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Multiply connected minimal surfaces and the geometric annulus theorem

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§ 0. Introduction.

The annulus theorem, which is significant in 3-manifold topology, was first announced by Waldhausen [15] and proved by Cannon-Feustel [1]. It is stated as follows:

Let M be a compact orientable P.L. manifold with boundary and let A be a P.L. annulus. Suppose $f:(A, \partial A)\to(M, \partial M)$ is an essential P.L. map. Then there exists an essential P.L. embedding $f^*:(A, \partial A)\to(M, \partial M)$.

In this paper we prove the above annulus theorem in the smooth category, realizing it by area-minimizing minimal surfaces on a Riemannian manifold.

To this end we take an arbitrary compact Riemannian manifold M of dimension m with convex boundary ($m\geqq 3$) and solve in § 3 the energy minimizing problem for essential (i.e. incompressible and boundary incompressible) maps from $(\Delta, \partial\Delta)$ into $(M, \partial M)$, where Δ is a k-ply connected compact planar domain ($2\leqq k<\infty$). Our variational problem is, as found from this setting, what is called a free boundary problem. Since the convergence in free boundary cases is attended with some troubles, we solve an appropriate fixed boundary problem in § 2 to make sure of some converging sequence whose limit is the required solution.

Next in § 4 we suppose $m=3$ and show that the above minimally immersed solution surface is an embedding or a double covering map of an embedded Möbius strip in case $k=2$. We find difficulties in fulfilling this, since the tower construction does not preserve the boundary incompressibility. We get over this obstacle by utilizing, with the characters of an annulus, a certain covering which is adequate for this situation. A simple example (Example 1) shows that the solution surfaces, in case $k\geqq 3$, are neither embeddings nor double covering maps of embedded surfaces in general. Our main result or the geometric annulus theorem is achieved in § 5:

THEOREM (Geometric Annulus Theorem). *Let M be a compact orientable Riemannian 3-manifold with convex incompressible boundary and let A be a smooth annulus. Suppose that there is an essential smooth map $f:(A, \partial A)\to(M, \partial M)$. Then:*

(1) *There exists an essential smooth immersion* $f^*:(A, \partial A)\to(M, \partial M)$ *which has least area among all such essential smooth maps.*

(2) *Any such immersion of least area is either an embedding, or a double covering map onto an embedded Möbius strip.*

(3) *The image of any two such extremal maps either are disjoint, or are identical or intersect each other along a single essential arc. Furthermore the distinct images of the double covering maps, which happen to appear, are all mutually disjoint.*

Meeks-Yau [**9**], [**10**] gave the geometric versions for a triple of theorems (Dehn's lemma, the loop theorem and the sphere theorem) in 3-manifold topology, realized by area-minimizing minimal surfaces. While these theorems deal with simply connected surfaces, our result corresponds to the case of an annulus, i.e. the doubly connected surface.

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§1. Preliminaries.

The following notations will be used throughout this paper:

M: an m-dimensional compact Riemannian manifold with convex boundary ($m\geqq 3$),

Δ: a k-ply connected compact domain in $\boldsymbol{C}$ with a smooth boundary ($2\leqq k<\infty$),

where the boundary ∂M is said to be *convex* if its mean curvature is non-negative with respect to the inward normal.

We use the Sobolev space $W^{1,2}(\mathrm{int}\,\Delta, M)$, where $\mathrm{int}\,\Delta$ denotes the interior of Δ; this is possible, because we may suppose M to be an embedded Riemannian submanifold of a Euclidean space $\boldsymbol{R}^p$ with some $p\in\boldsymbol{Z}^+$ owing to Nash's isometric embedding. We refer to Friedman [**4**] for the Sobolev space. For simplicity we abbreviate $W^{1,2}(\mathrm{int}\,\Delta, M)$ as $W^{1,2}(\Delta, M)$. For $f\in W^{1,2}(\Delta, M)$, the area $A(f, \Delta)$ and the energy $E(f, \Delta)$ are respectively given by

$$A(f, \Delta):=\iint_\Delta\sqrt{|\partial f/\partial x|^2|\partial f/\partial y|^2-\langle\partial f/\partial x, \partial f/\partial y\rangle^2}\,dx\,dy\,,$$

$$E(f, \Delta):=\iint_\Delta(|\partial f/\partial x|^2+|\partial f/\partial y|^2)dx\,dy\,.$$

