H_k(\mathfrak{c}) : H]/[H_k(\mathfrak{c}(S)) : \tilde{H}].$$

For each $c\in H_k(\mathfrak{c})/H$, set

(5) $$W_{\mathfrak{c}}(c, H) = \prod_S Z_{\mathfrak{c}(S)}(\tilde{c}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1}, \tilde{H})^{1/n(S,H)},$$

where S runs over all subsets of $\mathfrak{P}(\mathfrak{c})$ with $\tilde{H}\neq H_k(\mathfrak{c}(S))$. In Lemma 2, we assumed $\mathfrak{c}\neq(1)$, but $Z_{\mathfrak{c}}(c)$, $Z_{\mathfrak{c}}(c, H)$ and $W_{\mathfrak{c}}(c, H)$ are defined for any integral ideal $\mathfrak{c}$ of k. So we formally define $|E_{\mathfrak{c}}(c, H)|$ by (3) for $\mathfrak{c}=(1)$. Under this notation, $W_{\mathfrak{c}}(c, H)$ is described in terms of $E_{\mathfrak{c}}(c, H)$'s as follows:

(6) $$W_{\mathfrak{c}}(c, H) = \prod_S |E_{\mathfrak{c}(S)}(\tilde{c}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1}, \tilde{H})|^{2/w(K_S)[K:K_S]},$$

where K_S is the intersection of K with the ray class field modulo $\mathfrak{c}(S)$ over k, and S runs over all subsets of $\mathfrak{P}(\mathfrak{c})$ with $K_S\neq k$.

§3. Investigation of m for $Y_{\mathfrak{f}}(c, G)$.

Let F be a real quadratic field embedded in $\boldsymbol{R}$. Let M be a finite abelian extension of F in which exactly one of the two infinite primes of F, corresponding to the prescribed embedding of F into $\boldsymbol{R}$, splits. We assume that M is also embedded in $\boldsymbol{R}$. Let $\mathfrak{f}$ be a multiple of the conductor of M/F. Let G be the subgroup of $H_F(\mathfrak{f})$ corresponding to M. For each $x \in F$, denote by x' the conjugate of x. Take a totally positive integer ν of F with $\nu+1 \in \mathfrak{f}$. Let μ be an integer of F satisfying $\mu<0$, $\mu'>0$ and $\mu-1 \in \mathfrak{f}$. Denote by $\nu(\mathfrak{f})$ (resp. $\mu(\mathfrak{f})$) the element of $H_F(\mathfrak{f})$ represented by the principal ideal (ν) (resp. (μ)). Sometimes we write simply as ν and μ instead of $\nu(\mathfrak{f})$ and $\mu(\mathfrak{f})$. Then the order of ν in $H_F(\mathfrak{f})$ is 2 and the order of μ is at most 2. Further, it follows from our assumption on M that G contains μ but does not contain ν.

In § 1, we introduced the invariant $X_{\mathfrak{f}}(c, G)$ for each $c \in H_F(\mathfrak{f})/G$. Shintani introduced in [**5**] another invariant $Y_{\mathfrak{f}}(c, G)$ which does not depend on a special choice of $\mathfrak{f}$ (cf. Lemma 2 of [**5**]). Denote by $\mathfrak{P}(\mathfrak{f})$ the set of prime divisors of $\mathfrak{f}$. For each subset S of $\mathfrak{P}(\mathfrak{f})$, denote by $\mathfrak{f}(S)$ the intersection of all divisors of $\mathfrak{f}$ which are prime to any $\mathfrak{p} \in \mathfrak{P}(\mathfrak{f})-S$. Denote by $\tilde{c}$ (resp. $\tilde{G}$) the image of c (resp. G) under the natural homomorphism of $H_F(\mathfrak{f})$ onto $H_F(\mathfrak{f}(S))$. Further, denote by M_S the intersection of M with the narrow ray class field modulo $\mathfrak{f}(S)$ over F. So M_S coincides with the class field over F corresponding to $\tilde{G}$. Since $X_{\mathfrak{f}}(c\nu) = X_{\mathfrak{f}}(c)^{-1}$, we obtain the following formula:

$$Y_{\mathfrak{f}}(c, G)=\prod_{S} X_{\mathfrak{f}(S)}(\tilde{c} \prod_{\mathfrak{p} \in \mathfrak{P}(\mathfrak{f})-S} (\mathfrak{p})^{-1}, \tilde{G})^{1/[M:M_S]}, \tag{7}$$

where S runs over all subsets of $\mathfrak{P}(\mathfrak{f})$ with $M_S \not\subset M^+$. In particular, if M is a quadratic extension of F, and if $\mathfrak{f}$ is the conductor of M/F, then $Y_{\mathfrak{f}}(c, G) = X_{\mathfrak{f}}(c, G)$ $(\forall c \in H_F(\mathfrak{f})/G)$.

