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Projective plane curves and the automorphism groups of their complements

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1. Introduction.

Let C be an irreducible algebraic curve of degree d on $\boldsymbol{P}^2=\boldsymbol{P}^2(\boldsymbol{C})$ and put $V=\boldsymbol{P}^2\setminus C$. Let $\mathcal{G}$ be the automorphism group of the algebraic surface V and $\mathcal{L}$ the linear part of $\mathcal{G}$, i.e., $\mathcal{L}=\{T\in\mathrm{Aut}(\boldsymbol{P}^2)\mid T(C)=C\}$. If $d=1$, then $\mathcal{G}$ is generated by linear transformations and de Jonquières transformations of V (Nagata [**5**]); if $d=2$, then generators of the similar kind have been found by Gizatullin and Danilov [**2**]. In this paper we shall study the structure of $\mathcal{G}$ and at the same time the property of C in the case when $d\geqq 3$.

We shall use the following notations in addition to the above ones. Let (X, Y, Z) be a set of homogeneous coordinates on $\boldsymbol{P}^2$ and put $x=X/Z$ and $y=Y/Z$. Usually we do not treat the line $Z=0$, so we say that for an irreducible polynomial f, the curve $Z^d f(X/Z, Y/Z)=0$ is defined by f, where $d=\deg f$. Especially we denote by Δ [resp. Δ_e] the curve defined by $xy-x^3-y^3$ [resp. y^e-x^d, where $(e, d)=1$ and $1\leqq e\leqq d-2$]. Let M be the number of the singular points $\{P_1, \cdots, P_M\}$ of C and $\mu:\tilde{C}\to C$ the normalization of C. Then let N denote the number of elements of $\mu^{-1}(\{P_1, \cdots, P_M\})$ and g the genus of $\tilde{C}$. In case $N=1$, let $(e_1, \cdots, e_p)$ be the sequence of the multiplicities of all successive infinitely near singular points of P_1, and put

$$R=d^2-\sum_{i=1}^{p} e_i^2-e_p+1\,.$$

Let $\boldsymbol{G}_a$ and $\boldsymbol{G}_m$ be the additive and the multiplicative groups respectively.

First we shall prove the following with the help of the Plücker relations.

PROPOSITION 1. *Suppose that* $d\geqq 3$. *Then the following three conditions are equivalent.*

(1) *The order of* $\mathcal{L}$ *is infinite.*
(2) *The linear part* $\mathcal{L}$ *is isomorphic to* $\boldsymbol{G}_m$.
(3) *The curve* C *is projectively equivalent to* Δ_e.

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Note that $\mathcal{L}$ *is a finite group if* C *is not projectively equivalent to* Δ_e.

Next we shall consider $\mathcal{G}$. Applying the Castelnuovo's criterion for contracting a curve, we shall give the condition that $\mathcal{G}=\mathcal{L}$. In case $\mathcal{G}\neq\mathcal{L}$, let φ be an element of $\mathcal{G}\setminus\mathcal{L}$. Then there is a composition σ of blow-ups such that the induced map $\varphi\sigma$ is a morphism. Considering the total transform $\sigma^{-1}(C)$ in detail, we shall prove the following main result.

THEOREM A. *The order of* $\mathcal{G}$ *is finite if and only if* C *satisfies any one of the following conditions* (1), (2) *and* (3).

(1) $g\geqq 1$.

(2) $N\geqq 2$ *and* C *is projectively equivalent to neither* Δ *nor* Δ_e.

(3) $N=1$ *and* $R\leqq -1$.

On the contrary, for the remaining curves with $d\geqq 3$, $\mathcal{G}$ *has the following properties.*

(4) *If* C *is projectively equivalent to* Δ_e, *where* $e\geqq 2$, *then* $\mathcal{G}=\mathcal{L}\cong \boldsymbol{G}_m$.

(5) *If* C *is projectively equivalent to* Δ, *then the order of* $\mathcal{G}$ *is countably infinite and* $\mathcal{L}$ *is the dihedral group of order* 6.

(6) *If* C *is a curve with* $g=0$, $N=1$ *and* $R\geqq 0$, *then* $\mathcal{G}\supset(\boldsymbol{G}_a)^n$ *for every positive integer* n. *In this class of curves the order of* $\mathcal{L}$ *is infinite if and only if* C *is projectively equivalent to* Δ_1.

REMARK. We do not know whether or not the curve with the properties $g=0$, $N=1$, $R=0$ and $e_{p-1}>e_p$ exists.

The structure of $\mathcal{G}$ seems to be complicated for the curve $g=0$, $N=1$ and $R\geqq 0$. According to Abhyankar-Moh [**1**], if C satisfies that $C\setminus L\cong \boldsymbol{A}^1$ for a line L, then there is an automorphism of $\boldsymbol{P}^2\setminus L\cong \boldsymbol{A}^2$ by which C is transformed to a line L_1. Hence $\boldsymbol{P}^2\setminus(C\cup L)$ is isomorphic to $\boldsymbol{P}^2\setminus(L_1\cup L)$, this implies that C is a curve with $R\geqq 2$ if $d\geqq 3$. Moreover, if the logarithmic Kodaira dimension $\bar{\kappa}(V)$ is $-\infty$ and $d\geqq 3$, then C is a curve with $g=0$ and $N=1$ (Iitaka [**4**]). By these facts it seems interesting to study the curves of this class.

Now these curves have similarly the following properties, which will be shown in the course of the proof of Theorem A.

THEOREM B. *If* C *is a curve with* $g=0$, $N=1$ *and* $R\geqq 0$, *then there are one or two irreducible curves* C' *and* C'' *and two or three lines* L_i, *where* $i=1, 2, 3$, *which have the following properties.*

(1) *In case* $R\neq 1$, *there is an isomorphism*

$$\boldsymbol{P}^2\setminus(C\cup C')\cong \boldsymbol{P}^2\setminus(L_1\cup L_2).$$

(2) *In case* $R=1$, *there is an isomorphism*

$$\boldsymbol{P}^2\setminus(C\cup C'\cup C'')\cong \boldsymbol{P}^2\setminus(L_1\cup L_2\cup L_3),$$

where $L_1\cap L_2=L_1\cap L_3$.

We observe that the above curves C' and C'' have the same properties as the following C^*.

PROPOSITION 2. *If C satisfies the conditions $C\setminus\{P\}\cong \boldsymbol{A}^1$ and $R\geqq 0$, then there is a curve C^* having the properties $C\cap C^*=\{P\}$ and $C^*\setminus\{P\}\cong \boldsymbol{A}^1$.*

Note that, in case $d\geqq 3$, the condition $C\setminus\{P\}\cong \boldsymbol{A}^1$ is equivalent to the one $g=0$ and $N=1$. Especially Theorem B implies the following

COROLLARY. *If C satisfies the conditions $g=0$, $N=1$ and $R\geqq 0$, then $\bar{\kappa}(V)=-\infty$.*

This is a partial answer to the problem raised in [8]. Note that $\bar{\kappa}(V)$ is not necessarily $-\infty$ if $R\leqq -1$. Indeed, there exist curves with $g=0$, $N=1$, $R\leqq -1$ and $\bar{\kappa}(V)=1$ (Tsunoda [6] or Section 6).

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2. Structure of $\mathcal{L}$.

Thanks to the theory of the logarithmic Kodaira dimension, we have that $\mathcal{G}$ has a finite order if $\bar{\kappa}(V)=2$ (Iitaka [4]). Further, Wakabayashi [7] has shown the following

LEMMA 2.1. *If C satisfies none of the following conditions, then $\bar{\kappa}(V)=2$.*

(a) *$M=0$ and $d\leqq 3$.*

(b) *$g=0$ and $M=1$.*

(c) *$g=0$ and $M=N=2$.*

Later we shall make use of this lemma. Let $\sim$ denote the projective equivalence. Then the curves Δ and Δ_e have the following properties.

(1) If $C\sim\Delta_1$, then $M=N=1$, $R=d+1$ and $\bar{\kappa}(V)=-\infty$.

(2) If $C\sim\Delta_e$ and $e\geqq 2$, then $M=N=2$ and $\bar{\kappa}(V)=1$.

(3) If $C\sim\Delta$, then $M=1$, $N=2$ and $\bar{\kappa}(V)=0$.

Let $e(P, C)$ be the multiplicity of C at P, and $(D_1\cdot D_2)_P$ the intersection multiplicity of two curves D_1 and D_2 at P. If C_{ij} is an analytically irreducible branch of C at P_i, where $1\leqq i\leqq M$ and $j=1, 2, \cdots$, then we put $e_{ij}=e(P_i, C_{ij})$ and $\lambda_{ij}=(C_{ij}\cdot L_{ij})_{P_i}$, where L_{ij} is the tangent line to C_{ij} at P_i. If P is a flex, then we put $W=\sum_{P\in\{\text{flexes}\}}\rho_P$, where $\rho_P=(C\cdot L_P)_P-2$ and L_p is the tangent line to C at P. Then we have the following formula which is one of the Plücker relations (Iitaka [4]).

LEMMA 2.2. $$W=3d+6(g-1)-\sum_{i,j}(\lambda_{ij}+e_{ij}-3)\,.$$

Applying this formula, we get the following

LEMMA 2.3. *Suppose that $N=1$ and $e(P_1, C)=d-1$. Then C has one flex if* *and only if* $C\sim\Delta_1$.

PROOF. If C has one flex Q, then by a suitable projective transformation we may assume that $P_1=(0, 1, 0)$, $Q=(0, 0, 1)$ and that C is defined by

$$f=y+\sum_{i=1}^{d} a_i x^i.$$

Since we have that $W=d-2$ by Lemma 2.2, we get $a_2=\cdots=a_{d-1}=0$, i.e., $f=y+a_d x^d+a_1 x$, where $a_d\neq 0$. This implies that C is projectively equivalent to Δ_1. The "if" part is proved easily by direct computation. Q.E.D.

Now, let us prove Proposition 1.

