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Automorphism groups of multilinear mappings

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1. Introduction.

The relation between the nonsingularity of a multilinear mapping and the finiteness of its automorphism group was recently studied by the second author [2]. In particular, it was shown ([2, Theorem A]) that the nonsingularity implies the finiteness under some restriction on the characteristic of the underlying field. In this paper we shall prove the same result without this restriction.

If V is a vector space over a field, a multilinear mapping

$$\theta : V\times \overset{r}{\cdots\cdots} \times V \longrightarrow V$$

from the direct product of r copies of V into V itself is called simply a multilinear mapping of degree r on V. The subgroup $\mathrm{Aut}(\theta)$ of the general linear group $\mathrm{GL}(V)$ is defined by

$$\mathrm{Aut}(\theta)=\{\varphi\in\mathrm{GL}(V) \mid \theta(x_1, x_2, x_3, \cdots, x_r)^{\varphi}=\theta(x_1^{\varphi}, x_2^{\varphi}, x_3^{\varphi}, \cdots, x_r^{\varphi}) \quad \text{for all} \quad x_1, x_2, x_3, \cdots, x_r\in V\}.$$

We say that θ is nonsingular, if $\theta(x, x, x, \cdots, x)\neq 0$ for all $0\neq x\in V$.

Our main result is:

THEOREM A. *Let θ be a nonsingular multilinear mapping of degree $r\geqq 2$ on a vector space V of dimension n over an algebraically closed field F of characteristic $p>0$. Then* $\mathrm{Aut}(\theta)$ *is a finite group.*

Theorem A can be derived from the following Theorem B.

THEOREM B. *Let F, p, V, n, θ, r be as in Theorem* A. *Then for every unipotent subgroup Q of* $\mathrm{Aut}(\theta)$,

$$|Q|\leqq p^{\sum_{i\geqq 1}[n/p^i]},$$

where [ ] *denotes the greatest integer not exceeding the number inside.*

That Theorem B implies Theorem A follows from the following two propositions which appear as Propositions 1 and 6 in [2]. (In these two propositions, the characteristic is arbitrary. In Proposition D, the field need not be algebraically closed.)

PROPOSITION C. *Let θ be a nonsingular multilinear mapping of degree $r\geqq 2$ on a vector space of dimension n over an algebraically closed field. Then for every element $\sigma\in \mathrm{Aut}(\theta)$, σ^{m} is unipotent for some m at most $(r^{n}-1)^{n}$.*

PROPOSITION D. *Let V be a vector space of dimension n over a field, and G be a subgroup of $\mathrm{GL}(V)$. Suppose that the exponent of G is finite (i.e., there exists some number m such that $\sigma^{m}=1$ for all $\sigma\in G$), and that the order of any unipotent normal subgroup of G is finite. Then G is a finite group.*

As for the proof of Theorem B, we want to proceed by induction on n. In order to do that, we need to weaken the hypothesis concerning the nonsingularity of θ. More specifically, we prove the following theorem by induction on n.

THEOREM E. *Let θ be a multilinear mapping of degree $r\geqq 2$ on a vector space V of dimension n over an algebraically closed field F of characteristic p, and let Q be a unipotent subgroup of $\mathrm{Aut}(\theta)$. Suppose that for each $1\neq\tau\in Q$, the restriction of θ to $C_{V}(\tau)\cap[V, \tau]$ is non-singular, (i.e., $\theta(x, x, x, \cdots, x)\neq 0$ for all $0\neq x\in C_{V}(\tau)\cap[V, \tau]$). Then*

$$|Q|\leqq p^{\sum_{i\geq 1}[n/p^{i}]}.$$

(Here, $C_{V}(\tau)=\{x\in V \mid x^{\tau}=x\}$, $[V, \tau]=\{x^{\tau}-x \mid x\in V\}$, by definition.)

The following proposition is a key result to the induction.

PROPOSITION F. *Let F, p, V, n, θ, r, Q be as in Theorem E. Let σ be an element of order p of the center of Q, and let ρ be the linear mapping defined by*

$$x^{\rho}=x^{\sigma}-x, \qquad x\in V.$$

Let

$$m_{2}=\dim V^{\rho^{p-1}}.$$

Then

$$|C_{Q}(V^{\rho^{p-1}})|\leqq p^{m_{2}},$$

where

$$C_{Q}(V^{\rho^{p-1}})=\{\tau\in Q \mid x^{\tau}=x \text{ for all } x\in V^{\rho^{p-1}}\}$$

by definition.

The organization of this paper is as follows. In Section 2, we collect several general results concerning multilinear mappings. In Section 3, we fix our notation. Sections 4 and 5 are devoted to the proof of Proposition F. We complete the proof of Theorem E in Section 6.

REMARK 1. A multilinear mapping for which the equality holds in Theorem B can easily be constructed as follows. Let $\{e_{j}\}_{1\leq j\leq n}$ be a base of V, and define θ by

$$\theta(e_{j_1}, e_{j_2}, e_{j_3}, \cdots, e_{j_r})=\begin{cases} e_{j_1}, & \text{if } j_1=j_2=j_3=\cdots=j_r, \\ 0, & \text{otherwise.}\end{cases}$$

Then $\mathrm{Aut}(\theta)$ contains a subgroup isomorphic to the symmetric group of degree n, whose Sylow p-subgroups attain the upper bound.

REMARK 2. If the characteristic of the underlying field is 0, then $Q=1$ under the hypotheses and the notation of Theorem E. This can be verified by the argument we shall use to prove Theorem E for the case $n<p$. (See the remark following Lemma 3.6.) Thus Theorem A is true also when the characteristic is 0. Those two cases, (i.e., the case $n<p$ and the case in which the characteristic is 0), were already settled in [**2**, Proposition 4], and we could have quoted some results from [**2**] to shorten our proof. But considering that our proof is elementary, we decided to arrange this article so that it could be read without any outside references, except for the proof of Propositions C and D, for which the reader is referred to [**2**], and possibly Krull-Remak-Schmidt's theorem (Lemma 2.3), which is well-known but may not be regarded as part of elementary linear algebra.

REMARK 3. In [**2**, Proposition 5], it was also shown that if θ is a "nonsingular" multilinear form of degree $r+1$ with $r\geqq 2$ on a vector space of dimension n over an algebraically closed field of characteristic p and if either $p=0$ or $n<p$, then $\mathrm{Aut}(\theta)$ contains no unipotent element except 1. On the other hand, it was proved by H. Matsumura - P. Monsky [**1**, Theorem 1] that the automorphism group of a nonsingular symmetric multilinear form of degree greater than or equal to 3 on a finite-dimensional vector space over an algebraically closed field is finite. In view of those results, we suspect that the "multilinear-form version" of Theorem B also holds. But we have no clue to this problem.

2. Multilinear mapping.

Let X be a vector space of finite dimension over an algebraically closed field F. If φ is an element of $\mathrm{GL}(X)$, we denote $x^{\varphi}-x$ by $[x, \varphi]$. If Y is a subspace of X, we denote by $[Y, \varphi]$ the subspace spanned by $\{[x, \varphi]\}$ where x ranges over Y. If H is a subgroup of $\mathrm{GL}(X)$, we denote by $[Y, H]$ the subspace spanned by $\{[Y, \varphi]\}$ where φ ranges over H.

Let θ be a multilinear mapping of degree r on X. We denote by θ_t, $1\leqq t\leqq r$, the multilinear mapping defined by

$$\theta_t(x_1, x_2, x_3, \cdots, x_r)=\theta(x_2, x_3, \cdots, x_t, x_1, x_{t+1}, \cdots, x_r).$$

For subspaces Y, Z of X, we define the subspace $\Theta(Y, Z)$ by

$$\Theta(Y, Z)=\langle\theta_t(x_1, x_2, x_3, \cdots, x_r) \mid 1\leqq t\leqq r,\ x_1\in Y,\ x_2, x_3, \cdots, x_r\in Z\rangle.$$

If $\Theta(Y, Z)\subseteqq Y$, then we take $\Theta^0(Y, Z)=Y$ by convention and define $\Theta^j(Y, Z)$, inductively, by

$$\Theta^{j}(Y, Z)=\Theta(\Theta^{j-1}(Y, Z), Z).$$

We have

$$\Theta^{0}(Y, Z)\supseteqq\Theta^{1}(Y, Z)\supseteqq\Theta^{2}(Y, Z)\supseteqq\Theta^{3}(Y, Z)\supseteqq\cdots.$$

We let

$$\Theta^{\infty}(Y, Z)=\bigcap_{j=0}^{\infty}\Theta^{j}(Y, Z).$$

We remark that Y need not be contained in Z in the above definition.

If subspaces Y, Z of X are such that $Y\subseteqq Z$ and $\Theta(Z, Z)\subseteqq Z$ and $\Theta(Y, Z)\subseteqq Y$, then we say that Y is an ideal of Z. If Y is an ideal of Z, then a linear mapping φ from Y into Z is called normal with respect to Z if it satisfies the condition

$$\theta_{t}(x_{1}^{\varphi}, x_{2}, x_{3}, \cdots, x_{r})=(\theta_{t}(x_{1}, x_{2}, x_{3}, \cdots, x_{r}))^{\varphi}$$

for all $x_{1}\in Y$ and $x_{2}, x_{3}, \cdots, x_{r}\in Z$ and for all $1\leqq t\leqq r$.