In the remaining part of this paper, we assume that M^+ is abelian over $\boldsymbol{Q}$. We may assume that $\mathfrak{f}$ is a self conjugate integral ideal of F. In fact, if $\mathfrak{f}' \neq \mathfrak{f}$, we may replace $\mathfrak{f}$ by $\mathfrak{f} \cap \mathfrak{f}'$ and G by its inverse image under the natural homomorphism of $H_F(\mathfrak{f} \cap \mathfrak{f}')$ onto $H_F(\mathfrak{f})$. Denote by ι the non-trivial automorphism of F. Then ι acts naturally on the group $H_F(\mathfrak{f})$. We put $\iota(c)=c'$ for any $c \in H_F(\mathfrak{f})$. Then it follows from Lemma 3 of [**5**] that $\nu=\mu\mu'$ in $H_F(\mathfrak{f})$. Since M^+ is abelian over $\boldsymbol{Q}$, there exists a subgroup G_1 of G with index 2 which is invariant under ι and satisfies the following conditions (see p. 141 of [**5**]):

(8) The group G is generated by G_1 and μ;

$$[H_F(\mathfrak{f})/G_1 : (H_F(\mathfrak{f})/G_1)_0]=2, \tag{9}$$

where

$$(H_F(\mathfrak{f})/G_1)_0 = \{c \in H_F(\mathfrak{f})/G_1 \ ; \ \iota(c)=c\} .$$

Let K be the class field over F corresponding to G_1. Then K is the normal closure of M over $\boldsymbol{Q}$, and $[K: M]=2$. Let L be the class field over F corresponding to $\langle G_1, \nu\rangle$, where $\langle G_1, \nu\rangle$ is the subgroup of $H_F(\mathfrak{f})$ generated by G_1 and ν. Then L is the maximal absolutely abelian subfield of K, and $[K: L]=2$. We assume that K is embedded in $\boldsymbol{C}$ by extending the prescribed embedding of

M into $\boldsymbol{R}$. Denote by σ_F the Artin isomorphism of $H_F(\mathfrak{f})/G_1$ onto $\mathrm{Gal}(K/F)$. Let L_0 be the subfield of $\sigma_F((H_F(\mathfrak{f})/G_1)_0)$-fixed elements of K. In view of (9), L_0 is a quadratic extension of F. Further, it follows from Lemma 4 of [**5**] that L_0 is a composition of F with a suitable imaginary quadratic field k, and K is abelian over k. We note that any one of the two imaginary quadratic fields contained in L_0 can be taken for k. Let $\mathfrak{c}$ be the conductor of K/k and let H be the subgroup of $H_k(\mathfrak{c})$ corresponding to K. Since K is normal over $\boldsymbol{Q}$, both $\mathfrak{c}$ and H are invariant under the non-trivial automorphism κ of k. Set

$$(H_k(\mathfrak{c})/H)_0=\{c\in H_k(\mathfrak{c})/H\ ;\ \kappa(c)=c\}\,.$$

It follows from Lemma 6 of [**5**] that the subfield of $\sigma_k((H_k(\mathfrak{c})/H)_0)$-fixed elements of K coincides with L_0, and $\sigma_k^{-1}\sigma_F$ induces an isomorphism of the group $(H_F(\mathfrak{f})/G_1)_0$ onto the group $(H_k(\mathfrak{c})/H)_0$. For each $c\in(H_F(\mathfrak{f})/G_1)_0$, put

$$\dot{c}=\sigma_k^{-1}\sigma_F(c)\,. \tag{10}$$

The following lemma plays a key role in our argument.

LEMMA 3. *If* $\mathfrak{c}\neq(1)$, *then*

$$|E_{\mathfrak{c}}(\dot{c}\dot{\nu},H)/E_{\mathfrak{c}}(\dot{c},H)|^{2/w(K)}\in M\,,\qquad \textit{for}\quad \forall c\in(H_F(\mathfrak{f})/G_1)_0\,.$$