The implication (3)$\Rightarrow$(2) is checked by direct computation and the implication (2)$\Rightarrow$(1) is trivial. So let us prove the implication (1)$\Rightarrow$(3). Suppose that C is not projectively equivalent to Δ_e. Then, in case C is smooth, we have that $W=3d(d-2)$. Hence there are at least four non-collinear flexes, which implies the order of $\mathcal{L}$ is finite. Next let us consider the non-smooth case. Then by Lemma 2.1 we have only to treat the curves in the cases (b) and (c). Thus we have that $g=0$. Let K be the number of flexes of C. Then we note that the order of $\mathcal{L}$ is finite if $N+K\geqq 3$. In fact, an element T of $\mathcal{L}$ induces the automorphism $\tilde{T}$ of $\tilde{C}\cong \boldsymbol{P}^1$ and this correspondence $T\mapsto\tilde{T}$ is injective. First let us consider the case (b). In this case, if $N\leqq 2$, then $W\geqq 1$, i.e., $K\geqq 1$. Suppose that $N+K\leqq 2$. Then $K\geqq 1$. Hence we have that $N=K=1$, thus $W\geqq d-2$. Since $K=1$, it follows that $W=d-2$, which implies that $e(P_1, C)=d-1$. This is a contradiction by Lemma 2.3. Next let us consider the case (c). Let L_i be the tangent line to C at P_i, where $i=1, 2$. If L_1 contains P_2, then $e_2+\lambda_1\leqq d$, $e_1\leqq d-1$ and $\lambda_2\leqq d$, hence $W\geqq 1$. So that we may assume that $L_1\not\ni P_2$ and $L_2\not\ni P_1$. Moreover we may assume that L_i intersects C only in P_i, and that $W=0$, otherwise there is a third fixed point for $\mathcal{L}$. Thus we have only to consider the case when $\lambda_1=\lambda_2=d$ and $e_1+e_2=d$. In case $e_1=e_2=e$, seeing from that C is analytically irreducible at P_1 and P_2, we infer that the multiplicities of infinitely near singular points to P_1 and P_2 of order one must be both e. Then we have that $(d-1)(d-2)<4e(e-1)$, this contradicts the genus formula for plane curves. Therefore T fixes L_1, L_2 and the line passing through P_1 and P_2. By a suitable projective transformation these lines are assumed to be $Y=0$, $Z=0$ and $X=0$ respectively. Then the equation of C is

$$f=y^e+\sum{}^* c_{ij}x^i y^j-x^d,$$

where $e=e_1$ and $\sum^*$ denotes the summation for i and j satisfying $d>i+j>e>j>0$. Suppose that $(d-i)(e-j)=ij$ for all i, j in this equation. Then putting $b=(e, d)$, $e=be'$ and $d=bd'$, we have that $bd'e'=ie'+jd'$. Moreover putting $i=d'i'$ and $j=e'j'$, where $i'+j'=b$, we see that f is a homogeneous polynomial with variables $x^{d'}$ and $y^{e'}$. Hence f is factored as

$$\prod_{i=1}^{b}(y^{e'}+\alpha_i x^{d'})$$

for some α_i, where $1 \leqq i \leqq b$. Since C is irreducible, b is 1, hence $C \sim \Delta_e$. This contradicts our hypothesis, so there are i and j such that $k=(d-i)(e-j)-ij \neq 0$. Since T is represented as a diagonal matrix

$$\begin{pmatrix} \alpha & & \\ & \beta & \\ & & 1 \end{pmatrix},$$

we have that $\beta^e=\alpha^i\beta^j=\alpha^d$, i.e., $\alpha^k=1$. Whence $\mathcal{L}$ is a finite group. Q.E.D.

3. Relation between $\mathcal{G}$ and $\mathcal{L}$.

Since an element φ of $\mathcal{G}$ is the restricted mapping of a birational transformation of $\boldsymbol{P}^2$, let us denote by φ also the birational transformation. Let

$$S_r \xrightarrow{\sigma_r} S_{r-1} \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S_0=\boldsymbol{P}^2$$

be a finite sequence of blow-ups σ_i with successive centers Q_i in S_{i-1}, where $1 \leqq i \leqq r$ and $S_0=\boldsymbol{P}^2$, and put $\sigma=\sigma_1 \cdots \sigma_r$. For a birational transformation ψ we denote by $\psi(A)$ and $\psi[A]$ the total and the proper transforms of A respectively.

LEMMA 3.1. *Let φ be an element of $\mathcal{G}$. Then the following assertions hold true.*

(1) *Each birational transformation φ or $\varphi\sigma_1 \cdots \sigma_i$ has at most one fundamental point, where $1 \leqq i \leqq r$.*

(2) *The proper transform $\varphi[C]$ is C [resp. one point] if and only if φ belongs to $\mathcal{L}$ [resp. $\mathcal{G} \setminus \mathcal{L}$].*

PROOF. Note that if φ has no fundamental points, then φ is a birational morphism from $\boldsymbol{P}^2$ to $\boldsymbol{P}^2$ and so is an isomorphism. Since C is irreducible and φ is an automorphism of $\boldsymbol{P}^2 \setminus C$, both assertions are proved readily. Q.E.D.

Put $C_i=(\sigma_1 \cdots \sigma_i)^{-1}[C]$ and $C_0=C$, where $1 \leqq i \leqq r$. Let $D_1 \cdot D_2$ denote the intersection number of two curves D_1 and D_2 on some nonsingular complete surface. In case $D_1=D_2$, let us write D_1^2 instead of $D_1 \cdot D_1$ and call it the *weight* of D_1 for short. Then we have the following

LEMMA 3.2. *If C satisfies any one of the following conditions, then $\mathcal{G}=\mathcal{L}$.*

(1) *$g \geqq 1$.*

(2) *There is some i $(0 \leqq i \leqq r)$ such that C_i has at least two singular points.*

(3) *If C_i is smooth, then $C_i^2 \leqq -2$, where $1 \leqq i \leqq r$.*

(4) *$g=0$, $N=1$ and $R \leqq -e_p-1$.*

PROOF. From the above lemma and the Castelnuovo's criterion for contracting a curve (Hartshorne [3]), the assertions (1), (2) and (3) follow easily. Note that in the case (4) the weight C_p^2 is

$$d^2-\sum_{i=1}^{p} e_i^2=R+e_p-1,$$

if the center Q_{i+1} of the blow-up σ_{i+1} coincides with the singular point of C_i, where $0\leqq i\leqq p-1$. Hence this is a special case of (3). Q.E.D.

Combining Lemma 3.2 with Proposition 1, we get the following

COROLLARY 3.3. *If C satisfies any one of the following conditions, then $\mathcal{G}$ is a finite group.*

(1) $g\geqq 1$.

(2) *The curve C is not projectively equivalent to Δ_e and there is some i $(0\leqq i\leqq r)$ such that C_i has at least two singular points.*

(3) *If C_i is smooth, then $C_i^2\leqq -2$, where $1\leqq i\leqq r$.*

(4) $g=0$, $N=1$ *and* $R\leqq -e_p-1$.

PROOF. It suffices to check (3) and (4). If $C\sim\Delta_e$ and $e\geqq 2$, then let $(e_1, \cdots, e_a)$ and $(f_1, \cdots, f_b)$, where $e_1=e$ and $f_1=d-e$, be the sequences of the multiplicities of all infinitely near singular points of (0, 0, 1) and (0, 1, 0) respectively. Since $(e, d)=1$, we get

$$d^2-\sum_{i=1}^{a} e_i^2-\sum_{j=1}^{b} f_j^2=e_a+f_b$$

by the Euclidean algorithm and the genus formula for plane curves. If $e=1$, then the multiplicity of the singular point is $d-1$, so $R=d+1$. Hence the curves in (3) and (4) are not projectively equivalent to Δ_e. Q.E.D.

Now the assertions (1) and (4) in Theorem A follow from Lemma 3.2 and Proposition 1.

4. Representation of automorphisms by graphs.

In this section we follow the notations fixed in the previous sections. Hereafter we shall study the curves that have not been treated in Corollary 3.3 and that are not projectively equivalent to Δ_e, where $e\geqq 2$. Therefore we assume that C satisfies all of the following conditions (A_1) and $d\geqq 3$.

$$(A_1)\quad \begin{cases} (1)\ \ g=0. \\ (2)\ \ \text{The proper transform } C_i \text{ has at most one singular point for all } i, \\ \qquad \text{where } 0\leqq i\leqq r. \\ (3)\ \ \text{There is some } i\ (1\leqq i\leqq r) \text{ such that } C_i \text{ is smooth and } C_i^2\geqq -1. \\ (4)\ \ \text{If } N=1, \text{ then } R\geqq -e_p. \end{cases}$$

In particular M is 1. In view of the assertion (1) in Lemma 3.1 we may assume moreover the following condition (A_2).

(A_2) $\begin{cases} \text{For an element } \varphi \text{ of } \mathcal{G}\setminus\mathcal{L} \text{ the birational morphism } \sigma \text{ is a composition} \\ \text{of } r \text{ blow-ups such that } r \text{ is minimal in order that } \varphi\sigma \text{ is a morphism.} \end{cases}$

DEFINITION 4.1. We denote by $r(\varphi)$ the number of blow-ups defined in (A_2) and call it the *rank* of φ. Of course $r(\varphi)=0$ if and only if φ belongs to $\mathcal{L}$.

Put $E_i=\sigma_i^{-1}(Q_i)$, where $1\leqq i\leqq r$. Then the following facts hold true, of which we shall make frequent use later.

LEMMA 4.2. *If $\mathcal{G}\neq\mathcal{L}$ and φ belongs to $\mathcal{G}\setminus\mathcal{L}$, then we have the following.*

(i) *The center Q_{i+1} of σ_{i+1} coincides with the fundamental point of $\varphi\sigma_1\cdots\sigma_i$, where $0\leqq i\leqq r-1$ and $\varphi\sigma_0=\varphi$.*

(ii) *The center Q_{i+1} of σ_{i+1} belongs to E_i, where $1\leqq i\leqq r-1$.*

(iii) *If C_i is not smooth, then Q_{i+1} coincides with the singular point of C_i, where $0\leqq i\leqq r-1$.*

(iv) *If $C_i^2\geqq 0$, then Q_{i+1} belongs to $E_i\cap C_i$, where $1\leqq i\leqq r-1$.*

PROOF. First recall the assertion (1) in Lemma 3.1 and the conditions (A_1) and (A_2). Then (i) is clear. Similarly we get $\sigma_i(Q_{i+1})=\sigma_i(E_i)$. This proves (ii). Since φ has a fundamental point, the proper transform C_r must be contracted owing to the assertion (2) in Lemma 3.1. Then from the Castelnuovo's criterion we infer (iii) and (iv). Q.E.D.

Since there will be no danger of confusion, let E_i denote also the curve $(\sigma_{i+1}\cdots\sigma_r)^{-1}[E_i]$ on S_r. When we treat more than one automorphism at a time, let us write $\sigma_{i\varphi}$, $E_{i\varphi}$ and $S_{i\varphi}$ with the automorphism φ instead of σ_i, E_i and S_i respectively. For a divisor we shall not consider the multiplicities of the components, hence *we identify the divisor with the reduced one obtained from it.*

DEFINITION 4.3. Let φ be an element of $\mathcal{G}\setminus\mathcal{L}$ and put $r=r(\varphi)$. Since σ is determined by φ, let $\Gamma(\varphi)=\sigma^{-1}(C)$, i.e., $\Gamma(\varphi)=E_{1\varphi}+\cdots+E_{r\varphi}+C_\varphi$, where $C_\varphi=\sigma^{-1}[C]$. The total transform $\Gamma(\varphi)$ is called the *graph* of φ. Define the *orders* of $E_{i\varphi}$ and C_φ to be i and $r+1$ respectively, where $1\leqq i\leqq r$. For an element ϕ of $\mathcal{G}\setminus\mathcal{L}$, similarly $\Gamma(\phi)$ has the decomposition $E_{1\phi}+\cdots+E_{s\phi}+C_\phi$. If $r=s$ and there is an isomorphism $\alpha: S_{r\varphi}\to S_{s\phi}$ such that $\alpha(C_\varphi)=C_\phi$ and $\alpha(E_{i\varphi})=E_{i\phi}$ for all i, where $1\leqq i\leqq r$, then $\Gamma(\varphi)$ is said to be *equivalent* to $\Gamma(\phi)$. Note that α preserves the orders of the components. Let this equivalence be denoted by $\approx$. This equivalence satisfies the axiom of equivalence relation.

REMARK 4.4. If φ and ϕ belong to $\mathcal{G}\setminus\mathcal{L}$, then the following hold true.

(1) $E_{r\varphi}^2=C_\varphi^2=-1$ and $E_{i\varphi}^2\leqq -2$ where $1\leqq i\leqq r-1$.

(2) In order to check that the isomorphism α in Definition 4.3 gives the equivalence, it suffices to verify $\alpha(\Gamma(\varphi))=\Gamma(\phi)$ and $\alpha(C_\varphi)=C_\phi$ [or $\alpha(E_{r\varphi})=E_{r\phi}$].