LEMMA 2.1. *Suppose θ is nonsingular. Let Y be an ideal of X, and let φ be a linear mapping from Y into Y which is normal with respect to X. Let α_{1}, $\alpha_{2}, \alpha_{3}, \cdots, \alpha_{s}\in F$ be the distinct eigenvalues of φ, and let*

$$U^{(k)}=\{x\in Y \mid x^{\varphi}=\alpha_{k}x\}, \qquad 1\leqq k\leqq s.$$

Then $Y=\bigoplus_{k=1}^{s}U^{(k)}$. Furthermore, each $U^{(k)}$ is an ideal of X.

PROOF. By way of contradiction, suppose $Y\neq\bigoplus_{k=1}^{s}U^{(k)}$. Then there exist nonzero elements x, y of Y and an element α of F such that $x^{\varphi}=\alpha x$ and $y^{\varphi}=\alpha y+x$. Then

$$\begin{aligned}\alpha\theta(x, y, x, x, \cdots, x)&=\theta(x, y, x, x, \cdots, x)^{\varphi}=\theta(x, y^{\varphi}, x, x, \cdots, x)\\&=\alpha\theta(x, y, x, x, \cdots, x)+\theta(x, x, x, x, \cdots, x).\end{aligned}$$

Namely, $\theta(x, x, x, \cdots, x)=0$, which contradicts the nonsingularity of θ. Thus $Y=\bigoplus_{k=1}^{s}U^{(k)}$. Now let $x\in U^{(k)}$. Then

$$\begin{aligned}\theta_{t}(x, x_{2}, x_{3}, \cdots, x_{r})^{\varphi}&=\theta_{t}(x^{\varphi}, x_{2}, x_{3}, \cdots, x_{r})\\&=\alpha_{k}\theta_{t}(x, x_{2}, x_{3}, \cdots, x_{r}).\end{aligned}$$

This means $\theta_{t}(x, x_{2}, x_{3}, \cdots, x_{r})\in U^{(k)}$. Since x, x_{j}, t were arbitrary, this shows $\Theta(U^{(k)}, X)\subseteqq U^{(k)}$.

In order to state our next lemma, we need the following notation. For a subspace Y which satisfies $\Theta(Y, Y)\subseteqq Y$, we let $\Theta^{0}(Y)=Y$, and define $\Theta^{j}(Y)$, inductively, by

$$\Theta^{j}(Y)=\Theta(\Theta^{j-1}(Y), \Theta^{j-1}(Y)).$$

We have

$$\Theta^{0}(Y)\supseteqq\Theta^{1}(Y)\supseteqq\Theta^{2}(Y)\supseteqq\Theta^{3}(Y)\supseteqq\cdots.$$

We let

$$\Theta^{\infty}(Y)=\bigcap_{j=1}^{\infty}\Theta^{j}(Y).$$

Clearly $\Theta^{\infty}(Y)\subseteqq\Theta^{\infty}(Y, Y)$, and the inclusion is in general proper. Those two notations, $\Theta^{\infty}(Y)$ and $\Theta^{\infty}(Y, Y)$, may appear confusing. But the notation $\Theta^{\infty}(Y)$ is used only when we apply the following lemma, and that will exclude any risk of confusion in practice.

LEMMA 2.2. *Let Y be an ideal of X, and let φ be a linear mapping from Y into X which is normal with respect to X. Suppose that $(\Theta^{\infty}(Y))^{\varphi}=0$ and that the restriction of θ to Y^{φ} is nonsingular. Then $Y^{\varphi}=0$.*

PROOF. Suppose there exists $x\in Y$ such that $x^{\varphi}\neq 0$. Let $x_0=x$, and let

$$x_{j+1}=\theta(x_j, x_j, x_j, \cdots, x_j), \qquad j=0, 1, 2, \cdots.$$

There exists some j_0 such that $x_j^{\varphi}=0$ for all $j>j_0$ and such that $x_{j_0}^{\varphi}\neq 0$. Thus

$$\theta(x_{j_0}^{\varphi}, x_{j_0}^{\varphi}, x_{j_0}^{\varphi}, \cdots, x_{j_0}^{\varphi})=(x_{j_0+1})^{\varphi^r}=0.$$

This contradicts the nonsingularity of the restriction of θ to Y^{φ}.

An ideal Y of X is called indecomposable, if it cannot be expressed as the direct sum of two nontrivial ideals of X.

LEMMA 2.3 (Krull-Remak-Schmidt). *Let $X=\bigoplus_{j=1}^{c}X^{(j)}$ and $X=\bigoplus_{j=1}^{d}Y^{(j)}$ be direct sum decompositions in which each $X^{(j)}$ and each $Y^{(j)}$ are indecomposable ideals. Then $c=d$ and there exists $\varphi\in\mathrm{GL}(X)$ which is normal with respect to X such that, for some permutation π of $\{1, 2, 3, \cdots, d\}$, $(X^{(j)})^{\varphi}=Y^{(j\pi)}$ for each j.*

PROOF. We view X as an additive group with an operation set, where the operation set consists of the natural action of F and the action of type

$$\theta_t(\cdot, x_2, x_3, \cdots, x_r).$$

Here $x_2, x_3, \cdots, x_r$ range over X, and t ranges over the interval $1\leqq t\leqq r$. From this viewpoint, the lemma is nothing but Krull-Remak-Schmidt's theorem.

LEMMA 2.4. *Let $\bigoplus_{j=1}^{d}X^{(j)}$ and $\bigoplus_{j=1}^{d}Y^{(j)}$ be as in Lemma 2.3. Then $\Theta(X^{(j)}, X)=\Theta(Y^{(j\pi)}, X)$ for each j, where π is as in Lemma 2.3.*

PROOF. First note that $\Theta(X^{(j)}, X)=\Theta(X^{(j)}, X^{(j)})$. Then, with φ as in Lemma 2.3,

$$\begin{aligned}\Theta(Y^{(j\pi)}, X)&=\Theta(X^{(j)}, X)^{\varphi}=\Theta(X^{(j)}, X^{(j)})^{\varphi}\\&=\Theta((X^{(j)})^{\varphi}, X^{(j)})\subseteqq\Theta(X, X^{(j)})\subseteqq\Theta(X^{(j)}, X).\end{aligned}$$

Since this inclusion holds for each j, and since

$$\bigoplus_{j=1}^{d}\Theta(X^{(j)}, X)=\Theta(X, X)=\bigoplus_{j=1}^{d}\Theta(Y^{(j)}, X),$$

the desired conclusion holds.

LEMMA 2.5. *Let* $X=\bigoplus_{j=1}^{d} X^{(j)}$ *be as in Lemma* 2.3, *and let* $X=\bigoplus_{k=1}^{s} U^{(k)}$ *be a direct sum decomposition in which each* $U^{(k)}$ *is an ideal. Then there exists a partition of* $\{1, 2, 3, \cdots, d\}$ *into* s *disjoint nonempty subsets* I_k, $1 \leqq k \leqq s$, *such that* $\Theta(U^{(k)}, X)=\bigoplus_{j \in I_k} \Theta(X^{(j)}, X)$ *for each* k.

PROOF. Since $\bigoplus_{k=1}^{s} U^{(k)}$ can further be decomposed into a direct sum of indecomposable ideals, the lemma follows immediately from Lemma 2.4.

3. Notation.

In the remainder of this paper, let F, p, V, n, θ, Q, σ be as in Proposition F. We use the notation and the terminology defined in Section 2. Furthermore we introduce the following notation.

Let ρ be the linear mapping defined by

$$x^{\rho}=[x, \sigma], \qquad x \in V.$$

Let

$$Z_h=\operatorname{Ker}(\rho^h), \qquad 0 \leqq h \leqq p. \tag{3.1}$$

In particular,

$$Z_0=0, \qquad Z_p=V.$$

The following two lemmas are easily verified by induction on h.

LEMMA 3.1. *If* $w \in V$ *and* $x_2, x_3, \cdots, x_r \in Z_1$, *then*

$$\theta_t(w, x_2, x_3, \cdots, x_r)^{\rho^h}=\theta_t(w^{\rho^h}, x_2, x_3, \cdots, x_r), \qquad h \geqq 0, \quad 1 \leqq t \leqq r.$$

LEMMA 3.2. *If* $y \in V$, $z \in Z_2$, *and* $x_3, x_4, \cdots, x_r \in Z_1$, *then*

$$\begin{aligned}\theta_t(y, z, x_3, \cdots, x_r)^{\rho^h}=\theta_t(y^{\rho^h}, z, x_3, \cdots, x_r)+h(\theta_t(y^{\rho^{h-1}}, z^{\rho}, x_3, \cdots, x_r)\\ +\theta_t(y^{\rho^h}, z^{\rho}, x_3, \cdots, x_r)), \qquad h \geqq 1, \quad 1 \leqq t \leqq r.\end{aligned}$$

Now let

$$W=V^{\rho} \cap Z_1.$$

The following facts will be used throughout the paper.