PROOF. Let $\mathfrak{q}$ be a prime ideal of k belonging to the class $\dot{\nu}$ with $(\mathfrak{q},\mathfrak{c}w(K))=1$. It follows from Lemma 2 that $E_{\mathfrak{c}}(\dot{c}\dot{\nu},H)/E_{\mathfrak{c}}(\dot{c},H)^{N(\mathfrak{q})}$ is a $w(K)$th power of a number of K. Since $\sigma_k(\dot{\nu})$ $(=\sigma_F(\nu))$ induces the identity map on L, $\mathfrak{q}$ splits completely in L. It is easy to see that $N(\mathfrak{q})-1$ is a multiple of $w(L)$, because it is the order of the multiplicative group of the residue field of $\mathfrak{q}$. Since L is the maximal absolutely abelian subfield of K, we have $w(L)=w(K)$. Hence $E_{\mathfrak{c}}(\dot{c}\dot{\nu},H)/E_{\mathfrak{c}}(\dot{c},H)$ is a $w(K)$th power of a number of K. Since K is normal over $\boldsymbol{Q}$, the complex conjugation of $E_{\mathfrak{c}}(\dot{c}\dot{\nu},H)/E_{\mathfrak{c}}(\dot{c},H)$ is also a $w(K)$th power of a number of K. Since $K\cap\boldsymbol{R}=M$, the lemma follows.

The first half of the next proposition is essentially Proposition 4 of [**5**]. We note that $H_F(\mathfrak{f})/\langle G,\nu\rangle$ is identified with $(H_F(\mathfrak{f})/G_1)_0/\langle\nu\rangle$ by the natural homomorphisms

$$(H_F(\mathfrak{f})/G_1)_0\hookrightarrow H_F(\mathfrak{f})/G_1\longrightarrow H_F(\mathfrak{f})/\langle G,\nu\rangle\,.$$

PROPOSITION 4. i) *For each* $c\in(H_F(\mathfrak{f})/G_1)_0$,

$$Y_{\mathfrak{f}}(c,G)=\prod_S\left|E_{\mathfrak{c}(S)}\Big(\widetilde{\dot{c}\dot{\nu}}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1},\tilde{H}\Big)\Big/E_{\mathfrak{c}(S)}\Big(\tilde{\dot{c}}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1},\tilde{H}\Big)\right|^{2/w(K_S)[K:K_S]},$$

where S *runs over all subsets of* $\mathfrak{P}(\mathfrak{c})$ *with* $K_S\not\subset L$.

ii) *Assume that* $K_\emptyset\subset L$ *for a suitable choice of* k. *If* m *is a multiple of all* $[K:K_S]$ $(S\subset\mathfrak{P}(\mathfrak{c}),\ K_S\not\subset L)$, *then* $Y_{\mathfrak{f}}(c,G)^m$ *is a unit of* M. *In particular,* $Y_{\mathfrak{f}}(c,G)^{[M:Q]}$ *is a unit of* M.

PROOF. By the same way as in the proof of Proposition 4 of [5], we obtain $Y_{\mathfrak{f}}(c, G)=W_{\mathfrak{c}}(\dot{c}\dot{\nu}, H)/W_{\mathfrak{c}}(\dot{c}, H)$. In view of (6), we have

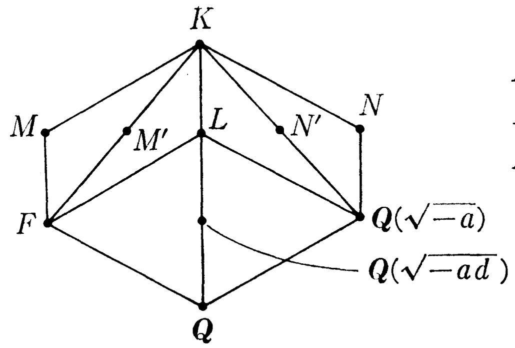
$$Y_{\mathfrak{f}}(c, G)=\prod_{S}\left|E_{\mathfrak{c}(S)}\Big(\widetilde{\dot{c}\dot{\nu}}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1}, \widetilde{H}\Big)\Big/E_{\mathfrak{c}(S)}\Big(\tilde{\dot{c}}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1}, \widetilde{H}\Big)\right|^{2/w(K_S)[K:K_S]},$$

where S runs over all subsets of $\mathfrak{P}(\mathfrak{c})$ with $K_S\neq k$. Since $\sigma_k(\dot{\nu})$ is the generator of $\mathrm{Gal}(K/L)$, $K_S\subset L$ is equivalent to $\dot{\nu}\in\widetilde{H}$. Hence the product is over all subsets S of $\mathfrak{P}(\mathfrak{c})$ with $K_S\not\subset L$. It follows from Lemma 3 and the first half of the proposition that $Y_{\mathfrak{f}}(c, G)^m$ belongs to M if m is a multiple of all $[K:K_S]$. Hence Proposition 5 of [5] implies that $Y_{\mathfrak{f}}(c, G)^m$ is a unit of M.

The following corollary is a special case of Theorem 3 of [7] but we give a different proof here.