PROOF. The first assertion is obtained from (ii) in Lemma 4.2 and the

Castelnuovo's criterion. In the second assertion we have that $\alpha(C_{\varphi})=C_{\psi}$ and $\alpha(E_{r\varphi})=E_{r\psi}$, since α preserves the weights of the components. By the blow-down $\sigma_{r\varphi}$ the isomorphism α defines also the isomorphism $\alpha_{r-1}: S_{r-1\,\varphi}\to S_{r-1\,\psi}$ such that $\alpha_{r-1}\sigma_{r\varphi}=\sigma_{r\psi}\alpha$. Since $\sigma_{r\varphi}(E_{i\varphi})$ and $\sigma_{r\psi}(E_{i\psi})$ have the weight -1 if and only if $i=r-1$, we have that $\alpha_{r-1}(\sigma_{r\varphi}(E_{r-1\,\varphi}))=\sigma_{r\psi}(E_{r-1\,\psi})$, i.e., $\alpha(E_{r-1\,\varphi})=E_{r-1\,\psi}$. In this way we complete the proof by induction. Q.E.D.

The following lemma is trivial, so its proof is omitted.

LEMMA 4.5. *We have that* $\Gamma(\varphi)\approx\Gamma(l\varphi)\approx\Gamma(\varphi l)$, *where* $\varphi\in\mathcal{G}\setminus\mathcal{L}$ *and* $l\in\mathcal{L}$.

Since $\varphi\sigma$ is a morphism, it can be expressed as $\sigma'\tilde{\varphi}$, applying the same factorization to φ^{-1}, where $\sigma'=\sigma_1'\cdots\sigma_r'$ and $\tilde{\varphi}: S_r\to S_r'$ is an isomorphism. In this expression σ_i' is a blow-up $S_i'\to S_{i-1}'$, where $1\leqq i\leqq r$ and $S_0'=\boldsymbol{P}^2$. Note that S_r, S_r' and $\tilde{\varphi}$ are determined uniquely by φ owing to (A_2).

DEFINITION 4.6. The above isomorphism $\tilde{\varphi}$ is called the *resolution* of φ.

Here we have that $\Gamma(\varphi^{-1})=\sigma'^{-1}(C)=E_{1\,\varphi^{-1}}+\cdots+E_{r\,\varphi^{-1}}+C_{\varphi^{-1}}$ and $\tilde{\varphi}(\Gamma(\varphi))=\Gamma(\varphi^{-1})$.

REMARK 4.7. (1) If φ belongs to $\mathcal{G}\setminus\mathcal{L}$, then $\tilde{\varphi}(E_{r\varphi})=C_{\varphi^{-1}}$ and $\tilde{\varphi}(C_{\varphi})=E_{r\varphi^{-1}}$.

(2) For two elements φ_1 and φ_2 of $\mathcal{G}\setminus\mathcal{L}$ we have that $\Gamma(\varphi_1)\approx\Gamma(\varphi_2)$ if and only if $\Gamma(\varphi_1^{-1})\approx\Gamma(\varphi_2^{-1})$.

PROOF. The first assertion follows from (2) in Remark 4.4. Let $\tilde{\varphi}_i$ be the resolution of φ_i, where $i=1, 2$, and $\alpha: S_{r\varphi_1}\to S_{r\varphi_2}$ give the equivalence between $\Gamma(\varphi_1)$ and $\Gamma(\varphi_2)$. Then from the same assertion (2) in Remark 4.4 we see that $\tilde{\varphi}_2\alpha\tilde{\varphi}_1^{-1}$ gives the equivalence between $\Gamma(\varphi_1^{-1})$ and $\Gamma(\varphi_2^{-1})$. The converse is proved similarly. Q.E.D.

First contract $\tilde{\varphi}^{-1}(E_{r\varphi^{-1}})=C_{\varphi}$, secondly $\tilde{\varphi}^{-1}(E_{r-1\,\varphi^{-1}})$, and so on. In this way, by using $\tilde{\varphi}$, we have the composition $\sigma''=\sigma_1''\cdots\sigma_r''$ of blow-downs $\sigma_i'': S_i''\to S_{i-1}''$ such that $S_r''=S_r$, $S_0''=S''$ and $\sigma''\tilde{\varphi}^{-1}=l\sigma'$, where $1\leqq i\leqq r$ and l is an isomorphism $\boldsymbol{P}^2\to S''$. Then we have that $\varphi\sigma=l^{-1}\sigma''$. Note that l is not necessarily the identity mapping and that the components of $\Gamma(\varphi)$ define two morphisms σ and σ''.

DEFINITION 4.8. The above morphism σ'' is called the *associate* of σ.

REMARK 4.9. Since $\sigma''(E_r)=l(C)$, the curve E_r is not contracted by the associate of σ.

LEMMA 4.10. *Let* φ_1 *and* φ_2 *be two elements of* $\mathcal{G}\setminus\mathcal{L}$. *Then there are two elements* l_1 *and* l_2 *of* $\mathcal{L}$ *satisfying* $\varphi_2=l_2\varphi_1 l_1$ *if and only if* $\Gamma(\varphi_1)\approx\Gamma(\varphi_2)$.

PROOF. The "only if" part is an easy consequence of Lemma 4.5. Suppose that $\Gamma(\varphi_1)\approx\Gamma(\varphi_2)$ and α is the isomorphism $S_{r\varphi_1}\to S_{r\varphi_2}$ defining its equivalence. Then α induces an element l of $\mathcal{L}$ which satisfies $\sigma_{1\varphi_2}\cdots\sigma_{r\varphi_2}\alpha=l\sigma_{1\varphi_1}\cdots\sigma_{r\varphi_1}$, where $\sigma_{i\varphi_1}$ and $\sigma_{i\varphi_2}$ are blow-ups, $1\leqq i\leqq r$. Let $\tilde{\varphi}_i$ be the resolution of φ_i,

$$
\begin{tikzcd}
& S_{r\varphi_1^{-1}} \arrow[rr] \arrow[ddd] & & S_{r\varphi_2^{-1}} \arrow[ddd, dashed] \\
S_{r\varphi_1} \arrow[ur, "\tilde{\varphi}_1"] \arrow[rr, "\alpha"] \arrow[d, "\sigma_{r\varphi_1}"'] & & S_{r\varphi_2} \arrow[ur, "\tilde{\varphi}_2"] \arrow[d, "\sigma_{r\varphi_2}"'] & \\
\vdots \arrow[dd, "\sigma_{1\varphi_1}"'] & & \vdots \arrow[dd, "\sigma_{1\varphi_2}"'] & \\
& \boldsymbol{P}^2 \arrow[rr] & & \boldsymbol{P}^2 \\
\boldsymbol{P}^2 \arrow[ur, "\varphi_1"] \arrow[rr, "l"'] & & \boldsymbol{P}^2 \arrow[ur, "\varphi_2"'] &
\end{tikzcd}
$$

where $i=1, 2$. Since $\alpha(C_{\varphi_1})=C_{\varphi_2}$, $\alpha(E_{r\varphi_1})=E_{r\varphi_2}$, $\tilde{\varphi}_i(C_{\varphi_i})=E_{r\varphi_i^{-1}}$ and $\tilde{\varphi}_i(E_{r\varphi_i})=C_{\varphi_i^{-1}}$, where $i=1, 2$, it follows that $\tilde{\varphi}_2\alpha\tilde{\varphi}_1^{-1}$ preserves the orders of the irreducible components of $\Gamma(\varphi_1^{-1})$ and $\Gamma(\varphi_2^{-1})$ by the assertion (2) in Remark 4.4. Hence we see that the isomorphism $\tilde{\varphi}_2\alpha\tilde{\varphi}_1^{-1}$ induces an isomorphism $\varphi_2 l\varphi_1^{-1}$ in $\mathcal{L}$. Q.E.D.

Let $\mathcal{C}$ be the set consisting of the equivalence class of the graphs of elements of $\mathcal{G}\setminus\mathcal{L}$, i.e.,

$$\mathcal{C}=\{\Gamma(\varphi)\mid\varphi\in\mathcal{G}\setminus\mathcal{L}\}/\approx .$$

Then the above lemma implies the following

COROLLARY 4.11.

$$\mathcal{G}\setminus\mathcal{L}=\bigcup_{\Gamma(\varphi)\in\mathcal{C}}\mathcal{L}\varphi\mathcal{L} .$$

Especially this means the following

LEMMA 4.12. *If the number of elements of $\mathcal{L}$ and that of $\mathcal{C}$ are both finite, then so is $\mathcal{G}$.*

Now, let us proceed to the proof of Theorem A. We shall find step by step the several conditions on which $\mathcal{G}$ is a finite group. Recalling the condition (A_1), we put

$$p=\min\{i\mid C_i \text{ is smooth}\}.$$

Then we have that $C_p^2\geqq-1$. If $j\geqq i$ and $C_i^2=-1$, then Q_{j+1} does not lie on C_j, hence we put

$$q=\min\{i\mid C_i^2=-1\}.$$

Note that the number q does not depend on φ and that $C_q=C_r$, where $r=r(\varphi)$. Since $C_p^2\geqq-1$, we have that $C_{p-1}^2\geqq3$, i.e., $q\geqq p$.

LEMMA 4.13. *If the intersection number $C_q\cdot E_q\geqq2$, then $p=q=r(\varphi)$ for all φ in $\mathcal{G}\setminus\mathcal{L}$.*

PROOF. Since $C_q\cdot E_q\geqq2$, the curve C_{q-1} is not smooth, i.e., we have that $q=p$. Suppose that $r>q$ for some φ, where $r=r(\varphi)$. Then the fundamental point of $\varphi\sigma_1\cdots\sigma_q$ exists on $E_q\setminus C_q$, since $C_q^2=-1$ and C_q has to be contracted.

When C_q is contracted, the image of E_q is not smooth. By Remark 4.9 the curve E_r is not contracted, hence E_q has to be contracted, since $q \neq r$. This is a contradiction. Q.E.D.

LEMMA 4.14. *If $q=p$, then $\mathcal{G}$ is a finite group.*

PROOF. Since $q=p$, we have that $C_q \cdot E_q \geqq 2$, then by Lemma 4.13 $\varphi\sigma_1 \cdots \sigma_p$ is a morphism for all $\varphi \in \mathcal{G} \setminus \mathcal{L}$. Since Q_i coincides with the singular point of C_{i-1} by Lemma 4.2, the blow-up σ_i does not depend on φ, where $1 \leqq i \leqq p$. Hence we see that $\mathcal{C}$ is a finite set (in fact it consists of at most one element). Since the order of $\mathcal{L}$ is finite by Proposition 1, so is $\mathcal{G}$ by Lemma 4.12. Q.E.D.

In view of the above lemma we may treat only the curves satisfying the following condition (A_3) hereafter.

$$(A_3): \qquad q > p.$$

LEMMA 4.15. *If E_p and C_p satisfy either one of the following conditions, then $\mathcal{G}=\mathcal{L}$ and it is a finite group.*

(i) *$E_p \cap C_p$ consists of at least three points.*

(ii) *$E_p \cap C_p$ consists of two points such that E_p and C_p meet transversally at none of them.*

PROOF. Suppose that $\mathcal{G} \neq \mathcal{L}$ and take an element φ of $\mathcal{G} \setminus \mathcal{L}$. Then by the condition (A_3) $\varphi\sigma_1 \cdots \sigma_p$ has one fundamental point Q_{p+1} in $E_p \cap C_p$. After C_r, $r=r(\varphi)$, is contracted, the image of E_p is not smooth on the condition of this lemma. Since $r \geqq q > p$ and E_r is not contracted, the curve E_p has to be contracted by the associate of σ. This is a contradiction. Of course $\mathcal{L}$ is a finite group by Proposition 1. Q.E.D.