We refer to Morrey ([**13**], Theorem 3.4.5) for the following compact operator

$$T: W^{1,2}(\Delta, M)\to L^2(\partial\Delta, M)$$

such that $T(f)=f|_{\partial\Delta}$ in $L^2(\partial\Delta, M)$ for $f\in W^{1,2}(\Delta, M)\cap C^0(\Delta, M)$.

While it is commonplace that the induced homomorphism

$$f_{\#}: \pi_1(\Delta) \longrightarrow \pi_1(M)$$

is defined for a continuous map $f:\Delta \to M$, we can consider this homomorphism even for possibly non-continuous maps as follows (cf. Schoen-Yau [**14**]):

LEMMA 1.1 (Schoen-Yau). *If f belongs to $W^{1,2}(\Delta, M)$, then $f_{\#}$ is well-defined.*

Although they dealt with the case in which Δ is a closed surface and M has an empty boundary, the proof is similar for the present situation.

On the other hand it is familiar that the induced map

$$f_{\#,\partial}: \pi_1(\Delta, \partial\Delta) \longrightarrow \pi_1(M, \partial M)$$

is defined for a continuous map $f:(\Delta, \partial\Delta) \to (M, \partial M)$, where $\pi_1(\ ,\ )$ is a 1-dimensional base-point-free relative homotopy. (We use the notation $f_{\#,\partial}$ in view of distinction from $f_{\#}$.) Corresponding to Lemma 1.1, we have

LEMMA 1.2. *If f belongs to $W^{1,2}(\Delta, M)$ and if $T(f)$ is equal to a continuous map ϕ from $\partial\Delta$ into ∂M in $L^2(\partial\Delta, M)$, then $f_{\#,\partial}$ is well-defined.*

PROOF. We first define $f_{\#,\partial}$ as follows:

Let ξ be any element of $\pi_1(\Delta, \partial\Delta)$. Take a smooth path $\beta:(I, \partial I) \to (\Delta, \partial\Delta)$ representing ξ. Without loss of generality, β meets $\partial\Delta$ at $\beta(I)\cap\partial\Delta=\beta(\partial I)$ non-tangentially. Choose a smooth immersion $\psi:(I, \partial I)\times(-\eta, \eta) \to (\Delta, \partial\Delta)$ $(\eta>0)$ such that $\psi(\ , 0)=\beta$. Then f is absolutely continuous on a path-image $\psi(I, t)$ for a.e. $t\in(-\eta, \eta)$, since f belongs to $W^{1,2}(\Delta, M)$. Taking such a path $\psi(\ , t_0)$ $(t_0\in(-\eta, \eta))$, we define $f_{\#,\partial}(\xi)$ as the homotopy class in $\pi_1(M, \partial M)$ represented by the continuous path $f\circ\psi(\ , t_0):(I, \partial I) \to (M, \partial M)$.

We will show that $f_{\#,\partial}$ is well-defined. Suppose $\beta_i:(I, \partial I) \to (M, \partial M)$ to be a smooth path $(i=1, 2)$ such that f is (absolutely) continuous on them and that both the paths represent the same homotopy class in $\pi_1(M, \partial M)$, i.e. there is a homotopy

$$H: (I, \partial I)\times I \longrightarrow (\Delta, \partial\Delta)$$

satisfying $H(\ , 0)=\beta_1$ and $H(\ , 1)=\beta_2$. We may suppose that $f\circ H(\ , t):(I, \partial I) \to (M, \partial M)$ is (absolutely) continuous for any $t\in J$, where J is a subset of equi-measure of the interval I.