COROLLARY. *If M is a quadratic extension of F of conductor $\mathfrak{f}$, then $X_{\mathfrak{f}}(c, G)$ is a unit of M for any $c\in H_F(\mathfrak{f})/G$.*

PROOF. Let $F=\boldsymbol{Q}(\sqrt{d})$ and let $M=F(\sqrt{\alpha})$, where d is a square free rational integer with $d>1$, and α is an integer of F such that $\alpha>0$, $\alpha'<0$. Put $\alpha\alpha'=-a$. Hence a is a positive rational integer. It is easy to see that $K=F(\sqrt{\alpha}, \sqrt{\alpha'})$, $L=L_0=F(\sqrt{-a})$ and $k=\boldsymbol{Q}(\sqrt{-a})$ or $\boldsymbol{Q}(\sqrt{-ad})$. Since $K/\boldsymbol{Q}$ is a non-abelian Galois extension of degree 8, and since M is not normal over $\boldsymbol{Q}$, $\mathrm{Gal}(K/\boldsymbol{Q})$ is isomorphic to the dihedral group D_4 of order 8. Hence the diagram of subfields of K is as follows:



$M'=F(\sqrt{\alpha'})$
$N=\boldsymbol{Q}(\sqrt{\alpha}+\sqrt{\alpha'})$
$N'=\boldsymbol{Q}(\sqrt{\alpha}-\sqrt{\alpha'})$

Figure 1.

Case 1. $K/\boldsymbol{Q}(\sqrt{-ad})$ is ramified at some primes. Put $k=\boldsymbol{Q}(\sqrt{-ad})$. Then it follows from Proposition 4 that $Y_{\mathfrak{f}}(c, G)$ is a unit of M. By the remark below (7), $Y_{\mathfrak{f}}(c, G)=X_{\mathfrak{f}}(c, G)$.

Case 2. $K/\boldsymbol{Q}(\sqrt{-ad})$ is unramified. Put $k=\boldsymbol{Q}(\sqrt{-a})$. Let $\mathfrak{d}$ be the conductor of N/k. Then it is easy to see that $\mathfrak{d}\neq(1)$ and the conductor of N'/k is $\mathfrak{d}'$, where $\mathfrak{d}'$ is the conjugate of $\mathfrak{d}$. Since $K=NN'$, the conductor $\mathfrak{c}$ of K/k

is the least common multiple of $\mathfrak{d}$ and $\mathfrak{d}'$. Hence $\mathfrak{P}(\mathfrak{c})=\mathfrak{P}(\mathfrak{d})\cup\mathfrak{P}(\mathfrak{d}')$. If $\mathfrak{P}(\mathfrak{d})=\mathfrak{P}(\mathfrak{d}')$, then it follows from Proposition 4 that $Y_{\mathfrak{f}}(c, G)$ $(=X_{\mathfrak{f}}(c, G))$ is a unit of M. If $\mathfrak{P}(\mathfrak{d})\neq\mathfrak{P}(\mathfrak{d}')$, put $S'=\{\mathfrak{p}' ; \mathfrak{p}\in S\}$ for each subset S of $\mathfrak{P}(\mathfrak{c})$. Then it follows from Proposition 4 that

$$Y_{\mathfrak{f}}(c, G)=|E_{\mathfrak{c}}(\dot{c}\dot{\nu}, H)/E_{\mathfrak{c}}(\dot{c}, H)|^{2/w(K)}$$

$$\times\prod_{\mathfrak{P}(\mathfrak{d})\subseteq S\subsetneq\mathfrak{P}(\mathfrak{c})}\left|E_{\mathfrak{c}(S)}\Big(\widetilde{c\nu}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1}, \tilde{H}\Big)\Big/E_{\mathfrak{c}(S)}\Big(\tilde{c}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1}, \tilde{H}\Big)\right|^{2/2w(N)}$$

$$\times\left|E_{\mathfrak{c}(S')}\Big(\widetilde{c\nu}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S'}(\mathfrak{p})^{-1}, \tilde{H}\Big)\Big/E_{\mathfrak{c}(S')}\Big(\tilde{c}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S'}(\mathfrak{p})^{-1}, \tilde{H}\Big)\right|^{2/2w(N')}.$$

It is easy to see that $w(N')=w(N)$ and $\kappa\Big(\widetilde{c\nu}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S}(\mathfrak{p})^{-1}\Big)=\widetilde{c\nu}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{c})-S'}(\mathfrak{p})^{-1}$. Lemma 3 and the next lemma now imply that $Y_{\mathfrak{f}}(c, G)$ $(=X_{\mathfrak{f}}(c, G))$ is a unit of M.

LEMMA 5. *For each* $c\in H_k(\mathfrak{c})$, $Z_{\kappa(\mathfrak{c})}(\kappa(c))=Z_{\mathfrak{c}}(c)$.