In view of the above lemma we may treat only the curves satisfying either one of the following conditions (1) and (2) hereafter.

(A_4)
(1) $E_p \cap C_p$ consists of one point.
(2) $E_p \cap C_p$ consists of two points $\{Q', Q''\}$ such that E_p and C_p meet transversally at Q' or Q''.

LEMMA 4.16. *If the total transform $(\sigma_1 \cdots \sigma_q)^{-1}(C)$ has not normal crossings, then we have that $r(\varphi)=q$ for every φ in $\mathcal{G} \setminus \mathcal{L}$, and hence $\mathcal{G}$ is a finite group.*

PROOF. First of all, on the condition (A_4) the possibilities for the centers of σ_{p+1} for every φ in $\mathcal{G} \setminus \mathcal{L}$ are at most two, while Q_j coincides with $E_{j-1} \cap C_{j-1}$ if $p+2 \leqq j \leqq q$. Thus Q_j depends on Q_{p+1}, where $p+2 \leqq j \leqq q$. Suppose that $r(\varphi) > q$ for some φ in $\mathcal{G} \setminus \mathcal{L}$. Then Q_{q+1} lies in $E_q \setminus C_q$, hence there is some such that $C_q \cdot E_i \geqq 2$ or C_q, E_i and E_j meet at one point, where $i, j \leqq q$. After the contraction of C_q the image of E_i has a singular point or E_i and E_j do not meet transversal. Hence there is a curve not contracted besides E_r. This is a contradiction. Thus we have that $r(\varphi)=q$ for every φ in $\mathcal{G} \setminus \mathcal{L}$. In view of the first consideration we conclude that C is a finite set (in fact it consists of at most two elements). Since, for the curve Δ_1, we have that $q=2d+1$ and that $(\sigma_1 \cdots \sigma_q)^{-1}(\Delta_1)$ has normal crossings, the linear part $\mathcal{L}$ has a finite order. Hence $\mathcal{G}$ is a finite group by Lemma 4.12. Q.E.D.

COROLLARY 4.17. *If $N=1$ and $R\leqq -1$, then $(\sigma_1 \cdots \sigma_q)^{-1}(C)$ has not normal crossings and hence $\mathcal{G}$ is a finite group.*

PROOF. Note that C_{p+j}, E_p and E_{p+j} meet at one point, where $1\leqq j\leqq e_p-1$. Since $C^2_{p+j}=R+e_p-1-j$, we have that $q=p+e_p+R$, hence $p+1\leqq q\leqq p+e_p-1$. This implies that $(\sigma_1 \cdots \sigma_q)^{-1}(C)$ has not normal crossings. Q.E.D.

We have just proved the assertion (3) in Theorem A. In view of Lemma 4.16 we may treat only the curves satisfying the following condition (A_5) hereafter.

(A_5) : $(\sigma_1 \cdots \sigma_q)^{-1}(C)$ has normal crossings.

Moreover the following lemma holds true.

LEMMA 4.18. *If $N\geqq 3$, then $\mathcal{G}$ is a finite group.*

PROOF. Suppose that $N\geqq 3$ and take an element φ of $\mathcal{G}\setminus\mathcal{L}$. Then C_r and $E_1+\cdots+E_r$ meet in at least 3 points, but E_r and $E_1+\cdots+E_{r-1}$ meet in at most 2 points, and $E_r=\tilde{\varphi}^{-1}(C_{\varphi-1})$ meets at least three other components of $\Gamma(\varphi)$. Hence E_r must meet C_r. This means that $r(\varphi)=q$. As was shown in the proof of Lemma 4.16, the possibilities for the centers of blow-ups are at most two, so that the order of $\mathcal{G}$ is finite. Q.E.D.

From the above results we may treat only the curves satisfying one of the following conditions hereafter.

(i) $N=2$.

(ii) $N=1$ and $R\geqq 0$.

In this section only the case (i) is considered. The other one will be treated in the next section.

In what follows the graphs will be represented as figures, where the following abbreviation will be used. The number r indicates the rank $r(\varphi)$ and a positive integer i beside a component indicates the curve E_i and a non-positive integer j beside a component indicates the weight j of the component and a component without a non-positive integer has the weight -2. Since we treat only the case of normal crossings, we often adopt *dual graphs,* where the symbols ○ and ● indicate the components whose weights are -2 and not -2 respectively.

DEFINITION 4.19. If a divisor on some surface S admits two ways of contractions $S\to\boldsymbol{P}^2$ such as σ and σ'', then it is said to be *contractible.* Of course the graph $\Gamma(\varphi)$ for φ in $\mathcal{G}\setminus\mathcal{L}$ is contractible.

DEFINITION 4.20. By using the resolution $\tilde{\varphi}$ of φ, we define a permutation

φ^* of the set $\{E_{1\varphi}, \cdots, E_{r\varphi}, C_\varphi\}$ as follows:

$$\varphi^*(C_\varphi)=E_{r\varphi}, \quad \varphi^*(E_{i\varphi})=\tilde{\varphi}^{-1}(E_{i\varphi-1}), \qquad \text{where} \quad 1\leqq i\leqq r.$$

Let us call φ^* the *permutation* of $\Gamma(\varphi)$ and denote by $D(\varphi)$ the divisor consisting of all the components of $\Gamma(\varphi)$ that are fixed by φ^*. In case all the components are moved, then we put $D(\varphi)=0$. Let us call $D(\varphi)$ the *center* of $\Gamma(\varphi)$. If φ^* has the order 2, then we may regard $\Gamma(\varphi)$ as being symmetrical about $D(\varphi)$.

Now, let us study the curves with $N=2$.

DEFINITION 4.21. A divisor $D=\sum_{i=1}^n D_i$ is said to contain a *loop* if a subset $\bigcup_{j\in I} D_j$ forms a closed path for some $I\subset\{1, 2, \cdots, n\}$. The divisor D is said to form a *simple loop* if the following two conditions are satisfied:

(1) D has normal crossings.

(2) Every irreducible component is nonsingular and rational and intersects others at two points.

Note that a divisor consists of just one loop if it forms a simple loop.

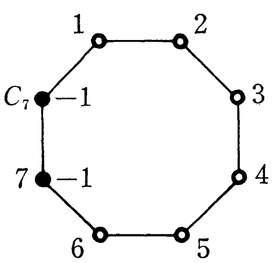
LEMMA 4.22. *The following three conditions are equivalent.*

(1) $\mathcal{G}\neq\mathcal{L}$ *and* $\Gamma(\varphi)$ *forms a simple loop for some* φ *in* $\mathcal{G}\setminus\mathcal{L}$.

(2) *The singular point* P_1 *of* C *is a node, i.e.,* $e(P_1, C)=N=2$ *and* $p=1$.

(3) $C\sim\Delta$.

PROOF. If $\Gamma(\varphi)$ forms a simple loop, then by contracting the components of $\Gamma(\varphi)$ we see that P_1 turns out to be a node. Next we assume that P_1 is a node. Since $g=0$, we have that $d=3$ by the genus formula for plane curves, which means that $C\sim\Delta$. Finally we assume that $C\sim\Delta$. Then by successive blow-ups we get the following figure.



In this figure, first contract C_7, secondly E_1, and so on. Eventually we get a curve which is projectively equivalent to Δ. From these blow-ups and blow-downs we obtain an element φ of $\mathcal{G}\setminus\mathcal{L}$. Q.E.D.

Let us study the case (i) by examining the following cases separately.

(i-1) The curve C is not projectively equivalent to Δ.

(i-2) The curve C is projectively equivalent to Δ.

First let us consider the case (i-1).

Then by the above lemma we have that $\mathcal{G}=\mathcal{L}$ or $\Gamma(\varphi)$ does not form a simple loop for all φ in $\mathcal{G}\setminus\mathcal{L}$. In the former case $\mathcal{G}$ has a finite order owing to Proposition 1 and the assumption that $M=1$ and $N=2$. Hence let us treat only the latter case. Put $\Gamma_i=(\sigma_1\cdots\sigma_i)^{-1}(C)$, where $i\geqq q$. By the condition (A_5) Γ_i has normal crossings. The divisor $E_1+\cdots+E_i$ is connected and does not contain a loop, but it connects two points in C_i. Hence Γ_i contains just a simple loop, to which C_i belongs. We denote it by Γ_i^0.

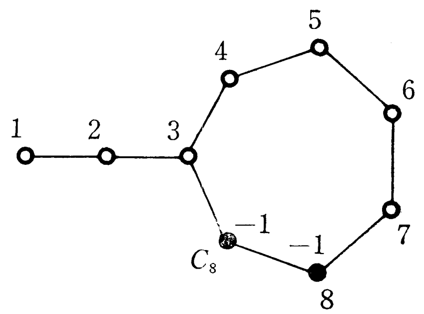
LEMMA 4.23. *If $\varphi\sigma_1\cdots\sigma_i$ has a fundamental point, where $i\geqq q$, then the center Q_{i+1} of σ_{i+1} is not a free point on E_i, hence E_{i+1} is a component of $\Gamma_r^0=\Gamma(\varphi)^0$. The curve E_q is a component of Γ_q^0, hence Γ_r^0 is obtained from Γ_q^0 by successive blow-ups.*

PROOF. Since $\Gamma(\varphi)^0$ [resp. $\Gamma(\varphi^{-1})^0$] is the unique simple loop contained in $\Gamma(\varphi)$ [resp. $\Gamma(\varphi^{-1})$], the resolution $\tilde{\varphi}$ maps $\Gamma(\varphi)^0$ onto $\Gamma(\varphi^{-1})^0$. As is mentioned above $\Gamma(\varphi)^0$ [resp. $\Gamma(\varphi^{-1})^0$] contains C_φ [resp. $C_{\varphi^{-1}}$], hence $\Gamma(\varphi)^0$ contains also $\tilde{\varphi}^{-1}(C_{\varphi^{-1}})=E_{r\varphi}$. Since Γ_q^0 has normal crossings, and E_q and C_q meets, and $N=2$, the curve E_q is a component of Γ_q^0. The above consideration proves the lemma. Q.E.D.

Since $\Gamma(\varphi)$ does not form a simple loop but contains the one Γ_r^0, there is a component E_k of Γ_r^0 which meets at least three other components. (For example, see the following figure, though it is not contractible, obtained from the curve

$$(y-x^2)^2+tx^2y^2+xy^3=0\,,\qquad \text{where}\quad t\neq 0.$$

In this case $k=3$.)