LEMMA 3.3.

(i) $$\Theta(Z_h, W) \subseteqq Z_h, \qquad 0 \leqq h \leqq p.$$

(ii) $$\Theta(V^{\rho^i}, W) \subseteqq V^{\rho^i}, \qquad 0 \leqq i \leqq p.$$

We shall prove a few more results concerning $\Theta(\cdot, W)$.

LEMMA 3.4.

$$\Theta(Z_1 \cap V^{\rho^i}, W) \subseteqq Z_1 \cap V^{\rho^{i+1}}, \qquad 0 \leqq i \leqq p-2.$$

PROOF. Let $x_1 \in Z_1 \cap V^{\rho^i}$, $x_2, x_3, \cdots, x_r \in W$. Let y, z be such that $y^{\rho^i} = x_1$, $z^{\rho} = x_2$. Since $y^{\rho^{i+1}} = 0$,

$$\theta_l(y, z, x_3, \cdots, x_r)^{\rho^{i+1}} = (i+1)\theta_l(x_1, x_2, x_3, \cdots, x_r)$$

by Lemma 3.2. Since $i+1 \neq 0$ in F, this means $\theta_l(x_1, x_2, x_3, \cdots, x_r) \in Z_1 \cap V^{\rho^{i+1}}$ by Lemma 3.3 (i). Since $x_1, x_2, x_3, \cdots, x_r$ were arbitrary, this proves the lemma.

LEMMA 3.5.

$$\Theta(Z_l \cap V^{\rho^i}, W) \subseteqq (Z_l \cap V^{\rho^{i+1}}) + (Z_{l-1} \cap V^{\rho^i}), \qquad 0 \leqq i \leqq p-2, \quad 1 \leqq l \leqq p-i-1.$$

PROOF. By Lemma 3.1,

$$\begin{aligned}\Theta(Z_l \cap V^{\rho^i}, W)^{\rho^{l-1}} &= \Theta((Z_l \cap V^{\rho^i})^{\rho^{l-1}}, W) \\ &= \Theta(Z_1 \cap V^{\rho^{i+l-1}}, W).\end{aligned}$$

Hence, by Lemma 3.4,

$$\Theta(Z_l \cap V^{\rho^i}, W)^{\rho^{l-1}} \subseteqq Z_1 \cap V^{\rho^{i+l}}.$$

Since the full inverse image of $Z_1 \cap V^{\rho^{i+l}}$ by ρ^{l-1} is $(Z_l \cap V^{\rho^{i+1}}) + Z_{l-1}$, we get

$$\Theta(Z_l \cap V^{\rho^i}, W) \subseteqq (Z_l \cap V^{\rho^{i+1}}) + Z_{l-1}.$$

Since $\Theta(Z_l \cap V^{\rho^i}, W) \subseteqq Z_l \cap V^{\rho^i}$ by Lemma 3.3, this implies the desired conclusion.

LEMMA 3.6.

$$\Theta^\infty(Z_h, W) \subseteqq V^{\rho^{p-h}}, \qquad 1 \leqq h \leqq p.$$

PROOF. Applying Lemma 3.5 with $i=0, 1, 2, \cdots, p-l-1$, successively, we get

$$\Theta^\infty(Z_l, W) \subseteqq V^{\rho^{p-l}} + Z_{l-1}, \qquad 1 \leqq l \leqq p. \tag{3.2}$$

Applying (3.2) with $l=h, h-1, \cdots, 1$, successively, we get the desired conclusion.

REMARK. If we let $h=1$ in the above lemma, we get $\Theta^\infty(Z_1, W) \subseteqq V^{\rho^{p-1}}$. From this and from the nonsingularity of the restriction of θ to W, it follows that $V^{\rho^{p-1}} \neq 0$, which implies that $n \geqq p$. This means that if $n < p$, then $Q=1$. Thus Proposition F and Theorems E, B, A are proved for $n < p$. Therefore we henceforth assume $n \geqq p$.

Now let

$$Y_h = \Theta^\infty(V^{\rho^h}, W), \qquad 0 \leqq h \leqq p. \tag{3.3}$$

We have

$$Y_h^{\rho} = Y_{h+1} \subseteqq Y_h, \qquad 0 \leqq h \leqq p-1. \tag{3.4}$$

Also we have

$$Y_h = \Theta^\infty(Z_{p-h}, W), \qquad 0 \leqq h \leqq p, \tag{3.5}$$

by Lemma 3.6.

Let

$$W_{1,i}=Z_1\cap Y_{p-i}\,, \qquad 1\leqq i\leqq p. \tag{3.6}$$

Now we want to choose $W_{h,i}$, $1\leqq h\leqq p$, $1\leqq i\leqq p+1-h$, so that the following three conditions (3.7), (3.8), (3.9) are satisfied:

$$W_{h,i}\supseteqq W_{h,i-1}\,; \tag{3.7}$$

$$(W_{h,i})^{\rho}=W_{h-1,i}\,; \tag{3.8}$$

$$W_{h,i} \text{ is a complement to } Y_{p-i+1-h}\cap Z_{h-1} \text{ in } Y_{p-i+1-h}\cap Z_h\,. \tag{3.9}$$

In order to do this, we assume that the subspaces $W_{h,i}$ for h with $1\leqq h\leqq l-1$ are already chosen. For each i with $1\leqq i\leqq p+1-l$, we have

$$Y_{p-i+1-l}\cap Z_l\subseteqq (W_{l-1,i})^{\rho^{-1}}+(Y_{p-i+1-l}\cap Z_{l-1})\,, \tag{3.10}$$

for (3.9) holds for $h=l-1$ and

$$(Y_{p-i+1-l})^{\rho}=Y_{p-i+2-l}\,.$$

(In this paragraph, we use the symbol ρ^{-1} to denote the full inverse image, whereas we shall find it convenient to use ρ^{-1} in a different manner later. See the paragraph immediately before Lemma 3.8.) Now we choose $W_{l,i}$, $i=1, 2, \cdots, p+1-l$, successively, as a complement to

$$(W_{l-1,i})^{\rho^{-1}}\cap Y_{p-i+1-l}\cap Z_{l-1}$$

in

$$(W_{l-1,i})^{\rho^{-1}}\cap Y_{p-i+1-l}$$

containing $W_{l,i-1}$. Then, by (3.10), (3.9) holds for $h=l$. Since

$$((W_{l-1,i})^{\rho^{-1}}\cap Y_{p-i+1-l})^{\rho}=W_{l-1,i}$$

and

$$((W_{l-1,i})^{\rho^{-1}}\cap Y_{p-i+1-l}\cap Z_{l-1})^{\rho}=0\,,$$

we have $(W_{l,i})^{\rho}=W_{l-1,i}$. Thus all of (3.7), (3.8), (3.9) hold for $h=l$.

Of course, there is more than one way to choose $W_{h,i}$ so that (3.7), (3.8), and (3.9) hold. We shall define a "convenient" choice in Lemma 4.11.

LEMMA 3.7. *$W_{1,i}$ is an ideal of W, $1\leqq i\leqq p$.*

PROOF. Since $\Theta(Y_{p-i}, W)=Y_{p-i}$ by the definition (3.3) and since $\Theta(Z_1, W)\subseteqq Z_1$, the lemma follows from the definition (3.6) of $W_{1,i}$.

We choose a complement V_h to Z_{h-1} in Z_h so that $V_h\supseteqq W_{h,p-h+1}$ and $V_h^{\rho}\subseteqq V_{h-1}$, $1\leqq h\leqq p$. We take $V_0=0$.

By ρ^{1-h}, $2\leqq h\leqq p$, we shall mean the inverse mapping of the bijection ρ^{h-1} from V_h onto $V_h^{\rho^{h-1}}$, unless otherwise stated. Using this notation, we can restate Lemmas 3.1 and 3.2 in the following form.

LEMMA 3.8. *Let* $1 \leqq i \leqq p-1$, $1 \leqq t \leqq r$. *Let* $x_1 \in Z_1 \cap V^{\rho^i}$, $x_2, x_3, \cdots, x_r \in Z_1$. *For each* m *with* $1 \leqq m \leqq i+1$, *write*

$$\theta_t(x_1^{\rho^{1-m}}, x_2, x_3, \cdots, x_r) = \sum_{l=1}^{m} a_{m,l}, \qquad a_{m,l} \in V_l .$$

Then

$$\text{(i)} \qquad a_{m,l} = (a_{m-l+1,1})^{\rho^{1-l}}.$$

Furthermore, suppose $x_2 \in W$, *and, for each* $1 \leqq m \leqq i+1$, *write*

$$\theta_t(x_1^{\rho^{1-m}}, x_2^{\rho^{-1}}, x_3, \cdots, x_r) = \sum_{l=1}^{m+1} b_{m,l}, \qquad b_{m,l} \in V_l .$$

(*When* $i=p-1$, *we take* $b_{p,p+1}=0$.) *Then*

$$\text{(ii)} \qquad b_{m,l} = (l-1)(a_{m-l+2,1} + a_{m-l+1,1})^{\rho^{1-l}} + (b_{m-l+1,1})^{\rho^{1-l}}.$$

(*We take* $a_{0,1} = b_{0,1} = a_{i+2,1} = 0$.)