PROOF. In view of (2), we have $Z_{\mathfrak{c}}(c)=\exp(-\omega(\mathfrak{c})\zeta_k'(0, c))$. Hence the lemma follows immediately from the definition of the partial zeta function $\zeta_k(s, c)$.

§4. Main results.

We use the same notation and assumption as in the previous section. Let Φ be a real abelian field, and put $M^*=M\Phi$. Since M^+ is abelian over $\boldsymbol{Q}$, $(M^*)^+=M^+\Phi$ is also abelian over $\boldsymbol{Q}$. Hence Conjecture 1 is true for M^* by Theorem A. Let p be an odd prime and let $\zeta_{p^{n+1}}$ be a primitive p^{n+1}th root of unity. Let $\boldsymbol{B}_n$ be the unique subfield of $\boldsymbol{Q}(\zeta_{p^{n+1}})^+$ of degree p^n over $\boldsymbol{Q}$. Then $\boldsymbol{B}_\infty=\bigcup_{n\geq 0}\boldsymbol{B}_n$ is the unique Z_p-extension of $\boldsymbol{Q}$. Put $M_\infty=M\boldsymbol{B}_\infty$. Hence M_∞ is the cyclotomic $\boldsymbol{Z}_p$-extension of M. Let M_n be the unique subfield of M of degree p^n over M. Then the above remark implies that Conjecture 1 is true for each M_n.

First we prove two propositions, and then we prove Theorem 1.

PROPOSITION 6. *Let α be an integer of F such that $\alpha>0$, $\alpha'<0$. Let $M=F(\sqrt{\alpha})$ and let $\mathfrak{f}$ be the conductor of M/F. Put $T=\{q$; q is an odd prime and each prime divisor of q in F divides $\mathfrak{f}\}$. We assume $T\neq\emptyset$. Let Φ be a real abelian field of odd degree such that each prime divisor of the conductor of $\Phi/\boldsymbol{Q}$ belongs to T. Let $M^*=M\Phi$. Then $X_{\mathfrak{f}^*}(c, G^*)$ is a unit of M^* for each $c\in H_F(\mathfrak{f}^*)/G^*$, where $\mathfrak{f}^*$ is the conductor of M^*/F and G^* is the subgroup of $H_F(\mathfrak{f}^*)$ corresponding to M^*.*

PROOF. Let $F=\boldsymbol{Q}(\sqrt{d})$ and put $\alpha\alpha'=-a$. Then a is a positive rational integer. We write $(\alpha)=\mathfrak{a}\mathfrak{b}^2$, where $\mathfrak{a}$ is a square free integral ideal of F, and $\mathfrak{b}$ is an integral ideal of F. Then a prime divisor $\mathfrak{p}$ of F with $(\mathfrak{p}, 2)=1$ ramifies in M, if and only if $\mathfrak{p}$ divides $\mathfrak{a}$. It follows from the assumption on the conductor of $\Phi/\boldsymbol{Q}$ that $\mathfrak{P}(\mathfrak{f}^*)=\mathfrak{P}(\mathfrak{f})$. Since $[\Phi:\boldsymbol{Q}]$ is odd, any intermediate field of M^*/F which is not contained in $(M^*)^+$ is a composition of M with a subfield of Φ. It follows from (7) that $Y_{\mathfrak{f}^*}(c, G^*)=X_{\mathfrak{f}^*}(c, G^*)$. Put $k=\boldsymbol{Q}(\sqrt{-ad})$, $K^*=K\Phi$ and $L^*=L\Phi$, where $K=F(\sqrt{\alpha}, \sqrt{\alpha'})$ and $L=F(\sqrt{-a})$. Then K/k is a cyclic extension of degree 4. Let $\mathfrak{c}$ and $\mathfrak{c}^*$ be the conductors of K/k and K^*/k respectively. Since $(\alpha)=\mathfrak{a}\mathfrak{b}^2$, $ad=N(\mathfrak{a})N(\mathfrak{b})^2d=a_0\left(\prod_{q\in T}q\right)^2N(\mathfrak{b})^2$, where a_0 is a positive rational integer which is prime to any $q\in T$. So $k=\boldsymbol{Q}(\sqrt{-a_0})$, and hence any $q\in T$ does not ramify in k. On the other hand, it follows from the definition of T that any $q\in T$ ramifies in M, hence in K. Since $K/\boldsymbol{Q}$ is normal, these facts imply that $\emptyset\neq T\subset\{q\,;\,q$ is an odd prime and each prime divisor of q in k divides $\mathfrak{c}\}$. It follows $\mathfrak{c}\neq(1)$ and $\mathfrak{P}(\mathfrak{c}^*)=\mathfrak{P}(\mathfrak{c})$. Since $[\Phi:\boldsymbol{Q}]$ is odd, any intermediate field of K^*/k which is not contained in L^* is a composition of K with a subfield of Φ. It follows from Proposition 4 that $Y_{\mathfrak{f}^*}(c, G^*)=|E_{\mathfrak{c}^*}(\dot{c}\dot{\nu}, H^*)/E_{\mathfrak{c}^*}(\dot{c}, H^*)|^{2/w(K^*)}$, where H^* is the subgroup of $H_k(\mathfrak{c}^*)$ corresponding to K^*. By Lemma 3, the right hand side of the equality is a unit of M^*.