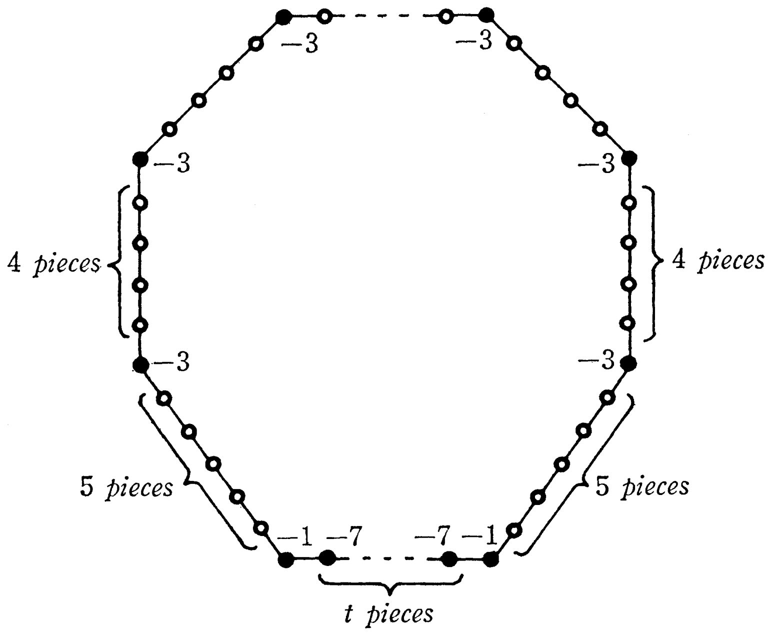
By Lemma 4.23 we have that $k\leqq q$, in other words, for any $i>q$, E_i is a component of Γ_r^0. Taking note of the position of E_k, we infer that the possibilities of graphs which represent automorphisms are finite. To explain this in detail, let us consider the one route in Γ_r^0 that connects C_r and E_k and that does not pass through E_r. Then let h be the sum of the weights of the divisors in the route, i.e., $h=C_r^2+\cdots+E_k^2$. For the sake of simplicity, let us take E_k that gives the minimum value for $-h$. Of course h is determined uniquely by C and is independent of φ. Suppose that $-h\ll r$. Then there are a great many components in the other route, because the center $Q_{q+1}\in E_q\setminus C_q$ and we have the facts in Lemma 4.23. Now, let us contract the graph from C_r, then, after the contraction of E_k, we will have a divisor with not normal crossings, but the number of its components will be still more than $q+1$, since $-h \ll r$. This is a contradiction, hence we have the conclusion mentioned above. Thus $\mathcal{C}$ consists of finitely many elements, i.e., $\mathcal{G}$ is a finite group.

Putting together all the results obtained above, we complete the proof of (2) in Theorem A.

Next let us consider the case (i-2).

Since C is projectively equivalent to Δ, the linear part $\mathcal{L}$ is the dihedral group of order 6. Let us consider non-linear elements.

LEMMA 4.24 (Wakabayashi). *For every element φ of* $\mathrm{Aut}(\boldsymbol{P}^2-\Delta)\setminus\mathcal{L}$, *the graph $\Gamma(\varphi)$ forms a simple loop and its figure is as follows.*



In this figure t is a non-negative integer determined by φ and the number of components of $\Gamma(\varphi)$ is $8+6t$, i.e., $r(\varphi)=6t+7$. Conversely, such a figure yields an element of $\mathrm{Aut}(\boldsymbol{P}^2-\Delta)$.

PROOF. We perform successive blow-ups of $\boldsymbol{P}^2$ in order that the total transform of Δ may become contractible. In the proof of Lemma 4.22, we have obtained that $q=7$ and the figure with $t=0$. Then, noting that Lemma 4.23 is also applicable to this case, we continue blow-ups if there is still a fundamental point. The center Q_8 must coincide with $E_6 \cap E_7$, and Q_9 with $E_7 \cap E_8$, and so on. When the weight of the proper transform of E_7 becomes -7, we may stop the blow-ups and get the contractible divisor with $t=1$ in the above figure. In case there is still a fundamental point, the center Q_{14} must coincide with $E_{12} \cap E_{13}$ and we proceed similarly. Finally we get all contractible divisors by such a manner, which are illustrated as above. Conversely, the above divisor is contractible and two curves which are images of the divisor are projectively equivalent. Hence from this divisor we get an element of $\mathcal{G} \setminus \mathcal{L}$. Q. E. D.

This lemma shows that the centers of blow-ups are always the intersection points of two curves, hence the order of $\mathcal{G}$ is countably infinite. Thus the assertion (5) in Theorem A is proved.

Concerning the assertion (2) in Theorem A we raise the following conjecture, which is true for all examples we now have.

CONJECTURE. Suppose that $N \geqq 2$ and that C is not projectively equivalent to Δ. Then $\mathcal{G} = \mathcal{L}$.

5. Curves with $g=0$, $N=1$ and $R \geqq 0$.

In this section also we follow the notations fixed in the previous sections and assume that C is a curve with $g=0$, $N=1$ and $R \geqq 0$. Recall that (A_2) is always assumed, so C satisfies all the conditions (A_i), $1 \leqq i \leqq 5$.

DEFINITION 5.1. Let D_i be a component of a divisor $D = \sum_{i=1}^{n} D_i$. Then a point Q on D_i is called a *free point* on D_i if Q does not belong to any other component D_j, where $j \neq i$. The divisor D is called a *zigzag* (or a divisor of type A_n) if its components are nonsingular and rational and it has the following expression:

$$D_i \cdot D_j = \begin{cases} -2 & \text{if } i=j. \\ 1 & \text{if } |i-j|=1, \\ 0 & \text{if } |i-j| \geqq 2. \end{cases} \qquad \text{where } 1 \leqq i, j \leqq n.$$

In the figures of graphs we sometimes use a dotted line which represents a zigzag. When we try to find the graphs of elements of $\mathcal{G} \setminus \mathcal{L}$, we shall make good use of the following facts freely.

REMARK 5.2. (1) The center Q_{i+1} of σ_{i+1} is contained in E_i, where $1 \leqq i \leqq r-1$.

(2) The graph $\Gamma(\varphi)$ is symmetrical about $D(\varphi)$ if φ^* has the order 2.

(3) There exists the associate σ'' of σ, i.e., a contraction can be started from C_r. Moreover the following fact holds true, which will not be used explicitly. Put $\Gamma_i'' = \sigma_{i+1}'' \cdots \sigma_r''(\Gamma(\varphi)) = (\sigma_1'' \cdots \sigma_i'')^{-1}(l(C))$. Then Γ_i'' contains one or two components with the weight -1. If Γ_i'' contains two such components, then $i \geqq q$, hence it has normal crossings.

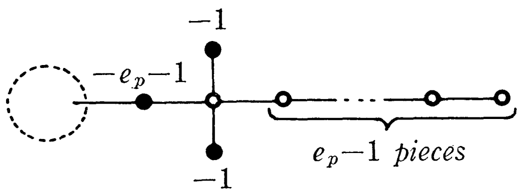
Then by considering blow-ups we get the following

PROPOSITION 5.3. *Suppose that $R=0$ and $\mathcal{G} \neq \mathcal{L}$. Then for every φ in $\mathcal{G} \setminus \mathcal{L}$*

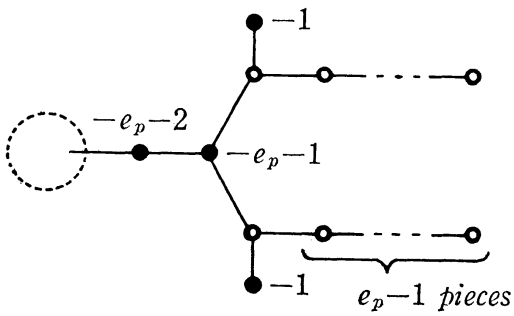
we have $q=p+e_p$. *Suppose moreover that* φ^* *has the order* 2 *and that the center* $D(\varphi)$ *of* $\Gamma(\varphi)$ *contains* $E_1+\cdots+E_{p-1}$. *Then the figure of* $\Gamma(\varphi)$ *is* (I) *or* (II) *in the following according as* $e_{p-1}=e_p$ *or* $e_{p-1}>e_p$. *In these figures the dotted circle represents the divisor* $E_1+\cdots+E_{p-1}$.

(I) *The case* $e_{p-1}=e_p$ *is separated into three subcases.*

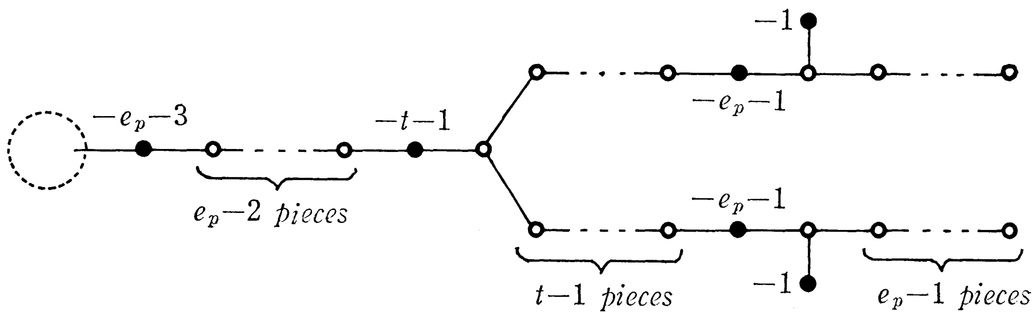
Subcase (a), $r=q+1$.



Subcase (b), $r=q+e_p+2$.

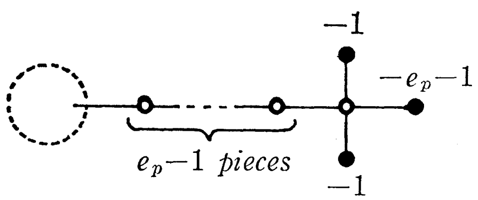


Subcase (c), $r=q+2e_p+2t+1$, *where* t *is an integer* $\geqq 1$.

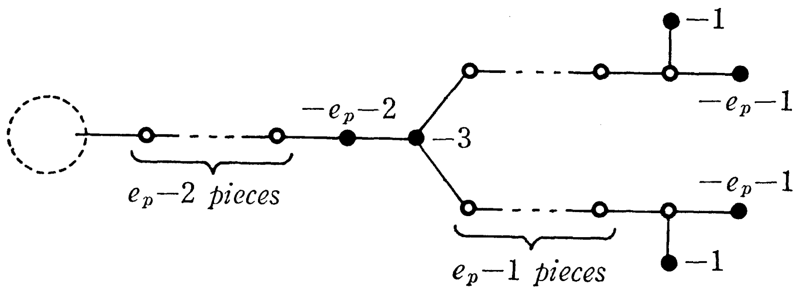


(II) *The case* $e_{p-1}>e_p$ *is also separated into three subcases.*

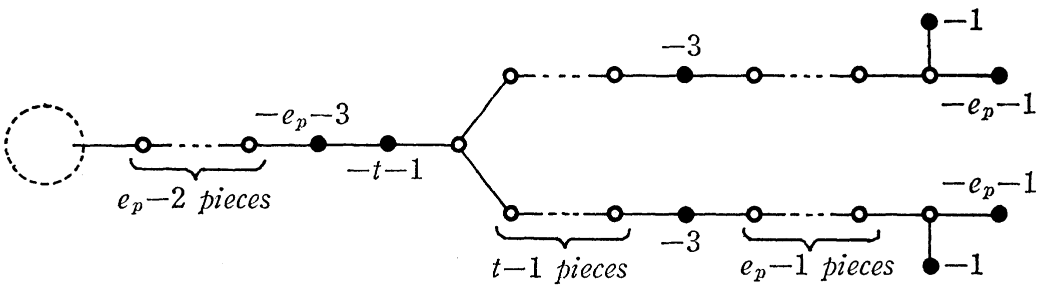
Subcase (a), $r=q+1$.



Subcase (b), $r=q+2e_p+2$.

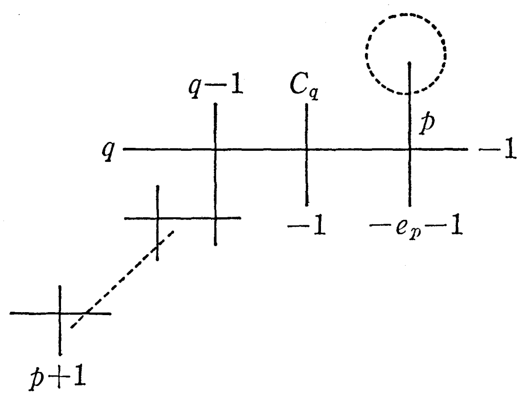


Subcase (c), $r=q+2e_p+2t+3$, *where* t *is an integer* $\geqq 1$.