4. Description of an element of $C_Q(Z_1)$.

Let τ be an arbitrary element of order p of $C_Q(Z_1)$. In this section, we shall give a somewhat explicit description of τ (Lemmas 4.12, 4.13).

LEMMA 4.1. $\qquad [Z_l, \tau] \subseteqq Z_{l-1}, \qquad 1 \leqq l \leqq p.$

PROOF. Since $\sigma\tau = \tau\sigma$, $\rho\tau = \tau\rho$. Consequently,

$$[Z_l, \tau]^{\rho^{l-1}} = [Z_l^{\rho^{l-1}}, \tau] = [Z_1, \tau] = 0,$$

as $\tau \in C_Q(Z_1)$. Therefore $[Z_l, \tau] \subseteqq Z_{l-1}$ by (3.1).

LEMMA 4.2. $\qquad [Y_h, \tau] \subseteqq Y_{h+1}, \qquad 0 \leqq h \leqq p-1.$

PROOF. Since $\tau \in C_Q(Z_1) \subseteqq C_Q(W)$, the lemma follows from (3.5) and Lemma 4.1.

LEMMA 4.3. $\qquad [W_{2,i}, \tau] \subseteqq W_{1,i}, \qquad 1 \leqq i \leqq p-1.$

PROOF. By Lemma 4.2 for $h=p-i-1$ and by Lemma 4.1 for $l=2$,

$$[Y_{p-i-1} \cap Z_2, \tau] \subseteqq Y_{p-i} \cap Z_1 .$$

Since $W_{2,i} \subseteqq Y_{p-i-1} \cap Z_2$ and $W_{1,i} = Y_{p-i} \cap Z_1$ by (3.9) for $h=2$ and by (3.6), respectively, this proves the lemma.

LEMMA 4.4. *There exist direct sum decompositions*

$$W_{h,i} = \bigoplus_{k=1}^{s} U_{h,i}^{(k)}, \qquad h=1, 2, \quad 1 \leqq i \leqq p-1,$$

such that

(i) $$(U_{2,i}^{(k)})^{\rho}=U_{1,i}^{(k)},$$

and such that

(ii) $$[x, \tau]=\alpha_{1,k}x^{\rho} \quad \textit{for all} \quad x\in U_{2,i}^{(k)},$$

where $\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \cdots, \alpha_{1,s}$ *are distinct elements of the field* F *which do not depend on* i. *Furthermore,*

(iii) $$U_{h,i}^{(k)}\supseteqq U_{h,i-1}^{(k)},$$

(iv) $$\Theta(U_{1,i}^{(k)}, W)\subseteqq U_{1,i}^{(k)},$$

(v) $$U_{1,1}^{(k)}=\Theta^{\infty}(U_{1,i}^{(k)}, W)\neq 0.$$

PROOF. For each i, $1\leqq i\leqq p-1$, let φ_i be the linear mapping on $W_{1,i}$ defined by

$$x^{\varphi_i}=[x^{\rho^{-1}}, \tau], \qquad x\in W_{1,i}.$$

Lemma 4.3 shows $(W_{1,i})^{\varphi_i}\subseteqq W_{1,i}$. Let $x_1\in W_{1,i}$, $x_2, x_3, \cdots, x_r\in W$, and let $a_{m,l}$ be as in Lemma 3.8. Then

$$\begin{aligned}\theta_t([x_1^{\rho^{-1}}, \tau], x_2, x_3, \cdots, x_r)&=[\theta_t(x_1^{\rho^{-1}}, x_2, x_3, \cdots, x_r), \tau]\\ &=[(a_{1,1})^{\rho^{-1}}+a_{2,1}, \tau]=[(a_{1,1})^{\rho^{-1}}, \tau]=\theta_t(x_1, x_2, x_3, \cdots, x_r)^{\varphi_i}.\end{aligned}$$

Since $x_1, x_2, x_3, \cdots, x_r$ were arbitrary, this means that φ_i is normal with respect to W. Now we apply Lemma 2.1 with $X=W$, $Y=W_{1,i}$ and $\varphi=\varphi_i$. Let $\alpha_{1,k}$, $1\leqq k\leqq s$, be the eigenvalues of φ_{p-1}. If we let $\bigoplus_{k=1}^{s}U_{1,i}^{(k)}$ be the decomposition described in Lemma 2.1 (At this stage $U_{1,i}^{(k)}$ may be trivial for some k, which we shall see is not the case by (v).), and if we let $U_{2,i}^{(k)}=(U_{1,i}^{(k)})^{\rho^{-1}}$, then (i), (ii) and (iv) hold. The property (ii) immediately implies (iii). By Lemma 3.6 for $h=1$, $\Theta^{\infty}(W_{1,i}, W)=W_{1,1}$. Hence

$$\bigoplus_{k=1}^{s}\Theta^{\infty}(U_{1,i}^{(k)}, W)=\Theta^{\infty}(W_{1,i}, W)=W_{1,1}$$

by (iv). Hence $\Theta^{\infty}(U_{1,i}^{(k)}, W)=U_{1,1}^{(k)}$. Finally, since the restriction of θ to $U_{1,i}^{(k)}$ $(\subseteqq W)$ is nonsingular, $\Theta^{\infty}(U_{1,i}^{(k)}, W)\neq 0$.

LEMMA 4.5. *For each* k,

$$\Theta(W_{1,1}, W_{1,1})\cap U_{1,1}^{(k)}=\Theta(U_{1,1}^{(k)}, W_{1,1})\neq 0.$$

PROOF. This is because of the nonsingularity of the restriction of θ to $W_{1,1}\subseteqq W$.

For each $1\leqq k\leqq s$, we define $\alpha_{h,k}$, inductively, by

$$\alpha_{h,k}=\frac{\alpha_{h-1,k}(\alpha_{1,k}-(h-1))}{h}, \qquad 2\leqq h\leqq p-1. \tag{4.1}$$

By convention, we take

$$\alpha_{0,k}=1 \tag{4.2}$$

for all k.

For each $1\leqq l\leqq p-1$ and for each $1\leqq k\leqq s$, let $\varphi_{l,k}$ be the linear mapping defined by

$$x^{\varphi_{l,k}}=x^{\tau}-x-\alpha_{1,k}x^{\rho}-\alpha_{2,k}x^{\rho^2}-\cdots-\alpha_{l,k}x^{\rho^l}, \qquad x\in V.$$

The following lemma is an immediate corollary to Lemma 4.4.

LEMMA 4.6. *If an element w of $W_{2,p-1}$ satisfies the condition*

$$w^{\varphi_{1,k}}\in U_{1,p-1}^{(k)}$$

for some k, then $w\in U_{2,p-1}^{(k)}$ and so $w^{\varphi_{1,k}}=0$.

We next prove several technical results concerning $\varphi_{l,k}$.

LEMMA 4.7. $$\rho\varphi_{l,k}=\varphi_{l,k}\rho\,.$$

PROOF. This follows from the fact $\rho\tau=\tau\rho$.

LEMMA 4.8. *If $w\in Z_{l+1}$, then*

$$w^{\varphi_{l,k}}=w^{\varphi_{l+1,k}}=w^{\varphi_{l+2,k}}=\cdots=w^{\varphi_{p-1,k}}\,.$$

PROOF. This is because $(Z_{l+1})^{\rho^{l+1}}=0$.

LEMMA 4.9. *Let $2\leqq v\leqq p-1$, $1\leqq k\leqq s$, $1\leqq t\leqq r$. Let $w\in V$, $x_2, x_3, \cdots, x_r\in Z_1$. Then*

$$\theta_t(w, x_2, x_3, \cdots, x_r)^{\varphi_{v,k}}=\theta_t(w^{\varphi_{v,k}}, x_2, x_3, \cdots, x_r)\,.$$

PROOF. This follows from Lemma 3.1.