On the other hand we may assume that M is an embedded submanifold of some Euclidean space $\boldsymbol{R}^p$. There exists a small number δ_0 such that we can find a retraction ρ of the δ_0-neighborhood M_{δ_0} of M in $\boldsymbol{R}^p$ onto M. Take and fix a positive number $\varepsilon_0<\min\{(\delta_0/2), E(f, \Delta)\}$. We can obtain $t_1\in J$ such that $E(f, I\times[0, t_1])=\varepsilon_0$. It loses no generality to suppose $f\circ H=\phi\circ H$ on $\partial I\times\partial[0, t_1]$. Then we define the closed curve γ as a map of the rectangle $\partial(I\times[0, t_1])$ into M by the relationship

$$\begin{array}{ll} f\circ H & \text{on } I\times\partial[0, t_1], \\ \phi\circ H & \text{on } \partial I\times[0, t_1]. \end{array}$$

It is neither simple nor smooth in general. Deform it to a smooth Jordan curve γ', in the $\delta_0/2$-neighborhood $M_{\delta_0/2}$ of M: This deformation may be written by the continuous map

$$\sigma:(I\times[0, t_1])\setminus D \longrightarrow M_{\delta_0/2}$$

such that

$$\sigma=\gamma \quad \text{on} \quad \partial(I\times[0, t_1]),$$
$$\sigma=\gamma' \quad \text{on} \quad \partial D,$$

where D is an open disk contained in $\operatorname{int}(I\times[0, t_1])$. We solve the Plateau problem for γ' in $\boldsymbol{R}^p$ and obtain the solution $g: D\rightarrow \boldsymbol{R}^p$. We see $g(D)\subset M_{\delta_0}$ by a theorem of Alexander-Osserman. We may suppose $g=\gamma'$ on ∂D up to a diffeomorphism over D. Then we define a continuous map h of $I\times[0, t_1]$ into M_{δ_0} by σ and g on each domain of definition. The continuous map $\rho\circ h$ provides a homotopy joining the two paths $f\circ\beta_1=f\circ H(\ ,0)$ and $f\circ H(\ ,t_1)$ as maps of pairs from $(I, \partial I)$ into $(M, \partial M)$. Thus we conclude that they both represent the same class in $\pi_1(M, \partial M)$.

Next we take $t_2\ (>t_1)\in J$ such that $E(f\circ H, I\times[t_1, t_2])=\varepsilon_0$. (We set $t_2=1$ if $E(f\circ H, I\times[t_2, 1])<\varepsilon_0$.) We can verify similarly as above that both the two paths $f\circ H(\ ,t_1)$ and $f\circ H(\ ,t_2)$ represent the same class in $\pi_1(M, \partial M)$. After repeating this process $q=([E(f\circ H, I\times I)/\varepsilon_0]+1)$ times, where [ ] is the Gauss' symbol, we have $t_q=1$. Therefore we conclude that both the two paths $f\circ\beta_1=f\circ H(\ ,0)$ and $f\circ\beta_2=f\circ H(\ ,1)$ represent the same class in $\pi_1(M, \partial M)$. q.e.d.

Keeping in mind the above facts, we introduce the following definition (cf. Jaco [**6**]):

DEFINITION 1. (i) For $f\in C^0(\Delta, M)$ or $W^{1,2}(\Delta, M)$ we say f is *incompressible* (in M), if $f_\#$ is injective.

(ii) For a continuous map $f:(\Delta, \partial\Delta)\rightarrow(M, \partial M)$ or a map f satisfying the assumptions of Lemma 1.2, we say f is *boundary incompressible* (in M), if $f_{\#,\partial}$ is injective.

§ 2. Some fixed boundary problem.