PROOF. Since $N=1$ and $d^2-\sum_{i=1}^{p}e_i^2-e_p=-1$, we have that $q=p+e_p$ and that the center Q_{i+1} of the blow-up σ_{i+1} coincides with the point $E_i \cap C_i$ if $i \leqq q-1$.

First let us take up the case (I). In this case the figure of $\Gamma_q=(\sigma_1 \cdots \sigma_q)^{-1}(C)$ is as follows, since C_p meets only E_p.



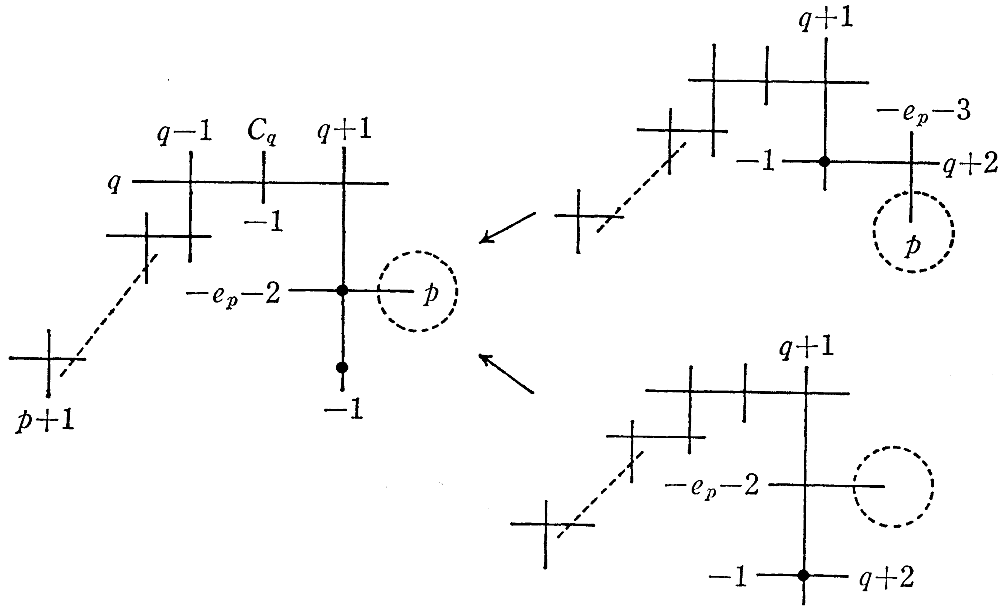
Note that Γ_q is independent of φ and φ^* is assumed to have the order 2. Hence by the hypothesis the divisor $E_1+\cdots+E_p$ is contained in the center $D(\varphi)$. We prove first $\{Q_{q+1}\}\neq E_q\cap E_{q-1}$. Suppose the contrary. Then we have that $\varphi^*(E_q)=E_q$. Since $\varphi^*(C_q)=E_r$, the divisor E_r must meet E_q. Then $\Gamma(\varphi)$ cannot be symmetrical about $D(\varphi)$. This is a contradiction. Thus we have that $\{Q_{q+1}\}\neq E_q\cap E_{q-1}$. Hence let us study this case (I) by examining the following cases separately.

(1) Q_{q+1} is a free point on E_q.

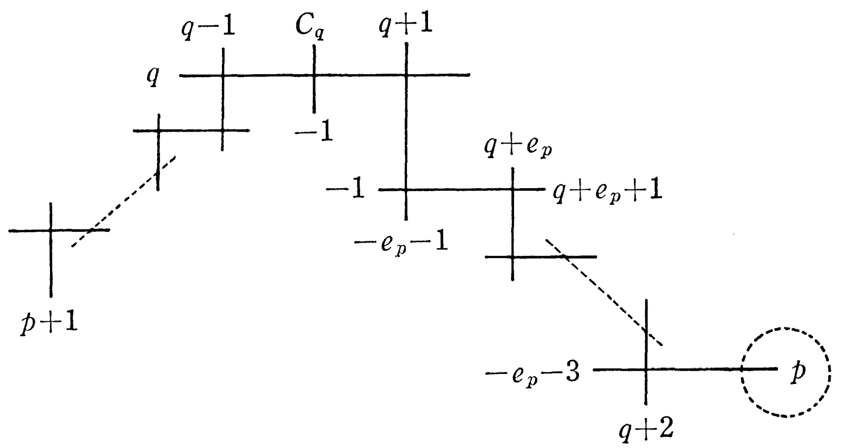
(2) $\{Q_{q+1}\}=E_p\cap E_q$.

In the first case (1) the divisor $E_1+\cdots+E_p+E_q$ is contained in the center $D(\varphi)$. Since $\varphi^*(C_q)=E_r$, we get $r=q+1$ and the figure of subcase (a).

In the second case (2) the center Q_{q+2} is a free point on E_{q+1} or coincides with the point $E_p\cap E_{q+1}$. Since the contraction can be started from C_q, the weight of the proper transform of E_{q+1} must be $-e_p-1$. Now, in the former case, taking note that $\Gamma(\varphi)$ is symmetrical about $E_1+\cdots+E_p+E_{q+1}$, we get the figure of subcase (b).

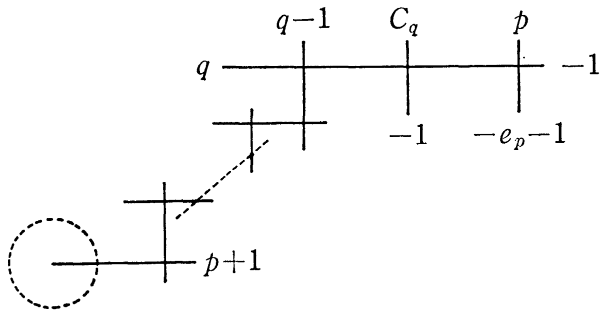


On the other hand, in the latter case, we get the following figure.



Then Q_{q+e_p+2} is a free point on E_{q+e_p+1} or coincides with the point $E_{q+e_p+1} \cap E_{q+e_p}$. In the former case, taking note that $\Gamma(\varphi)$ is symmetrical about $E_1 + \cdots + E_p + E_{q+2} + \cdots + E_{q+e_p+1}$, we get the figure (c) with $t=1$. In the latter case, Q_{q+e_p+3} is a free point on E_{q+e_p+2} or coincides with the point $E_{q+e_p} \cap E_{q+e_p+2}$. Then we get the figure (c) with $t=2$ by the same reason as above or we proceed similarly. In this way we get the figures of subcase (c). From the above procedure we see that every possible case is exhausted.

Next let us take up the case (II). Since $C_{p-1} \cdot E_{p-1} = e_{p-1} > e_p$, the curve E_{p-1} is a tangent to C_{p-1}, hence C_p, E_p and E_{p-1} meet at Q_{p+1}. So that the figure of $\Gamma_q = (\sigma_1 \cdots \sigma_q)^{-1}(C)$ is as follows.

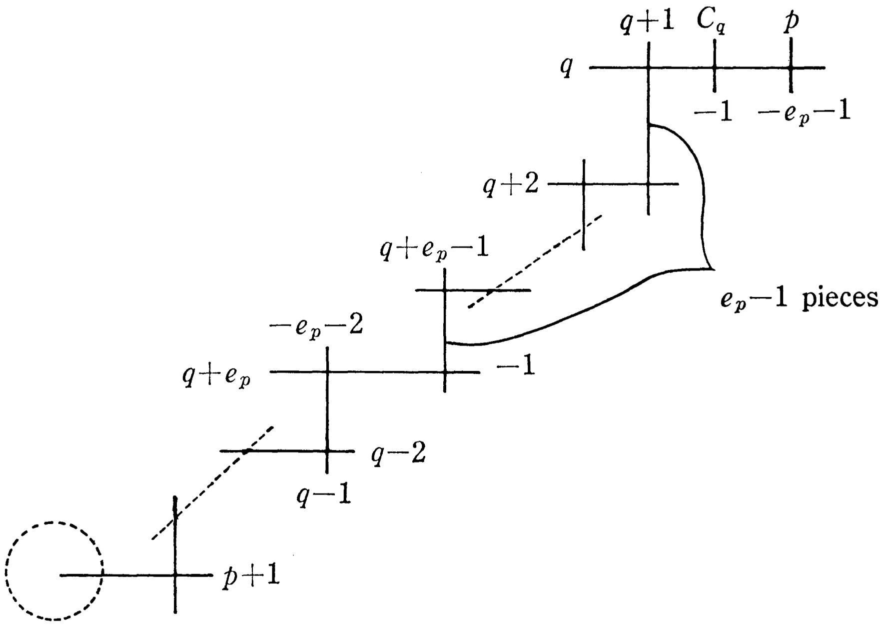


Note that Γ_q is independent of φ and φ^* is assumed to have the order 2. Hence by the hypothesis the divisor $E_1 + \cdots + E_{p-1} + E_{p+1} + \cdots + E_{q-1}$ is contained in the center $D(\varphi)$. Then we conclude similarly that $\{Q_{q+1}\} \neq E_p \cap E_q$. Hence let us study this case (II) by examining the following cases separately.

(1) Q_{q+1} is a free point on E_q.

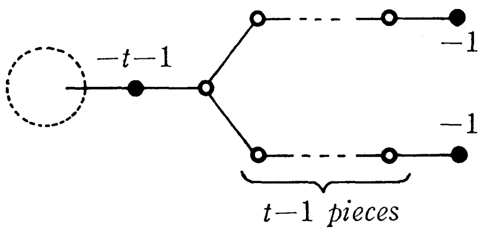
(2) $\{Q_{q+1}\} = E_{q-1} \cap E_q$.

In the first case (1) E_q is also contained in the center $D(\varphi)$. Since $\varphi^*(C_q) = E_r$, we get $r=q+1$ and the figure of subcase (a). In the second case (2) the weight of the proper transform of E_{q-1} must be $-e_p-2$, since the contraction can be started from C_q and E_p must be contracted. Then we get the following figure.

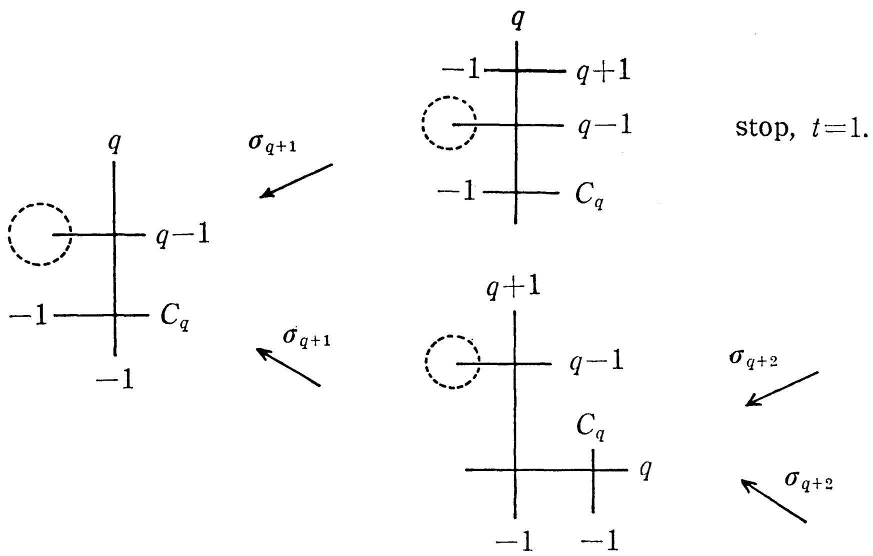


Similarly, since E_p must be contracted, we have that $\{Q_{q+e_p+1}\} \neq E_{q+e_p} \cap E_{q+e_p-1}$. Then Q_{q+e_p+1} is a free point on E_{q+e_p} or coincides with the point $E_{q-1} \cap E_{q+e_p}$. In the former case, taking note that $\Gamma(\varphi)$ is symmetrical about $E_1+\cdots+E_{p-1}+E_{p+1}+\cdots+E_{q-1}+E_{q+e_p}$, we get $r=q+2e_p+2$ and the figure of subcase (b). In the latter case, since E_p must be contracted, we have that $\{Q_{q+e_p+2}\}=E_{q+e_p} \cap E_{q+e_p+1}$. Then similarly $\{Q_{q+e_p+3}\} \neq E_{q+e_p} \cap E_{q+e_p+2}$, i.e., Q_{q+e_p+3} is a free point on E_{q+e_p+2} or coincides with $E_{q+e_p+1} \cap E_{q+e_p+2}$. In the former case, taking note that $\Gamma(\varphi)$ is symmetrical, we get the figure (c) with $t=1$. In the latter case Q_{q+e_p+4} is a free point on E_{q+e_p+3} or coincides with $E_{q+e_p+1} \cap E_{q+e_p+3}$. Then we get the figure (c) with $t=2$ by the same reason as above or proceed similarly. In this way we get the figures of subcase (c). From the above procedure we see that every possible case is exhausted. Q.E.D.