LEMMA 4.10. *Let $2\leqq v\leqq p$, $1\leqq k\leqq s$, $1\leqq t\leqq r$. Let $y\in Z_v$, $z\in U_{2,p-1}^{(k)}$, $x_3, x_4, \cdots, x_r\in Z_1$. Then the following hold:*

(i) *If $2\leqq v\leqq p-1$, then*

$$\begin{aligned}&\theta_t(y, z, x_3, \cdots, x_r)^{\varphi_{v,k}}\\&=\theta_t(y^{\varphi_{v-1,k}}, z, x_3, \cdots, x_r)+\alpha_{1,k}\theta_t(y^{\varphi_{v-1,k}}, z^{\rho}, x_3, \cdots, x_r)\,.\end{aligned}$$

(ii) *For $v=p$, we have*

$$\begin{aligned}&\theta_t(y, z, x_3, \cdots, x_r)^{\varphi_{p-1,k}}\\&=\theta_t(y^{\varphi_{p-1,k}}, z, x_3, \cdots, x_r)+\alpha_{1,k}\theta_t(y^{\varphi_{p-1,k}}, z^{\rho}, x_3, \cdots, x_r)\\&\quad+((p-1)\alpha_{p-1,k}-\alpha_{1,k}\alpha_{p-1,k})\theta_t(y^{\rho^{p-1}}, z^{\rho}, x_3, \cdots, x_r)\,.\end{aligned}$$

PROOF. First suppose $2\leqq v\leqq p-1$. We have

$$\begin{aligned}&\theta_t(y, z, x_3, \cdots, x_r)^{\varphi_{v,k}}\\&=\theta_t(y^{\tau}, (z+\alpha_{1,k}z^{\rho}), x_3, \cdots, x_r)-\theta_t(y, z, x_3, \cdots, x_r)\end{aligned}$$

$$-\sum_{h=1}^{v}\alpha_{h,k}\theta_t(y^{\rho^h}, z, x_3, \cdots, x_r)$$

$$-\sum_{h=1}^{v}\alpha_{h,k}h\theta_t(y^{\rho^{h-1}}, z^\rho, x_3, \cdots, x_r)$$

$$-\sum_{h=1}^{v}\alpha_{h,k}h\theta_t(y^{\rho^h}, z^\rho, x_3, \cdots, x_r) \qquad \text{(by Lemmas 3.1 and 4.4 (ii))}$$

$$=\theta_t(y^\tau, z, x_3, \cdots, x_r)-\theta_t(y, z, x_3, \cdots, x_r)$$

$$-\sum_{h=1}^{v-1}\alpha_{h,k}\theta_t(y^{\rho^h}, z, x_3, \cdots, x_r)$$

$$+\alpha_{1,k}\theta_t(y^\tau, z^\rho, x_3, \cdots, x_r)$$

$$-\sum_{h=0}^{v-1}\alpha_{h+1,k}(h+1)\theta_t(y^{\rho^h}, z^\rho, x_3, \cdots, x_r)$$

$$-\sum_{h=0}^{v-1}\alpha_{h,k}h\theta_t(y^{\rho^h}, z^\rho, x_3, \cdots, x_r) \qquad (\text{for } y^{\rho^v}=0)$$

$$=\theta_t(y^{\varphi_{v-1,k}}, z, x_3, \cdots, x_r)+\alpha_{1,k}\theta_t(y^\tau, z^\rho, x_3, \cdots, x_r)$$

$$-\sum_{h=0}^{v-1}((h+1)\alpha_{h+1,k}+h\alpha_{h,k})\theta_t(y^{\rho^h}, z^\rho, x_3, \cdots, x_r).$$

Since $(h+1)\alpha_{h+1,k}+h\alpha_{h,k}=\alpha_{1,k}\alpha_{h,k}$ for $0\leqq h\leqq p-2$ by (4.1) and (4.2), this proves (i). For $v=p$, the above calculation is valid if we interpret $\varphi_{p,k}$ as $\varphi_{p-1,k}$, except that in the last sum the coefficient for

$$\theta_t(x^{\rho^{p-1}}, y^\rho, x_3, \cdots, x_r)$$

is $(p-1)\alpha_{p-1,k}$. Thus (ii) is proved.

Now let

$$U_{h,i}^{(k)}=(U_{1,i}^{(k)})^{\rho^{1-h}}, \qquad 1\leqq k\leqq s, \quad 2\leqq h\leqq p, \quad 1\leqq i\leqq p-h+1. \tag{4.3}$$

If $h=2$, this coincides with the description of $U_{2,i}^{(k)}$ in Lemma 4.4. Since

$$W_{1,i}=\bigoplus_{k=1}^{s}U_{1,i}^{(k)},$$

we have

$$W_{h,i}=\bigoplus_{k=1}^{s}U_{h,i}^{(k)}. \tag{4.4}$$

Of course, $U_{h,i}^{(k)}$ depends on ρ^{1-h}, which depends on the choice of $W_{h,i}$. In fact, a suitable choice of $W_{h,i}$ enables $U_{h,i}^{(k)}$ to have the following nice property.

LEMMA 4.11. *The subspaces $W_{h,i}$ can be chosen so that if we define $U_{h,i}^{(k)}$ by* (4.3), *then the inclusion*

$$(U_{h,i}^{(k)})^{\varphi_{1,k}} \subseteqq \bigoplus_{l=1}^{h-2} U_{l,h+i-1-l}^{(k)}$$

holds for all h, i with $2 \leqq h \leqq p+1-i$ and for all k.

PROOF. We prove by induction on v that $W_{h,i}$ can be chosen so that

$$(U_{v,m}^{(k)})^{\varphi_{1,k}} \subseteqq \bigoplus_{l=1}^{v-2} U_{l,v+m-1-l}^{(k)} \tag{4.5}$$

for all $m \leqq p+1-v$ and for all k. This certainly holds for $v=2$ by Lemma 4.4 (ii). Now suppose that (4.5) holds for $v \leqq t-1$. For each k, let

$$n_{i,k} = \dim U_{1,i}^{(k)}, \qquad 1 \leqq i \leqq p-1,$$

and let

$$\{e_j^{(k)}\}_{1 \leq j \leq n_{p+1-t,k}}$$

be a base of $U_{t,p+1-t}^{(k)}$ such that

$$\{e_j^{(k)}\}_{1 \leq j \leq n_{m,k}}$$

is a base of $U_{t,m}^{(k)}$ for $1 \leqq m \leqq p+1-t$. Let

$$x \in \{e_j^{(k)}\}_{n_{m-1,k}+1 \leq j \leq n_{m,k}}.$$

By (4.3), $x^{\rho} \in U_{t-1,m}^{(k)}$. Hence, by the inductive hypothesis and Lemma 4.7,

$$(x^{\varphi_{1,k}})^{\rho} = (x^{\rho})^{\varphi_{1,k}} \in \bigoplus_{l=1}^{t-3} U_{l,t+m-2-l}^{(k)}.$$

Therefore we may write $x^{\varphi_{1,k}} = w+z$, where

$$w \in \bigoplus_{l=1}^{t-2} U_{l,t+m-1-l}^{(k)} \quad \text{and} \quad z \in Z_1.$$

Since $x \in U_{t,m}^{(k)} \subseteqq W_{t,m} \subseteqq Y_{p+1-t-m}$ by (4.4) and (3.9),

$$x^{\varphi_{1,k}} = [x, \tau] - \alpha_{1,k} x^{\rho} \in Y_{p+2-t-m}$$

by (3.4) and Lemma 4.2. Also

$$w \in \bigoplus_{l=1}^{t-2} W_{l,t+m-1-l} \subseteqq Y_{p+2-t-m}$$

by (4.4) and (3.9). Consequently

$$z \in Z_1 \cap Y_{p+2-t-m} = W_{1,t+m-2}.$$

Therefore we may write

$$z = w' + z',$$

where

$$w' \in U_{1,t+m-2}^{(k)} \quad \text{and} \quad z' \in \bigoplus_{u \neq k} U_{1,t+m-2}^{(u)}.$$

Hence, replacing w by $w+w'$ and z by z', we may assume

$$z\in\bigoplus_{u\neq k}U_{1,t+m-2}^{(u)}.$$

Since $2+(t+m-2)\leqq p+1$, $W_{2,t+m-2}$ is well-defined. Also note that $\varphi_{1,k}$ is a bijection from $\bigoplus_{u\neq k}U_{2,t+m-2}^{(u)}$ onto $\bigoplus_{u\neq k}U_{1,t+m-2}^{(u)}$, for $\alpha_{1,u}\neq\alpha_{1,k}$, $u\neq k$. Hence there exists $y\in U_{2,t+m-2}^{(u)}$ such that

$$z=y^{\varphi_{1,k}}.$$

If we let $x'=x-y$, then

$$x'\in Y_{p+1-t-m} \tag{4.6}$$

and

$$(x')^{\varphi_{1,k}}=w. \tag{4.7}$$

Since m, k were arbitrary and x was also arbitrary, we may replace all $e_j^{(k)}$ by $(e_j^{(k)})'$ following the above procedure. Let

$$(U_{t,i}^{(k)})'=\langle(e_j^{(k)})'\rangle_{1\leq j\leq n_{i,k}}$$

and

$$W'_{t,i}=\bigoplus_{k=1}^{s}(U_{t,i}^{(k)})'.$$

Then, by (4.6), the $W'_{t,i}$ satisfy (3.9) for $h=t$. Hence if we let

$$(U_{t-1,i}^{(k)})'=(U_{t,i}^{(k)})'^{\rho},\qquad 1\leqq i\leqq p+1-t,$$

and

$$(U_{t-1,p+2-t}^{(k)})'=(U_{t-1,p+1-t}^{(k)})'\oplus X_k,$$

where X_k is a complement to $U_{t-1,p+1-t}^{(k)}$ in $U_{t-1,p+2-t}^{(k)}$, and let

$$W'_{t-1,i}=\bigoplus_{k=1}^{s}(U_{t-1,i}^{(k)})',\qquad 1\leqq i\leqq p+2-t,$$

and if we simply let

$$(U_{h,i}^{(k)})'=U_{h,i}^{(k)}\quad\text{and}\quad W'_{h,i}=W_{h,i},\qquad 1\leqq m\leqq p+1-h,$$

for $h\leqq t-2$, then the $W'_{h,i}$ satisfy (3.7), (3.8), (3.9) for $h\leqq t$, and the $(U_{h,i}^{(k)})'$ satisfy (4.5) for $v=t$ by (4.7). Of course (4.3) is also satisfied for $h\leqq t$. Also note that this "replacement" does not hurt the property that (4.5) holds for $v\leqq t-1$. Now the only thing we have to do is to choose $W'_{h,i}$ for $h\geqq t+1$ following the method described in Section 3.