In this section we will consider an energy minimizing problem (and consequently an area minimizing one) in a fixed boundary setting. Just as mentioned in the introduction, this is convenient to solve the free boundary problem desired.

We begin with explanation of some terms necessary for describing an admissible class.

From the viewpoint of conformal equivalence we have only to consider compact planar domains all of whose boundary consist of k circles and which are contained in the unit disk centered at the origin, where one of the boundary components is the unit circle. In case $k=2$, we may assume further that they are concentric annuli. We call them *normalized domain*, which we adopt as our parameter domain.

Next we will define the ε-boundary incompressibility which is weaker than the boundary incompressibility. We use this weaker form in this section, since an approximate step is required for solving the free boundary problem in §3. Let ε be an arbitrary fixed non-negative number and let $N(\partial M, \varepsilon)$ be the closed ε-neighborhood of ∂M. For any continuous map $f:(\varDelta, \partial\varDelta)\to(M, N(\partial M, \varepsilon))$, we have the induced map

$$f_{\#,\partial,\varepsilon}: \pi_1(\varDelta, \partial\varDelta) \longrightarrow \pi_1(M, N(\partial M, \varepsilon)),$$

where we use the notation $f_{\#,\partial,\varepsilon}$ in view of distinction from $f_{\#,\partial}$. With respect to the induced map $f_{\#,\partial,\varepsilon}$, we have similarly to Lemma 1.2:

LEMMA 2.1. *If f belongs to $W^{1,2}(\varDelta, M)$ and if $T(f)$ is equal to a continuous map from $\partial\varDelta$ into $N(\partial M, \varepsilon)$ in $L^2(\partial\varDelta, M)$, then $f_{\#,\partial,\varepsilon}$ is well-defined.*

Then we give the following definition:

DEFINITION 2. For a continuous map $f:(\varDelta, \partial\varDelta)\to(M, N(\partial M, \varepsilon))$ or a map f satisfying the assumptions of Lemma 2.1, we say f is *ε-boundary incompressible* (in M), if $f_{\#,\partial,\varepsilon}$ is injective.

Now the terms are ready for our variational problem. Let Γ be a collection of k mutually disjoint Jordan curves in $N(\partial M, \varepsilon)$. We define an admissible class $\mathcal{F}_\varepsilon(\Gamma)$ of pairs $(f, \varDelta)$ of maps and domains satisfying the following four conditions:

1° $\varDelta$ is a normalized domain,

2° f belongs to $W^{1,2}(\varDelta, M)$,

3° $T(f)$ in $L^2(\partial\varDelta, M)$ is equal to a continuous map of $\partial\varDelta$ into M, the restriction of which to each component is a parametrization of the corresponding Jordan curve of Γ,

4° f is incompressible and ε-boundary incompressible.

Then we set

PROBLEM I. *Minimize the energy $E(\ ,\ )$ in $\mathcal{F}_\varepsilon(\Gamma)$.*

The standard argument assures the following lemma (cf. Morrey [**13**], Courant [**2**]): It provides the usual reason why we deal with the energy minimizing problem for the purpose of the area minimizing one:

LEMMA 2.2. *If $(f, \varDelta)$ minimizes the energy $E(\ ,\ )$ in $\mathcal{F}_\varepsilon(\Gamma)$, then it also minimizes the area $A(\ ,\ )$. Both minimum values are equal to each other.*

We introduce the following natural topology to $\mathcal{F}_\varepsilon(\Gamma)$:

DEFINITION 3. Let $(f_n, \varDelta_n)$ and $(f, \varDelta)$ be members of $\mathcal{F}_\varepsilon(\Gamma)$ $(n=1, 2, \cdots)$. We say $(f_n, \varDelta_n)$ converges to $(f, \varDelta)$ as n increases to infinity if the following two conditions are satisfied:

1° $\varDelta_n$ converges to $\varDelta$ in the sense of Fréchet, 2° there exist diffeomorphisms $\chi_n : \Delta \to \Delta_n$ $(n=1, 2, \cdots)$ such that χ_n and $d\chi_n$ converge to the respective identities uniformly on Δ and that $f_n \circ \chi_n$ converges to f weakly in $W^{1,2}(\Delta, M)$.