PROPOSITION 7. *Let M, $\mathfrak{f}$, T and Φ be as in Proposition* 6 (*we allow $T=\emptyset$*). *Assume that there is an odd prime p such that p splits in F and one of the two prime divisors of p in F ramifies and the other remains prime in M. Further, assume that p splits completely in Φ. Let $\Psi\neq\boldsymbol{Q}$ be a real abelian field with p-power conductor whose degree over $\boldsymbol{Q}$ is prime to* $2[\Phi:\boldsymbol{Q}]$. *Put $M_1^*=M\Phi\Psi$. Then $X_{\mathfrak{f}_1^*}(c, G_1^*)$ is a unit of M_1^* for each $c\in H_F(\mathfrak{f}_1^*)/G_1^*$, where $\mathfrak{f}_1^*$ is the conductor of M_1^*/F and G_1^* is the subgroup of $H_F(\mathfrak{f}_1^*)$ corresponding to M_1^*.*

PROOF. We use the same notation as in the proof of Proposition 6. We write $p=\mathfrak{p}\mathfrak{p}'$ with $\mathfrak{p}\,|\,\mathfrak{f}$, $\mathfrak{p}'\nmid\mathfrak{f}$. It follows from our assumption on p that $\sigma_F(\mathfrak{p}')$ induces the identity on $(M^*)^+$ though it is not the identity map of M^*. Hence $\mathfrak{p}'=\nu(\mathfrak{f}^*)$ in $H_F(\mathfrak{f}^*)/G^*$. Since $[M:F]$ $(=2)$, $[\Phi:\boldsymbol{Q}]$ and $[\Psi:\boldsymbol{Q}]$ are co-prime with each other, any intermediate field of M_1^*/F which is not contained in $(M_1^*)^+$ is a composition of M with a subfield of Φ and a subfield of Ψ. Since $\mathfrak{P}(\mathfrak{f}_1^*)=\mathfrak{P}(\mathfrak{f}^*)\cup\{\mathfrak{p}'\}$ and $\mathfrak{P}(\mathfrak{f}^*)=\mathfrak{P}(\mathfrak{f})$, for such an intermediate field of M_1^*/F the set of prime divisors of its conductor over F is either $\mathfrak{P}(\mathfrak{f}_1^*)$ or $\mathfrak{P}(\mathfrak{f}^*)$. Now it follows from (7) that $Y_{\mathfrak{f}_1^*}(c, G_1^*)=X_{\mathfrak{f}_1^*}(c, G_1^*)\times X_{\mathfrak{f}^*}(c(\mathfrak{p}')^{-1}, G^*)^{1/[M_1^*:M^*]}$. Since $\mathfrak{p}'=\nu(\mathfrak{f}^*)$ in $H_F(\mathfrak{f}^*)/G^*$, we have $X_{\mathfrak{f}^*}(c(\mathfrak{p}')^{-1}, G^*)=X_{\mathfrak{f}^*}(c, G^*)^{-1}$. Further, we have seen in the proof of Proposition 6 that $Y_{\mathfrak{f}^*}(c, G^*)=X_{\mathfrak{f}^*}(c, G^*)$. Hence we obtain

$$X_{\mathfrak{f}_1^*}(c, G_1^*)=Y_{\mathfrak{f}_1^*}(c, G_1^*)Y_{\mathfrak{f}^*}(c, G^*)^{1/[M_1^*:M^*]}. \tag{11}$$