In the remainder of this section, we assume that the $W_{h,i}$ are chosen so that the conclusion of Lemma 4.11 holds.

LEMMA 4.12. (i) *Let* $2\leqq h\leqq p$. *Then* $(U_{h,p-h+1}^{(k)})^{\varphi_{h-1,k}}=0$ *for all* k.
(ii) *Let* $1\leqq h\leqq p-1$. *Then*

$$\Theta(U_{h,p-h}^{(k)},W)\subseteq\bigoplus_{l=1}^{h}U_{l,p-l}^{(k)}$$

for all k.

(iii) *Let* $2\leqq h\leqq p$. *Then if* $x\in Z_h$ *satisfies the condition* $x^{\varphi_{h-1,k}}\in U_{1,p-1}^{(k)}$ *for some* k, *then* $x^{\varphi_{h-1,k}}=0$.

PROOF. We proceed by induction on h. Statement (ii) for $h=1$ follows from Lemma 4.4 (iv), and (i) for $h=2$ follows from Lemma 4.4 (ii). Now assume that (i) is proved for $2\leqq h\leqq v$, and that (ii) and (iii) are proved for $1\leqq h\leqq v-1$.

First we prove (iii) for $h=v$. Let

$$A=\{x\in Z_v \mid x^{\varphi_{v-1,k}}\in U_{1,p-1}^{(k)}\}.$$

By Lemma 4.9 and Lemma 4.4 (iv),

$$\Theta(A, W)\subseteqq A. \tag{4.8}$$

Let y be an element of $A\cap Z_{v-1}$. By Lemma 4.8, $y^{\varphi_{v-2,k}}=0$. By (iii) for $h=v-1$, $y^{\varphi_{v-2,k}}=0$ and so $y^{\varphi_{v-1,k}}=0$. Although the induction breaks down when $v=2$, yet we can still prove $y^{\varphi_{v-1,k}}=0$ as $(Z_1)^{\varphi_{1,k}}=0$. Thus $(A\cap Z_{v-1})^{\varphi_{v-1,k}}=0$, which means that we may regard $\varphi_{v-1,k}$ as a mapping from $A/(A\cap Z_{v-1})$ into $U_{1,p-1}^{(k)}$. We regard ρ^{v-1} as a bijection from $A/(A\cap Z_{v-1})$ onto $A^{\rho^{v-1}}$ and denote its inverse by ρ^*. We know that $A^{\rho^{v-1}}$ is an ideal of W by (4.8) and by Lemma 4.9. Now let $x_1\in A^{\rho^{v-1}}$, $x_2, x_3, \cdots, x_r\in W$. Note that

$$\theta_t(x_1^{\rho^*}, x_2, x_3, \cdots, x_r)$$

is well-defined as an element of $A/(A\cap Z_{v-1})$, for $x_1^{\rho^*}\subseteqq x_1^{\rho^{1-v}}+Z_{v-1}$ and $\Theta(A\cap Z_{v-1}, W)\subseteqq A\cap Z_{v-1}$ by (4.8). Also

$$\theta_t(x_1^{\rho^*}, x_2, x_3, \cdots, x_r)^{\rho^{v-1}}=\theta_t(x_1, x_2, x_3, \cdots, x_r)$$

by Lemma 3.1, and so

$$\theta_t(x_1^{\rho^*}, x_2, x_3, \cdots, x_r)=\theta_t(x_1, x_2, x_3, \cdots, x_r)^{\rho^*}.$$

Hence

$$\begin{aligned}&\theta_t(x_1^{\rho^*\varphi_{v-1,k}}, x_2, x_3, \cdots, x_r)\\&=\theta_t(x_1^{\rho^*}, x_2, x_3, \cdots, x_r)^{\varphi_{v-1,k}} \qquad \text{(by Lemma 4.9)}\\&=\theta_t(x_1, x_2, x_3, \cdots, x_r)^{\rho^*\varphi_{v-1,k}}.\end{aligned}$$

Since $x_1, x_2, x_3, \cdots, x_r$ were arbitrary, this means that $\rho^*\varphi_{v-1,k}$ is normal with respect to W. On the other hand, we have $(U_{1,1}^{(k)})^{\rho^*\varphi_{v-1,k}}=0$, for $(U_{v,1}^{(k)})^{\varphi_{v-1,k}}=0$ by (i) for $h=v$ and

$$(U_{v,1}^{(k)}+(A\cap Z_{v-1}))/(A\cap Z_{v-1})=(U_{1,1}^{(k)})^{\rho^*}.$$

Moreover, since $\alpha_{1,u}\neq\alpha_{1,k}$ for $u\neq k$, we have $A^{\rho^{v-1}}\cap W_{1,1}=U_{1,1}^{(k)}$. Since $\Theta^\infty(Z_1, W)=W_{1,1}$, this means $\Theta^\infty(A^{\rho^{v-1}}, W)=U_{1,1}^{(k)}$. Now since $\Theta^\infty(A^{\rho^{v-1}})\subseteqq\Theta^\infty(A^{\rho^{v-1}}, W)$ and since the restriction of θ to $(A^{\rho^{v-1}})^{\rho^*\varphi_{v-1,k}}$ $(\subseteqq U_{1,p}^{(k)}\subseteqq W_{1,p}\subseteqq W)$ is nonsingular, we may apply Lemma 2.2 with $X=W$ and $Y=A^{\rho^{v-1}}$ and $\varphi=\rho^*\varphi_{v-1,k}$ to get $A^{\varphi_{v-1,k}}=0$.

Next we prove (ii) for $h=v$. Let $x\in U_{v,p-v}^{(k)}$ and $x_2, x_3, \cdots, x_r\in W$. We want to show

$$\theta_t(x, x_2, x_3, \cdots, x_r)\in \bigoplus_{l=1}^{v} U_{l,p-l}^{(k)}.$$

Let $x_1=x^{\rho^{v-1}}$, and let $a_{m,l}$ be as in Lemma 3.8. Since $x^{\rho}\in U_{v-1,p-v}^{(k)}=(U_{v+1,p-v}^{(k)})^{\rho^2}\subseteqq Y_0^{\rho^2}=Y_2$ by (3.4),

$$\theta_t(x^{\rho}, x_2, x_3, \cdots, x_r)\in\Theta(Y_2, W)=Y_2$$

by (3.3). On the other hand,

$$\theta_t(x^{\rho}, x_2, x_3, \cdots, x_r)\subseteqq \bigoplus_{l=1}^{v-1} U_{l,p-l}^{(k)}$$

by (ii) for $h=v-1$. Hence

$$\theta_t(x^{\rho}, x_2, x_3, \cdots, x_r)\in Y_2\cap \bigoplus_{l=1}^{v-1} U_{l,p-l}^{(k)}=\bigoplus_{l=1}^{v-1} U_{l,p-1-l}^{(k)}.$$

Since

$$\theta_t(x^{\rho}, x_2, x_3, \cdots, x_r)=\sum_{l=1}^{v-1}(a_{v-l,1})^{\rho^{1-l}}$$

by Lemma 3.8 (i) for $m=v-1$, this implies that

$$a_{m,1}\in U_{1,p-1-v+m}^{(k)} \tag{4.9}$$

for $1\leqq m\leqq v-1$. Hence

$$\sum_{l=2}^{v}(a_{v-l+1,1})^{\rho^{1-l}}\in \bigoplus_{l=2}^{v} U_{l,p-l}^{(k)}.$$