PROPOSITION 5.4. *Suppose that $R \geqq 1$ and $\mathcal{G} \neq \mathcal{L}$. Then for every φ in $\mathcal{G} \setminus \mathcal{L}$ we have that $q=p+e_p+R$. Suppose moreover that φ^* has the order 2 and the center $D(\varphi)$ of $\Gamma(\varphi)$ contains $E_1+\cdots+E_{q-2}$. Then $r=q+2t-1$, where t is a positive integer determined by φ, and the figure of $\Gamma(\varphi)$ is as follows, where the dotted circle represents the divisor $E_1+\cdots+E_{q-2}$.*



PROOF. By Lemma 4.2 we have that $\{Q_i\}=E_{i-1}\cap C_{i-1}$ for $i\leqq q$. Since $C_q^2=-1$, the center Q_{q+1} lies in $E_q\setminus C_q$. First we consider the case when Q_{q+1} is a free point on E_q. Then $E_1+\cdots+E_q$ is the center $D(\varphi)$, since $E_1, \cdots, E_{q-2}$ are contained in the center, hence we get $r=q+1$ and the figure with $t=1$.



Next we consider the case when $\{Q_{q+1}\}=E_{q-1}\cap E_q$. Then we see that Q_{q+2} lies in $E_{q+1}\setminus E_q$. In case Q_{q+2} is a free point on E_{q+1}, then $E_1+\cdots+E_{q-1}+E_{q+1}$ is the center $D(\varphi)$. Hence we get $r=q+3$ and the figure with $t=2$. On the other hand in case $\{Q_{q+2}\}=E_{q-1}\cap E_{q+1}$, then we proceed similarly in order to get contractible graphs. Seeing from this procedure, we infer that every possible case is exhausted. Q.E.D.

REMARK 5.5. It seems that the above two propositions hold true without the assumptions about φ^* and $D(\varphi)$.

Let Γ denote a divisor on S_r with a figure in Proposition 5.3 or 5.4. As we see from the manner of the proof of them, we have the following result independently of the automorphisms.

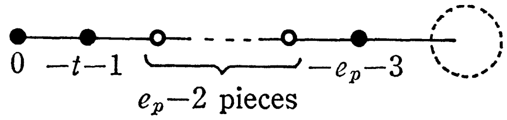
REMARK 5.6. For a curve with $g=0$, $N=1$ and $R\geqq 0$ we can perform blow-ups in order to get Γ. Conversely, for any divisor Γ, there is a curve C with $g=0$, $N=1$ and $R\geqq 0$ such that Γ is obtained from C by some blow-ups.

Now, let us begin the proof of (6) in Theorem A. It suffices to find a set of elements of $\mathcal{G}$ which defines the group $(\boldsymbol{G}_a)^n$. By the above remark we shall use the divisors Γ to find such elements. In the case when $R=0$ we consider only the graphs of subcase (c), since they turn out to yield the wanted elements.

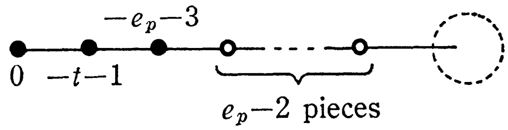
Since Γ is contractible, we have two curves on $\boldsymbol{P}^2$ which are images of Γ, but we do not know whether they are projectively equivalent. So we cannot conclude immediately that we can get an element of $\mathcal{G}$ by the blow-ups and the blow-downs (compare this with the case of Lemmas 4.22 and 4.24).

Starting contractions from C_r and E_r at the same time and contracting components of Γ symmetrically, we get the following figures (here r is already irrelevant to the rank).

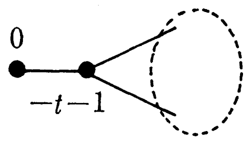
(1) The case when $R=0$ and $e_{p-1}=e_p$.



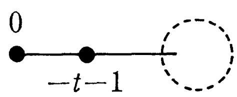
(2) The case when $R=0$ and $e_{p-1}>e_p$.



(3) The case when $R=1$.



(4) The case when $R\geqq 2$.



Note that in case $R=1$, the component E_{q-1} meets E_p, E_{q-2} and E_{q+t-1}, hence the figure is as above. Let F be the surface obtained from S_r by the above contractions and τ_1 be the morphism $S_r\to F$. In the above figures let us denote by E_f and E_s the components with the weight 0 and $-t-1$ respectively. Since $E_f^2=0$ and $E_f\cdot K=-2$, we have, by the Riemann-Roch theorem, that $\dim|E_f|=1$. Hence we have a fiber space $\rho: F\to \boldsymbol{P}^1$ with a fiber E_f and a section E_s. Let $\mathcal{E}$ be the divisor on F consisting of the components of $\tau_1(\Gamma)$ except E_f and E_s, i.e., $\mathcal{E}=\tau_1(\Gamma)_{\mathrm{red}}-E_f-E_s$, where $\tau_1(\Gamma)_{\mathrm{red}}$ is the reduced divisor obtained from $\tau_1(\Gamma)$. Since $\mathcal{E}$ does not meet E_f, the divisor $\mathcal{E}$ is contained in fibers of ρ. Note that $\mathcal{E}$ is disconnected if and only if $R=1$. In case $R\neq 1$, let $\mathcal{F}$ be the unique fiber of ρ containing $\mathcal{E}$, on the other hand in case $R=1$, let $\mathcal{E}_i$ be a connected component of $\mathcal{E}$ and let $\mathcal{F}_i$ be the fiber of ρ containing $\mathcal{E}_i$, where $i=1, 2$. In this case $\mathcal{E}=\mathcal{E}_1+\mathcal{E}_2$, and put also $\mathcal{F}=\mathcal{F}_1+\mathcal{F}_2$.

Before proceeding further we fix some notations. Let F_n denote the rational ruled surface $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}(n))$, where $n\geqq 0$, and $\pi: F_n\to\boldsymbol{P}^1$ the natural morphism. Let B_n be the base line, i.e., it is the unique irreducible curve on F_n with the negative weight $-n$ in case $n\neq 0$.

Now, let us return to the proof.

LEMMA 5.7. *In case $R\neq 1$, there is only one singular fiber $\mathcal{F}$. On the other hand, in case $R=1$, there are just two singular fibers $\mathcal{F}_1$ and $\mathcal{F}_2$.*

PROOF. Contracting components of singular fibers of ρ, we get a birational morphism $\tau_2: F\to F_n$ for some $n\geqq 0$. The number of components of $\tau_2\{\tau_1(\Gamma)\cup\mathcal{F}\}$ is 3 [resp. 4], in case $R\neq 1$ [resp. $R=1$].

$$\begin{tikzcd}
& S_r \arrow[d, "\tau_1"] \arrow[ddl, "\sigma"'] & \\
& F \arrow[r, "\rho"] \arrow[d, "\tau_2"] & \boldsymbol{P}^1 \\
\boldsymbol{P}^2 & F_n \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Comparing the number of blow-ups to obtain S_r from $\boldsymbol{P}^2$ with that of blow-downs to obtain F_n from S_r, we infer that the latter number is $r-1$. Since the number of components of Γ is $r+1$, there is one curve D [resp. two curves D_1 and D_2] not belonging to $\mathcal{E}$ and contracted by τ_2. Note that $\mathcal{E}$ does not contain a curve with the weight -1 by the assertion (1) in Remark 4.4. So the curve D has the weight -1 and $\mathcal{F}$ consists of $\mathcal{E}$ and D in case $R\neq 1$. Similarly in the other case, since $\mathcal{E}_1$ and $\mathcal{E}_2$ do not contain curves with the weight -1, we infer that $\mathcal{F}_i$ consists of $\mathcal{E}_i$ and D_i such that $D_i^2=-1$, where $i=1$ and 2. Whence the lemma is proved. Q.E.D.

Moreover we have the following

LEMMA 5.8. *There is a birational morphism $\tau_2: F\to F_n$ such that*

(1) *τ_2 is a composition of blow-downs defined by contracting components of $\mathcal{F}$,*

(2) τ_2 *does not contract the component(s) of* $\mathcal{F}$ *which meet(s)* E_s, *hence* $n=t+1$.

PROOF. Contracting components of $\mathcal{F}$, we get a birational morphism τ_2': $F\to F_{n'}$. Since E_f and E_s meet transversally at one point, the curve E_s is carried to a section of the natural morphism $F_{n'}\to \boldsymbol{P}^1$. If the weight of $\tau_2'(E_s)$ is more than $-t-1$, then $\tau_2'^{-1}$ has a fundamental point at $\tau_2'(E_s\cap\mathcal{F})$. Then by the elementary transformation with the center at this point we have a new surface which F dominates, too. We can repeat this procedure to get the surface F_n on which the weight of the image of E_s becomes equal to $-t-1$. This proves the whole parts of the lemma. Q.E.D.

As we have shown in the proof of Lemma 5.7, there is an irreducible curve D [resp. two irreducible curves D_1 and D_2] on F such that $\mathcal{F}=\mathcal{E}+D$ [resp. $\mathcal{F}_i=\mathcal{E}_i+D_i$, where $i=1, 2$]. Put $C'=\sigma\tau_1^{-1}(D)$ [resp. $C'=\sigma\tau_1^{-1}(D_1)$ and $C''=\sigma\tau_1^{-1}(D_2)$]. Then performing suitable successive elementary transformations starting from F_{t+1}, we complete the proof of Theorem B. Since $\tau_1^{-1}(D)$, $\tau_1^{-1}(D_1)$ and $\tau_1^{-1}(D_2)$ do not meet C_r, Proposition 2 follows easily.

By definition, F and Γ depend on t, so hereafter let us write $F(t)$ and $\Gamma(t)$ instead of F and Γ respectively, when we emphasize the parameter t. Put

$$\mathcal{G}_1(t)=\{\varphi\in \mathrm{Aut}(F(t))\mid \varphi(\tau_1(\Gamma(t)))=\tau_1(\Gamma(t))\}.$$

Since $\boldsymbol{P}^2\setminus C\cong F(t)\setminus\tau_1(\Gamma(t))$, we have the following

LEMMA 5.9. *For any integer* $t\geqq 1$, *the group* $\mathcal{G}_1(t)$ *may be regarded as a subgroup of* $\mathcal{G}$.