Then the energy $E(\ ,\ )$ is lower semi-continuous on $\mathcal{F}_\varepsilon(\Gamma)$. Furthermore we have

LEMMA 2.3. *For an arbitrary positive number C,*

$$\mathcal{F}_\varepsilon(\Gamma) \cap \{E(f, \Delta) \leqq C\}$$

is sequentially compact.

PROOF. In the following, $D(z, r)$ denotes the open disk centered at z with radius r in $\boldsymbol{C}$. Let $\{(f_n, \Delta_n)\}_{n=1}^{\infty}$ be an arbitrary sequence in $\mathcal{F}_\varepsilon(\Gamma) \cap \{E(f, \Delta) \leqq C\}$. We may suppose that $\{\Delta_n\}_{n=1}^{\infty}$, passing to a subsequence if necessary, converges to a compact domain Δ_∞ by their normalization. Suppose that Δ_∞ is not normalized. Then there are two cases to be considered:

Case 1. A boundary component C_n of Δ_n converges to a point.

Case 2. There exist (at least) two components C_n^1 and C_n^2 such that $\mathrm{dist}(C_n^1, C_n^2)$ tends to 0 as $n \to \infty$.

We first consider Case 1. Let z_n denote the center of the circle C_n. The standard estimate shows that for any η such that $0<\eta<1$, there exists an $r_0 = r_0(\eta) \in [\eta^2, \eta]$ satisfying the following two conditions (cf. Morrey [**13**]):

(i) Taking a subsequence of $\{(f_n, \Delta_n)\}_{n=1}^{\infty}$, we see that f_n is absolutely continuous on $\partial D(z_n, r_0)$ contained in Δ_n,

(ii) $$\int_0^{2\pi} \left\| \frac{\partial f_n}{\partial \theta}(z_n + r_0 e^{i\theta}) \right\| d\theta \leqq \sqrt{\frac{2\pi C}{|\log \eta|}}.$$

Hence the length of the curve $f_n|_{\partial D(z_n, r_0)}$ becomes as small as one pleases. Then by the compactness of the manifold M, this curve is zero-homotopic in M for any sufficiently small η. This contradicts the incompressibility of f_n.

We next examine Case 2. We may assume that the radii of the circle C_n^1 and C_n^2 are uniformly bounded below by a positive constant, since otherwise we can reduce to Case 1. We may suppose that for all n, C_n^i corresponds to a Jordan curve γ_i of Γ $(i=1, 2)$, i.e. $f_n|_{C_n^i}$ is a parametrization of γ_i. The assumption makes us find a point z_n^1 of C_n^1 for each n such that $\mathrm{dist}(z_n^1, C_n^1)$ tends to 0 as $n \to \infty$. Then it follows similarly to Case 1 that for any η such that $0<\eta<1$, there exists an $r_0 = r_0(\eta) \in [\eta^2, \eta]$ satisfying the following two conditions:

(i) Passing to a subsequence of $\{(f_n, \Delta_n)\}_{n=1}^{\infty}$ if necessary, f_n is absolutely continuous on $\partial D(z_n, r_0) \cap \Delta_n$ which consists of two circular arcs joining C_n^1 and C_n^2,

(ii) $$\int \left\| \frac{\partial f_n}{\partial \theta}(z_n + r_0 e^{i\theta}) \right\| d\theta \leqq \sqrt{\frac{2\pi C}{|\log \eta|}},$$

where the integral is over the set $\{\theta\in[0, 2\pi]\,;\, z_n+r_0e^{i\theta}\in\Delta_n\}$. $\mathrm{dist}(\gamma_1, \gamma_2)$ is majorized by the above integral which becomes as small as one pleases. Hence $\mathrm{dist}(\gamma_1, \gamma_2)=0$, i.e. these two Jordan curves are not mutually disjoint. This is a contradiction. Therefore we conclude that Δ_∞ is a normalized domain.