Put $k=\boldsymbol{Q}(\sqrt{-ad})$. Then we can write $ad=a_1pb^2$, where a_1 is a positive rational integer prime to p and any $q\in T$. Hence p ramifies in k. So we write $p=\mathfrak{p}_k^2$. Put $K_1^*=K^*\Psi$ and $L_1^*=L^*\Psi$. Let $\mathfrak{c}_1^*$ be the conductor of K_1^*/k and let H_1^* be the subgroup of $H_k(\mathfrak{c}_1^*)$ corresponding to K_1^*. Since the ramification index of $\mathfrak{p}$ in K/F is 2, the ramification index of p in $K/\boldsymbol{Q}$ is also 2. Since $p=\mathfrak{p}_k^2$, $\mathfrak{p}_k$ does not ramify in K/k, hence $\mathfrak{p}_k \nmid \mathfrak{c}$. We have seen in the proof of Proposition 6 that $\mathfrak{P}(\mathfrak{c}^*)=\mathfrak{P}(\mathfrak{c})$. Since the conductor of $\Psi/\boldsymbol{Q}$ is a power of p, $\mathfrak{p}_k|\mathfrak{c}_1^*$ and $\mathfrak{P}(\mathfrak{c}_1^*)=\mathfrak{P}(\mathfrak{c}^*)\cup\{\mathfrak{p}_k\}$. It follows from our assumption on $[\Psi:\boldsymbol{Q}]$ that any intermediate field of K_1^*/k which is not contained in L_1^* is a composition of K with a subfield of Φ and a subfield of Ψ. Thus for such an intermediate field of K_1^*/k the set of prime divisors of its conductor over k is either $\mathfrak{P}(\mathfrak{c}_1^*)$ or $\mathfrak{P}(\mathfrak{c}^*)$. Now it follows from Proposition 4 that

$$Y_{\mathfrak{f}_1^*}(c, G_1^*)=|E_{\mathfrak{c}_1^*}(\dot{c}\dot{\nu}, H_1^*)/E_{\mathfrak{c}_1^*}(\dot{c}, H_1^*)|^{2/w(K_1^*)} \tag{12}$$

$$\times|E_{\mathfrak{c}^*}(\widetilde{\dot{c}\dot{\nu}}(\mathfrak{p}_k)^{-1}, H^*)/E_{\mathfrak{c}^*}(\tilde{\dot{c}}(\mathfrak{p}_k)^{-1}, H^*)|^{2/w(K^*)[K_1^*:K^*]},$$

$$Y_{\mathfrak{f}^*}(c, G^*)=|E_{\mathfrak{c}^*}(\dot{c}\dot{\nu}, H^*)/E_{\mathfrak{c}^*}(\dot{c}, H^*)|^{2/w(K^*)}. \tag{13}$$

It follows from our assumption on p that the number of the prime divisors of K lying over p is two. This implies that the decomposition field of $\mathfrak{p}_k$ in K/k is L. Since p splits completely in Φ, the decomposition field of $\mathfrak{p}_k$ in K^*/k is L^*. Hence $\mathfrak{p}_k=\dot{\nu}$ in $H_k(\mathfrak{c}^*)/H^*$. Hence the equalities (11), (12) and (13) imply that

$$X_{\mathfrak{f}_1^*}(c, G_1^*)=|E_{\mathfrak{c}_1^*}(\dot{c}\dot{\nu}, H_1^*)/E_{\mathfrak{c}_1^*}(\dot{c}, H_1^*)|^{2/w(K_1^*)}.$$

Since $\mathfrak{c}_1^*\neq(1)$, we can apply Lemma 3, so $X_{\mathfrak{f}_1^*}(c, G_1^*)$ is a unit of M_1^*.

PROOF OF THEOREM 1. Let $\mathfrak{f}$ be the conductor of M/F. About p and $\mathfrak{f}$, only the following three cases are possible: Case 1. Each prime divisor of p in F divides $\mathfrak{f}$; Case 2. p splits in F ($p=\mathfrak{p}\mathfrak{p}'$), $\mathfrak{p}|\mathfrak{f}$ and $\mathfrak{p}'\nmid\mathfrak{f}$; Case 3. p is prime to $\mathfrak{f}$. Case 1 is a special case of Proposition 6 ($\Phi=\boldsymbol{B}_n$). Case 2 is a special case of Proposition 7 ($\Phi=\boldsymbol{Q}$, $\Psi=\boldsymbol{B}_n$). The proof of Case 3 goes similarly to that of Propositions 6 and 7.

In Propositions 6 and 7, we assumed that $[\Phi:\boldsymbol{Q}]$ is odd, so that it was easy to study the set of prime divisors of the conductor of each intermediate field of M^*/F. Now we state a few results on the case of $[\Phi:\boldsymbol{Q}]=2\times$(odd) without proofs. They can be proved by repeating the arguments of the proof of Proposition 7. In the remaining part of this paper, we denote by (P) the property of M in Theorem 1:

(P) Let $M=\bigcup_{n\geq 0}M_n$ be the cyclotomic $\boldsymbol{Z}_p$-extension of M. Then $X_{\mathfrak{f}_n}(c, G_n)$ is a unit of M_n for each $c\in H_F(\mathfrak{f}_n)/G_n$ ($\forall n\geqq 0$), where $\mathfrak{f}_n$ is the conductor of M_n/F and G_n is the subgroup of $H_F(\mathfrak{f}_n)$ corresponding to M_n.