Let $\tau_2: F(t)\to F_{t+1}$ be the morphism obtained in Lemma 5.8 and put $\Sigma=E_f+E_s+\mathcal{F}$.

First we consider the case when $R\neq 1$. Note that $\tau_2(D)=A$ is a point on the fiber $\tau_2(\mathcal{F})$ not lying on the base line. Then put

$$\mathcal{G}_2(t)=\{\psi\in \mathrm{Aut}(F_{t+1})\mid \psi(\tau_2(\Sigma))=\tau_2(\Sigma) \text{ and } \psi(A)=A\}.$$

We shall find elements ψ of $\mathcal{G}_2(t)$ such that $\tau_2^{-1}\psi\tau_2\in\mathcal{G}_1(t)$. Let us call such ψ a *liftable* element. Before stating how to find liftable elements, we make some preparation.

Let B be a free point on D, i.e., B does not lie on $\tau_1(\Gamma)$. Let T' be an irreducible curve intersecting D transversally at B, and U be a small neighbourhood of B such that $U\cap\tau_1(\Gamma)=\emptyset$. We assume that $T=T'\cap U$ and D meet only at B. Let $\tau_2=\tau_{21}\cdots\tau_{2k}$ be the factorization of τ_2 into blow-downs τ_{2i}. Recall that τ_{2k} is defined by contracting D. (Here $k=p+e_p-2$ or $q-1$ according as $R=0$ or $R\geqq 2$). Let m_i be the multiplicity of $\tau_{2i}\cdots\tau_{2k}(T)$ at $\tau_{2i}\cdots\tau_{2k}(D)$, where $1\leqq i\leqq k$. Then we put $m_0=\sum_{i=1}^k m_i^2$. Note that m_0 is independent of t, since $\mathcal{F}$ is independent of t, more precisely, $F(t)\setminus(E_f\cup E_s)$ and $F(1)\setminus(E_f\cup E_s)$ are isomorphic for each t.

Now, for an element ψ of $\mathcal{G}_2(t)$ we define $I(\psi)$ to be the intersection multiplicity $(\tau_2(T)\cdot\psi(\tau_2(T)))_A$ [if $\psi(\tau_2(T))=\tau_2(T)$, then we put $I(\psi)=\infty$]. Then we have the following

LEMMA 5.10. *An element ψ of $\mathcal{G}_2(t)$ is liftable if $I(\psi)\geqq m_0$.*

PROOF. Since $\psi(A)=A$, we have an isomorphism $\tau_{21}^{-1}\psi\tau_{21}=\psi_1$ such that $\psi_1(\tau_{21}^{-1}(A))=\tau_{21}^{-1}(A)$. Since ψ_1 also fixes the center of τ_{22} by the hypothesis, we have an isomorphism $\tau_{22}^{-1}\psi_1\tau_{22}=\psi_2$ such that $\psi_2(\tau_{22}^{-1}\tau_{21}^{-1}(A))=\tau_{22}^{-1}\tau_{21}^{-1}(A)$. In view of the above preparation this procedure can be continued to get the lift of ψ.

Q.E.D.

By the way, we consider generally $\mathrm{Aut}(F_n)$ for a while. Let (u_0, u_1) be a set of homogeneous coordinates on $\boldsymbol{P}^1$ and (v_0, v_1) affine coordinates on $\boldsymbol{A}^2$. The elements (u_0, u_1, v_0, v_1) and (u_0', u_1', v_0', v_1') of $\boldsymbol{P}^1\times(\boldsymbol{A}^2\setminus\{(0, 0)\})$ determine the same point on F_n if and only if there are α and β in $\boldsymbol{C}^*=\boldsymbol{C}\setminus\{0\}$ such that

$$u_0'=\alpha u_0\,,\quad u_1'=\alpha\beta^n u_1\,,\quad v_0'=\beta v_0\,,\quad v_1'=\beta v_1\,.$$

For $n>0$ automorphisms of F_n are of the form

$$\begin{cases} u_0'=u_0 \\ u_1'=\gamma u_1+u_0\Phi(v_0, v_1)\,, \end{cases}\qquad \begin{cases} v_0'=av_0+bv_1 \\ v_1'=cv_0+dv_1\,, \end{cases}$$

where $\gamma\in\boldsymbol{C}^*$ and Φ is a form of degree n, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{PGL}_2\,.$$

Now, let us resume the proof. In the coordinates given above, the equation of $\tau_2(E_s)$ is $u_0=0$. Let the equations of $\tau_2(\mathcal{F})$ and $\tau_2(E_f)$ be $v_0=0$ and $v_1=0$ respectively. Since A is a free point on $\tau_2(\mathcal{F})$, we may assume that $A=(1, 0, 0, 1)$. Let (η, θ) be a set of local coordinates near A, where $\eta=u_1/u_0$ and $\theta=v_0/v_1$. Since an element ψ of $\mathcal{G}_2(t)$ fixes the curves $\tau_2(\mathcal{F})$, $\tau_2(E_f)$ and the point A, it is of the form

$$\begin{cases} \eta'=\gamma\eta+g(\theta)\,, \\ \theta'=(a/d)\theta\,. \end{cases}\qquad \text{where } g(\theta)=v_1^{t+1}\Phi(\theta, 1) \text{ and } g(0)=0\,.$$

Note that $g(\theta)$ is a polynomial of degree$\leqq t+1$. Let the local equation of $\tau_2(T)$ be $h(\eta, \theta)=0$. Since the local equation of $\tau_2(\mathcal{F})$ is $\theta=0$, we have that $h(\eta, 0)\not\equiv 0$. Then $h(\eta, \theta)$ can be expressed as

$$\eta^m+w_1(\theta)\eta^{m-1}+\cdots+w_m(\theta)=0\,,$$

where $w_i(\theta)$, $1\leqq i\leqq m$, is a convergent power series with $w_i(0)=0$. Let us take the automorphism ψ with $a=d$ and $\gamma=1$. Then the equation of $\psi(\tau_2(T))$ is

$$(\eta-g(\theta))^m+w_1(\theta)(\eta-g(\theta))^{m-1}+\cdots+w_m(\theta)=0\,.$$

Whence we infer that $I(\psi)\geqq sm$ if $g(\theta)=a_{t+1}\theta^{t+1}+\cdots+a_s\theta^s$. Thus by Lemma 5.10 we see that ψ is liftable to $\mathcal{G}_1(t)$ if $s\geqq m_0$. Taking such automorphisms, we have that $\mathcal{G}\supset(G_a)^{t+2-s}$ by Lemma 5.9. Recalling that m_0 is independent of t, this relation holds good for every sufficiently large t, hence we complete the proof of (6) in Theorem A when $R\neq 1$.

Next we consider the case when $R=1$. The point $A_i=\tau_2(D_i)$ lies on the fiber $\tau_2(\mathcal{F}_i)$, but not on the base line, where $i=1, 2$. Then put

$$\mathcal{G}_2(t)=\{\psi\in\mathrm{Aut}(F_{t+1})\mid\psi(\tau_2(\Sigma))=\tau_2(\Sigma)\text{ and }\psi(A_i)=A_i\text{ for }i=1, 2\}\,.$$

Similarly let T_i be a part of a curve on $F(t)$ intersecting D_i at a free point, where $i=1, 2$. For an element ψ of $\mathcal{G}_2(t)$ we define $I_i(\psi)$ to be $(\tau_2(T_i)\cdot\psi(\tau_2(T_i)))_{A_i}$. Then similarly there are integers m_{0i} such that ψ is liftable if $I_i(\psi)\geqq m_{0i}$ for $i=1$ and 2. Since ψ fixes three fibers, it is of the form $a=d$ and $b=c=0$. Let the equations of three fibers be $v_0=0$, $v_0=v_1$ and $v_1=0$ respectively. We may assume that $A_1=(1, 0, 0, 1)$, $A_2=(\alpha_1, \alpha_2, 1, 1)$ and $\alpha_1\neq 0$. In case $\gamma=1$ and $a=1$, the automorphism fixes A_1 and A_2 if and only if $\Phi(0, 1)=\Phi(1, 1)=0$. Then take the automorphism of the form

$$\begin{cases}\eta'=\eta+g(\theta)\,, & g(0)=g(1)=0\,,\\ \theta'=\theta\,,\end{cases}$$

where the notations are the same as above.

Let the first condition $I_1(\psi)\geqq m_{01}$ be similarly described as above. The second condition $I_2(\psi)\geqq m_{02}$ decreases the number of the free coefficients of $g(\theta)$, i.e., the order of $g(\theta+1)=a_{t+1}(\theta+1)^{t+1}+\cdots+a_{m_{01}}(\theta+1)^{m_{01}}$ must be at least m_{02}, but as we have mentioned above t can take every sufficiently large integer. Hence we conclude similarly (6) in Theorem A. Thus we have finished the proof of the whole parts of Theorem A.

Lastly we prove the corollary, which is an immediate consequence of Theorem B. In fact, if $R\neq 1$, then we have that $\bar{\kappa}(V)\leqq\bar{\kappa}(\boldsymbol{P}^2\setminus(C\cup C_1))=\bar{\kappa}(\boldsymbol{P}^2\setminus(L_1\cup L_2))=-\infty$. If $R=1$, then similarly we have that $\bar{\kappa}(V)\leqq\bar{\kappa}(\boldsymbol{P}^2\setminus(L_1\cup L_2\cup L_3))=-\infty$, since $L_1\cap L_2=L_1\cap L_3$.

6. Examples and problems.

Also in this section we follow the notations in the previous sections. We present examples of curves with $g=0$, $N=1$ and $R\leqq 1$. For the details, see [9]. Put $F(X, Y, Z)=a^n(YZ^{n-1}-X^n)^{mn+1}+\{aX(YZ^{n-1}-X^n)^m+\sum_{i=1}^m b_iZ^{ni+1}(YZ^{n-1}-X^n)^{m-i}\}^n$, where $a\neq 0$, $m\geqq 1$, $n\geqq 2$ and b_i are arbitrary for $i=1, \cdots, m$. Let C be the curve defined by F/Z^{n-1}. Then C has the following properties.

(1) $C\setminus\{(0, 1, 0)\}\cong \boldsymbol{A}^1$.

(2) $d=mn^2+1$, $p=2m+2n$, $e_1=mn^2-mn$, $e_2=\cdots=e_{2n}=mn$, $e_{2n+1}=\cdots=e_{2m+2n}=n$.

(3) $R=2-n$.

Moreover we have the following new one. Let Θ_λ be the conic defined by

$$YZ-X^2+\lambda Y^2=0.$$

Then Θ_0 and $\Theta_{\lambda'}$, where $\lambda'\neq 0$, meet at only one point $(0, 0, 1)$. Let φ be a non-linear automorphism of $\boldsymbol{P}^2\setminus\Theta_0$ inducing an automorphism on the line $Z=0$. Then putting $C=\varphi(\Theta_{\lambda'})$, we have that $C\setminus\{(0, 1, 0)\}\cong \boldsymbol{A}^1$, $e_p=4$ and $d^2-\sum_{i=1}^{p}e_i^2=4$, since the morphism $\varphi\sigma$ contracts first the proper transform $\sigma^{-1}[\Theta_0]$. This shows that $R=1$.

Finally we raise problems concerning curves with $g=0$ and $N=1$.

PROBLEM 1. Do there exist curves with $R=0$ and $e_{p-1}>e_p$?

PROBLEM 2. Find $\bar{\kappa}(V)$ in the case when $R\leqq -1$. Especially do there exist curves with $\bar{\kappa}(V)=2$?

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