Since

$$\theta_t(x, x_2, x_3, \cdots, x_r)=\sum_{l=1}^{v}(a_{v-l+1,1})^{\rho^{1-l}},$$

this means that we have only to show that $a_{v,1}\in U_{1,p-1}^{(k)}$. Since $x\in U_{v,p-v}^{(k)}$,

$$x_1^{\rho^{-v}}\in U_{v+1,p-v}^{(k)}\subseteqq Y_0. \tag{4.10}$$

We shall compute

$$\theta_t(x_1^{\rho^{-v}}, x_2, x_3, \cdots, x_r)^{\varphi_{v,k}}$$

in two manners. Since $(x_1^{\rho^{-v}})^{\rho}=x\in U_{v,p-v}^{(k)}$, $(x_1^{\rho^{-v}})^{\rho\varphi_{v-1,k}}=0$ by (i) for $h=v$, and so $(x_1^{\rho^{-v}})^{\varphi_{v,k}\rho}=(x_1^{\rho^{-v}})^{\rho\varphi_{v,k}}=(x_1^{\rho^{-v}})^{\rho\varphi_{v-1,k}}=0$ by Lemmas 4.7 and 4.8. Hence

$$(x_1^{\rho^{-v}})^{\varphi_{v,k}}\in Z_1. \tag{4.11}$$

On the other hand, since

$$(x_1^{\rho^{-v}})^{\rho^{v+1-l}}\in U_{l,p-v}^{(k)}\subseteqq U_{l,p-l}^{(k)}, \qquad 1\leqq l\leqq v-1,$$

we have

$$(x_1^{\rho^{-v}})^{\varphi_{v,k}} \in \bigoplus_{l=1}^{v-1} U_{l,p-l}^{(k)} \tag{4.12}$$

by Lemma 4.11. By (4.11) and (4.12),

$$(x_1^{\rho^{-v}})^{\varphi_{v,k}} \in Z_1 \cap (\bigoplus_{l=1}^{v-1} U_{l,p-l}^{(k)}) = U_{1,p-1}^{(k)}.$$

Hence

$$\theta_t(x_1^{\rho^{-v}}, x_2, x_3, \cdots, x_r)^{\varphi_{v,k}} = \theta_t((x_1^{\rho^{-v}})^{\varphi_{v,k}}, x_2, x_3, \cdots, x_r) \in U_{1,p-1}^{(k)} \tag{4.13}$$

by Lemma 4.9 and Lemma 4.4 (iv). On the other hand,

$$\theta_t(x_1^{\rho^{-v}}, x_2, x_3, \cdots, x_r) = \sum_{l=1}^{v+1} (a_{v-l+2,1})^{\rho^{1-l}} \tag{4.14}$$

by Lemma 3.8 (i). Since $a_{v-l+2,1} \in U_{1,p-l+1}^{(k)}$ for $3 \leqq l \leqq v$ by (4.9),

$$(\sum_{l=3}^{v} (a_{v-l+2,1})^{\rho^{1-l}})^{\varphi_{v,k}} = 0$$

by (i) for $3 \leqq h \leqq v$. Also $(a_{v+1,1})^{\varphi_{v,k}} = 0$, for $a_{v+1,1} \in Z_1$. Therefore

$$\theta_t(x_1^{\rho^{-v}}, x_2, x_3, \cdots, x_r)^{\varphi_{v,k}} = (a_{v,1})^{\rho^{-1}\varphi_{v,k}} + (a_{1,1})^{\rho^{-v}\varphi_{v,k}}. \tag{4.15}$$

Since $a_{1,1} \in U_{1,p-v}^{(k)}$, $(a_{1,1})^{\rho^{-v}\varphi_{v,k}} \in U_{1,p-1}^{(k)}$ by (i) for $h=v$ and by Lemma 4.11. Hence, comparing (4.13) and (4.15), we get

$$(a_{v,1})^{\rho^{-1}\varphi_{v,k}} = \theta_t(x_1^{\rho^{-v}\varphi_{v,k}}, x_2, x_3, \cdots, x_r) - (a_{1,1})^{\rho^{-v}\varphi_{v,k}} \in U_{1,p-1}^{(k)}. \tag{4.16}$$

By Lemma 4.8, $(a_{v,1})^{\rho^{-1}\varphi_{1,k}} \in U_{1,p-1}^{(k)}$. On the other hand, (4.10) implies

$$\theta_t(x_1^{\rho^{-v}}, x_2, x_3, \cdots, x_r) \in \Theta(Y_0, W) = Y_0,$$

and so

$$(a_{v,1})^{\rho^{-1}} \in Y_0 \cap V_2 = W_{2,p-1}$$

by (4.14). Thus $(a_{v,1})^{\rho^{-1}}$ satisfies the hypotheses of Lemma 4.6. Hence $(a_{v,1})^{\rho^{-1}} \in U_{2,p-1}^{(k)}$, and so $a_{v,1} \in U_{1,p-1}^{(k)}$. Note that this implies $(a_{v,1})^{\rho^{-1}\varphi_{v,k}} = 0$, and so we have

$$\theta_t(x_1^{\rho^{-v}\varphi_{v,k}}, x_2, x_3, \cdots, x_r) = \theta_t(x_1, x_2, x_3, \cdots, x_r)^{\rho^{-v}\varphi_{v,k}} \tag{4.17}$$

by (4.16).

It remains only to prove (i) for $h=v+1$. Note that (4.17) shows that the mapping $\rho^{-v}\varphi_{v,k}$ from $U_{1,p-v}^{(k)}$ into $U_{1,p-1}^{(k)}$ is normal with respect to W. Now we shall show

$$\Theta(U_{1,1}^{(k)}, U_{1,1}^{(k)})^{\rho^{-v}\varphi_{v,k}} = 0,$$

from which we can obtain the desired conclusion by applying Lemma 2.2 with $X=W$ and $Y=U_{1,p-v}^{(k)}$, for

$$\Theta^\infty(U_{1,p-v}^{(k)}) = \Theta^\infty(\Theta^\infty(U_{1,p-v}^{(k)}, W)) = \Theta^\infty(U_{1,1}^{(k)}).$$

Let $x_1, x_2, x_3, \cdots, x_r \in U_{1,1}^{(k)}$. Let $a_{m,l}$, $b_{m,l}$ be as in Lemma 3.8, and let

$$x=\sum_{l=1}^{v}(b_{v-l+1,1})^{\rho^{1-l}} \in Z_v .$$

Since $x_1^{\rho^{1-v}\varphi_{v-1,k}}=0$ by (i) for $h=v$,

$$\theta_t(x_1^{\rho^{1-v}}, x_2^{\rho^{-1}}, x_3, \cdots, x_r)^{\varphi_{v,k}}=0$$

by Lemma 4.10 (i). Namely,

$$\sum_{l=2}^{v+1}(l-1)(a_{v-l+2,1}+a_{v-l+1,1})^{\rho^{1-l}\varphi_{v,k}}+x^{\varphi_{v,k}}=0 \tag{4.18}$$

by Lemma 3.8 (ii). By (ii) for $h=v$, $a_{m,1}\in U_{1,p-1-v+m}^{(k)}$ for $1\leqq m\leqq v$. Hence

$$(a_{1,1})^{\rho^{-v}\varphi_{v,k}}\in U_{1,p-1}^{(k)} \tag{4.19}$$

by (i) for $h=v$ and by Lemma 4.11, and

$$\sum_{l=2}^{v}(l-1)(a_{v-l+2,1}+a_{v-l+1,1})^{\rho^{1-l}\varphi_{v,k}}=0$$

by (i) for $2\leqq h\leqq v$. Thus (4.18) can be written in the form

$$(va_{1,1})^{\rho^{-v}\varphi_{v,k}}+x^{\varphi_{v,k}}=0 . \tag{4.20}$$

Therefore $x^{\varphi_{v,k}}\in U_{1,p-1}^{(k)}$ by (4.19). Since $x\in Z_v$, this means that x satisfies the hypotheses of (iii) for $h=v$. Hence $x^{\varphi_{v,k}}=0$, and so

$$(a_{1,1})^{\rho^{-v}\varphi_{v,k}}=0$$

by (4.20). Namely,

$$\theta_t(x_1, x_2, x_3, \cdots, x_r)^{\rho^{-v}\varphi_{v,k}}=0 .$$

Since $x_1, x_2, x_3, \cdots, x_r$ were arbitrary, $\Theta(U_{1,1}^{(k)}, U_{1,1}^{(k)})^{\rho^{-v}\varphi_{v,k}}=0$ as desired.

LEMMA 4.13. *For each* k,

$$\alpha_{1,k}\in\{0, 1, 2, \cdots, p-1\} .$$

PROOF. Let $0\neq w\in U_{1,1}^{(k)}$. By Lemma 4.10 (ii) and Lemma 4.12 (i) for $h=p$,

$$\begin{aligned}&\theta(w^{\rho^{1-p}}, w^{\rho^{-1}}, w, \cdots, w)^{\varphi_{p-1,k}}\\ &=((p-1)\alpha_{p-1,k}-\alpha_{1,k}\alpha_{p-1,k})\theta(w, w, w, \cdots, w)\in U_{1,1}^{(k)} .\end{aligned}$$

This means that $x=\theta(w^{\rho^{1-p}}, w^{\rho^{-1}}, w, \cdots, w)$ satisfies the hypotheses of Lemma 4.12 (iii) for $h=p$. Hence $x^{\varphi_{p-1,k}}=0$. Since $\theta(w, w, w, \cdots, w)\neq 0$ by the non-singularity of the restriction of θ to W, this implies

$$(p-1)\alpha_{p-1,k}-\alpha_{1,k}\alpha_{p-1,k}=0 . \tag{4.21}$$

Substituting (4.1) for $h=p-1, p-2, \cdots, 3, 2$, successively, in (4.21), we get

$$(\alpha_{1,k}-(p-1))(\alpha_{1,k}-(p-2))\cdots(\alpha_{1,k}-1)\alpha_{1,k}=0\,,$$

and hence $\alpha_{1,k}$ is one of $0, 1, 2, \cdots, p-1$ as desired.