Since Δ_n converges to Δ_∞, we can find diffeomorphisms $\chi_n:\Delta_\infty\to\Delta_n$ such that χ_n and $d\chi_n$ converge to the respective identities uniformly on Δ_∞. $\{f_n\circ\chi_n\}_{n=1}^\infty$ is a bounded sequence in $W^{1,2}(\Delta_\infty, M)$, since it has a uniform upper bound with respect to the energy. Therefore its appropriate subsequence has a weak limit f_∞ in $W^{1,2}(\Delta_\infty, M)$. Then (f_n, Δ_n) converges to $(f_\infty, \Delta_\infty)$ in our topology. Hence we have

$$E(f_\infty, \Delta_\infty)\leqq\liminf_{n\to\infty} E(f_n, \Delta_n)\leqq C\,.$$

Thus $(f_\infty, \Delta_\infty)$ satisfies the defining properties 1° and 2° of $\mathcal{F}_\varepsilon(\Gamma)$ together with $E(f_\infty, \Delta_\infty)\leqq C$.

Next we will show that it satisfies the property 3° as well. Obviously $(f_n\circ\chi_n, \Delta_\infty)$ belongs to $\mathcal{F}_\varepsilon(\Gamma)$: Especially there exists a continuous map ϕ_n of $\partial\Delta_\infty$ into M satisfying the following two conditions:

(i) The restriction of ϕ_n to each component is a parametrization of the corresponding Jordan curve of Γ,

(ii) $T(f_n\circ\chi_n)$ is equal to ϕ_n in $L^2(\partial\Delta_\infty, M)$.

Then $\{\phi_n\}_{n=1}^\infty$ is equi-continuous on $\partial\Delta_\infty$ by the standard estimate (see Morrey [**13**], Chap. 9) together with the incompressibility of f_n. Hence by the theorem of Ascoli-Arzela, ϕ_n converges to a continuous map ϕ_∞ of $\partial\Delta_\infty$ into M uniformly there. (Evidently ϕ_∞ satisfies the above condition (i) in place of ϕ_n.) On the other hand $T(f_n\circ\chi_n)$ converges to $T(f_\infty)$ strongly in $L^2(\partial\Delta_\infty, M)$, since T is a compact operator. Therefore $T(f_\infty)$ is equal to ϕ_∞ in $L^2(\partial\Delta_\infty, M)$.

Finally we check the property 4°. We will show that f_∞ is incompressible. Suppose not. Then there exists a loop $\gamma: S^1\to\Delta_\infty$ such that γ is not zero-homotopic in Δ_∞ and that $f\circ\gamma$ is a zero-homotopic continuous loop in M. Without loss of generality we may assume that all f_n are absolutely continuous on $\gamma(S^1)$. Then taking a subsequence if necessary, we can see that $\{f_n\circ\gamma\}_{n=1}^\infty$ is equi-continuous, since $\{f_n\}_{n=1}^\infty$ has a uniform upper bound with respect to the energy. Consequently $f_n\circ\gamma$ is zero-homotopic in M for sufficiently large n, since $f_\infty\circ\gamma$ is so. This contradicts the incompressibility of f_n. Thus we conclude that f_∞ is incompressible. We can also verify that f_∞ is ε-boundary incompressible, by the similar argument with respect to paths instead of loops. q.e.d.

Now we take an energy minimizing sequence $\{(f_n, \Delta_n)\}_{n=1}^\infty$ in $\mathcal{F}_\varepsilon(\Gamma)$. We may assume that it has a uniform upper bound of the energy. Lemma 2.3 implies that passing to a subsequence, it converges to some member $(f_\infty, \Delta_\infty)$ of $\mathcal{F}_\varepsilon(\Gamma)$ as $n\to\infty$. Then by the lower continuity of the energy, we conclude that $(f_\infty, \Delta_\infty)$ minimizes the energy in $\mathcal{F}_\varepsilon(\Gamma)$.