PROPOSITION 8. *Let l be a prime number which is congruent to* 5 *modulo* 8, *and let $F=\boldsymbol{Q}(\sqrt{l})$. Let ε (>1) be the fundamental unit of F, and assume*

$N_{F/Q}(\varepsilon)=-1$. *If* p *is a prime number which is congruent to* 3 *modulo* 4 *and remains prime in* F, *then* $M=F(\sqrt{\varepsilon}, \sqrt{p})$ *has the property* (P)

PROPOSITION 9. *Let* l, F *and* ε *be as in Proposition* 8. *If* $T_{F/Q}(\varepsilon)$ *is a quadratic residue modulo* l, *then* $M=F(\sqrt{\varepsilon}, \sqrt[4]{l})$ *has the property* (P) *with* $p=l$.

PROPOSITION 10. *Let* F *be a real quadratic field. Let* p *be a prime number which is congruent to* 1 *modulo* 4 *and splits in* F. *We write* $p=\mathfrak{p}\mathfrak{p}'$. *Take an integer* α *of* F *such that* $\alpha>0$, $\alpha'<0$, $\alpha\in\mathfrak{p}$, $\alpha\notin\mathfrak{p}^2$ *and* $\alpha\notin\mathfrak{p}'$. *Put* $\alpha\alpha'=-ap$, *so* a *is a positive rational integer prime to* p. *If* a *is a quadratic residue modulo* p *and* $T_{F/Q}(\alpha)$ *is not, then* $M=F(\sqrt{\alpha}, \sqrt{p})$ *has the property* (P).

So far, we have constructed cyclotomic $\boldsymbol{Z}_p$-extensions with the property (P). The base field M of such a $\boldsymbol{Z}_p$-extension has been a composition of a quadratic extension of F with a real abelian field. Now we are going to construct certain quartic cyclic extension M of F with the property (P).

LEMMA 11. *Let* F *be a field of characteristic* $\neq 2$. *Let* E *be a quadratic extension of* F. *Then* E *is embedded in a quartic cyclic extension* P *of* F, *if and only if* $-1\in N_{E/F}(E)$. *If* $\beta\in E$ *and* $N_{E/F}(\beta)=-1$, *then we can take* $\alpha\in E$ *such that* $\alpha^t/\alpha=\beta^2$, *where* t *is the generator of* $\mathrm{Gal}(E/F)$, *and* $P=F(\sqrt{\alpha})$ *is a desired extension of* F.

This lemma is well-known. For example, see p. 124 of Jacobson [4]. Applying Lemma 11, we obtain the next lemma.

LEMMA 12. *Let* b $(\geqq 1)$ *be an odd integer, and put* $d=4+b^2$. *Assume that* d *is square free. Put* $F=\boldsymbol{Q}(\sqrt{d})$ *and* $\varepsilon=(b+\sqrt{d})/2$. *Let* θ *be a totally positive integer of* F, *and put* $\alpha=\theta(\sqrt{d}+\sqrt{\varepsilon\sqrt{d}})$, $M=F(\sqrt{\alpha})$. *Then* M *is a quartic cyclic extension of* F, *and* $M^+=\boldsymbol{Q}(\sqrt{\varepsilon\sqrt{d}})$ *is a quartic cyclic extension of* $\boldsymbol{Q}$. *Further, exactly one of the two infinite primes of* F *splits in* M.

PROPOSITION 13. *Let* F *and* M *be as in Lemma* 12. *Let* $\mathfrak{f}$ *be the conductor of* M/F. *Put* $T=\{q$; q *is an odd prime and each prime divisor of* q *in* F *divides* $\mathfrak{f}\}$. *Then each* $q|d$ *belongs to* T, *so* $T\neq\emptyset$.

i) *If* $p\in T$, *then* M *has the property* (P).

ii) *Let* p *be an odd prime which splits in* F $(p=\mathfrak{p}\mathfrak{p}')$. *If* $\mathfrak{p}|\mathfrak{f}$, $\mathfrak{p}'\nmid\mathfrak{f}$ *and the decomposition field of* $\mathfrak{p}'$ *in* M/F *is* M^+, *then* M *has the property* (P).

PROOF. Same as the proof of Propositions 6 and 7.

REMARK. If (i) $p\equiv 3 \pmod 4$ or (ii) $p\equiv 5 \pmod 8$ and $F=\boldsymbol{Q}(\sqrt{p})$, then we can replace M by $M\cdot\boldsymbol{Q}(\zeta_p)^+$ in Theorem 1, Propositions 8, 9 and 13. Similarly, if $p\equiv 5 \bmod 8$, we can replace M by $M\cdot\boldsymbol{Q}(\zeta_p)^+$ in Proposition 10.

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