5. Proof of Proposition F.

We finish the proof of Proposition F in this section.

LEMMA 5.1. *If $\tau\in C_Q(Z_1)$ acts trivially on*

$$\Theta(W_{1,1}, W_{1,1})\oplus\Theta(W_{1,1}, W_{1,1})^{\rho^{-1}},$$

then τ acts trivially on Y_0.

PROOF. Let s, $\alpha_{1,k}$ be as in Lemma 4.4. Then $s=1$ and $\alpha_{1,1}=0$ by Lemma 4.5. Thus the desired conclusion follows from Lemma 4.12 (i).

LEMMA 5.2. *If $\tau\in C_Q(Z_1)$ acts trivially on Y_0, then $\tau=1$.*

PROOF. We prove by induction on h that τ acts trivially on Z_h. If $h=1$, there is nothing to be proved. Now we assume that τ acts trivially on Z_{h-1}, and let φ be the mapping defined by

$$x^{\varphi}=[x^{\rho^{1-h}}, \tau]\,, \qquad x\in V_h^{\rho^{h-1}}.$$

By the inductive hypothesis, $(V_h^{\rho^{h-1}})^{\varphi}\subseteqq Z_1$. Let $x_1\in V_h^{\rho^{h-1}}$, $x_2, x_3, \cdots, x_r\in Z_1$, and let $a_{m,l}$ be as in Lemma 3.8. We have

$$\begin{aligned}&\theta_t([x_1^{\rho^{1-h}}, \tau], x_2, x_3, \cdots, x_r)=[\theta_t(x_1^{\rho^{1-h}}, x_2, x_3, \cdots, x_r), \tau]\\&=[\sum_{l=1}^{h}(a_{h-l+1,1})^{\rho^{1-l}}, \tau]=[(a_{1,1})^{\rho^{1-h}}, \tau]\\&=\theta_t(x_1, x_2, x_3, \cdots, x_r)^{\varphi}\end{aligned}$$

by the inductive hypothesis. This means that φ is normal with respect to Z_1. Also $(W_{1,p-h+1})^{\varphi}=0$ by the hypothesis of the lemma. Since $(V_h^{\rho^{h-1}})^{\varphi}\subseteqq Z_1\cap[V, \tau]$ $\subseteqq C_V(\tau)\cap[V, \tau]$, we may now apply Lemma 2.2 with $X=Z_1$ and $Y=V_h^{\rho^{h-1}}$ to get $(V_h^{\rho^{h-1}})^{\varphi}=0$. This means $[V_h, \tau]=0$, and so $[Z_h, \tau]=0$ as desired.

PROOF OF PROPOSITION F. Let $W_{1,1}=\bigoplus_{j=1}^{d}X^{(j)}$ be a fixed direct sum decomposition of $W_{1,1}$ in which each $X^{(j)}$ is an indecomposable ideal of $W_{1,1}$. By the nonsingularity of the restriction of θ to $W_{1,1}$, $\Theta(X^{(j)}, W_{1,1})\neq 0$ for each j. Hence, by Lemmas 4.4 and 4.13 and by Lemma 2.5,

$$\begin{aligned}&|C_Q(Z_1)/C_{C_Q(Z_1)}(\Theta(W_{1,1}, W_{1,1})\oplus\Theta(W_{1,1}, W_{1,1})^{\rho^{-1}})|\\&\leqq p^d\leqq p^{\dim W_{1,1}}\leqq p^{m_2}.\end{aligned}$$

Since

$$C_{C_Q(Z_1)}(\Theta(W_{1,1}, W_{1,1})\oplus\Theta(W_{1,1}, W_{1,1})^{\rho^{-1}})=1$$

by Lemmas 5.1 and 5.2, this yields the desired conclusion.

6. Proof of Theorem E.

In this section, we finish the proof of Theorem E by induction on n.

Let $\bar{Q}=Q/C_Q(Z_1)$. Also let $W_h=Z_1\cap V^{\rho^h}$. Thus $W_1=W$, and $W_0=Z_1$. Let $m_1=\dim(W_0/W_1)$.

LEMMA 6.1.
$$|C_{\bar{Q}}(W_{p-1})|\leqq p^{\sum_{i\geqq 1}[m_1/p^i]}.$$

PROOF. First we show
$$[W_1, C_{\bar{Q}}(W_{p-1})]=0. \tag{6.1}$$

In order to do this, we prove by downward induction on h that
$$[W_h, C_{\bar{Q}}(W_{p-1})]=0, \qquad 1\leqq h\leqq p-1.$$

Assume $[W_{h+1}, C_{\bar{Q}}(W_{p-1})]=0$, and, by way of contradiction, suppose $[W_h, C_{\bar{Q}}(W_{p-1})]\neq 0$. Then there exist $x, y\in W_h$ and $\tau\in C_{\bar{Q}}(W_{p-1})$ such that $x^\tau=x+y$, $y^\tau=y\neq 0$. Then
$$\theta(x, y, y, \cdots, y)^\tau=\theta(x, y, y, \cdots, y)+\theta(y, y, y, \cdots, y).$$

But since $\theta(x, y, y, \cdots, y)\in W_{h+1}$ by Lemma 3.4, this contradicts the nonsingularity of the restriction of θ to W_1. Thus (6.1) is proved. Next let $\widetilde{W}_0=W_0/W_1$. Let τ be an arbitrary element of $C_{\bar{Q}}(W_{p-1})$. Let x be an element of W_0 such that $\tilde{x}\in C_{\widetilde{W}_0}(\tau)$. Then $x^\tau=x+y$, $y\in W_1$. Arguing as above, we get $y=0$, and so $x\in C_{W_0}(\tau)$. Since x was arbitrary, this means
$$C_{\widetilde{W}_0}(\tau)=\widetilde{C_{W_0}(\tau)}. \tag{6.2}$$

Since τ was arbitrary, this, in particular, implies that $C_{\bar{Q}}(W_{p-1})$ acts faithfully on $\widetilde{W}_0$. On the other hand, we can easily verify $\Theta(W_1, W_0)\subseteqq W_1$ by Lemma 3.1 for $h=1$. Hence θ naturally induces a multilinear mapping on $\widetilde{W}_0$, which we shall denote by $\tilde{\theta}$. Then we may regard $C_{\bar{Q}}(W_{p-1})$ as a unipotent subgroup of $\mathrm{Aut}(\tilde{\theta})$. Now we show that for every $\tau\in C_{\bar{Q}}(W_{p-1})$ and for every y such that $0\neq\tilde{y}\in C_{\widetilde{W}_0}(\tau)\cap[\widetilde{W}_0, \tau]$, we have
$$\tilde{\theta}(\tilde{y}, \tilde{y}, \tilde{y}, \cdots, \tilde{y})\neq 0,$$

from which we get the desired conclusion by the inductive hypothesis. Let τ, y be as above, and suppose, by way of contradiction, that $\tilde{\theta}(\tilde{y}, \tilde{y}, \tilde{y}, \cdots, \tilde{y})=0$, namely, $\theta(y, y, y, \cdots, y)\in W_1$. Since $\tilde{y}\in[\widetilde{W}_0, \tau]$, we may assume that there exists $x\in W_0$ such that $x^\tau=x+y$. Then
$$\theta(x, y, y, \cdots, y)^\tau=\theta(x, y, y, \cdots, y)+\theta(y, y, y, \cdots, y), \tag{6.3}$$

for $y\in C_{W_0}(\tau)$ by (6.2). Since $\theta(y, y, y, \cdots, y)\in W_1$, (6.3) shows $\widetilde{\theta(x, y, y, \cdots, y)}$

$\in C_{\tilde{W}_0}(\tau)$. By (6.2), this implies $\theta(x, y, y, \cdots, y) \in C_{W_0}(\tau)$, which means $\theta(y, y, y, \cdots, y)=0$ by (6.3). Since $y \in C_{W_0}(\tau) \cap [W_0, \tau]$, this contradicts the hypothesis of Theorem E.

PROOF OF THEOREM E. By the inductive hypothesis,

$$|\bar{Q}/C_{\bar{Q}}(W_{p-1})| \leqq p^{\sum_{i\geqq 1}[m_2/p^i]}.$$

Combining this and Lemma 6.1, we get

$$|\bar{Q}| \leqq p^{\sum_{i\geqq 1}([m_2/p^i]+[m_1/p^i])}.$$

Combining this and Proposition F, we get

$$\begin{aligned}|Q| &\leqq p^{m_2+\sum_{i\geqq 1}[m_2/p^i]+\sum_{i\geqq 1}[m_1/p^i]} \\ &\leqq p^{\sum_{i\geqq 1}([pm_2/p^i]+[m_1/p^i])} \leqq p^{\sum_{i\geqq 1}[n/p^i]}\end{aligned}$$

as desired.

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