The regularity arguments of Morrey [**13**] prove that f_∞ belongs to $C^0(\Delta_\infty, M)\cap C^\infty(\mathrm{int}\,\Delta_\infty, M)$: We should note that his arguments preserve the incompressibility and the ε-boundary incompressibility. In view of Lemma 2.2, f_∞ is of vanishing mean curvature with possible exception of isolated branch points. Consequently we have

PROPOSITION 1. *If $\mathcal{F}_\varepsilon(\Gamma)$ is not empty, there exists a solution (f^*, Δ^*) of Problem I: (f^*, Δ^*) minimizes the energy and the area in $\mathcal{F}_\varepsilon(\Gamma)$. f^* is a branched minimal immersion of the interior of Δ^*, which is continuous on Δ^*.*

§ 3. Essential branched minimal immersions.

In this section we are going to consider the free boundary problem mentioned in the introduction. For the formulation we first define an admissible class $\mathcal{F}$ of pairs (f, Δ) of maps and domains satisfying the following three conditions:

1° Δ is a normalized domain,

2° $f:(\Delta, \partial\Delta)\to(M, \partial M)$ is a smooth map,

3° f is incompressible and boundary incompressible.

Then we intend to solve

PROBLEM II. *Minimize the energy $E(\ ,\ )$ in $\mathcal{F}$.*

Although for this purpose we would make a minimizing sequence converge to a member of $\mathcal{F}$ in some sense, it is not easy to find directly such a converging sequence. So we take another course:

In the paragraph A we define an *approximate sequence* $\{(f_n, \Delta_n)\}_{n=1}^\infty$ of pairs of maps and domains (Definition 4): $f_n(\partial\Delta_n)$ does not lie on ∂M, but it is k mutually disjoint Jordan curves in a neighborhood of ∂M which becomes closer and closer to ∂M as n grows to infinity.

The paragraph B is concerned with a convergence of the approximate sequence: In general, such a sequence $\{(f_n, \Delta_n)\}_{n=1}^\infty$ does not converge in any reasonable sense, even if it minimizes the value $\liminf_{n\to\infty} E(f_n, \Delta_n)$ in the class of all approximate sequences. So we replace each term (f_n, Δ_n) by the solution (f_n^*, Δ_n^*) of Problem I and adopt the new approximate sequence $\{(f_n^*, \Delta_n^*)\}_{n=1}^\infty$. The area minimality yields the convergence of the maps, which owes to the argument of Meeks-Yau [**10**].

The paragraph C deals with the boundary regularity of its limit map: We can verify that the limit $(f_\infty^*, \Delta_\infty^*)$ belongs not to $\mathcal{F}$ but to a wider class $\tilde{\mathcal{F}}$ indifferent to the definite boundary images of maps (Definition 5). On the other hand the argument of Jäger [**7**] shows that the energy minimizing map in $\tilde{\mathcal{F}}$ has the boundary regularity. Thus we conclude that $(f_\infty^*, \Delta_\infty^*)$ belongs to $\mathcal{F}$ and minimizes the energy (and consequently the area) in this class.

Now we carry out the details of the above steps.

A. We first take and fix a positive number ε_0 such that there is a deformation retraction ρ of $N(\partial M, \varepsilon_0)$ onto ∂M. We give the following definition:

DEFINITION 4. We say $\{(f_n, \Delta_n)\}_{n=1}^{\infty}$ is an *approximate sequence* if the following three conditions are satisfied:

1° Δ_n is a normalized domain,

2° f_n is a continuous map of Δ_n into M, which is smooth in the interior,

3° for any ε such that $0<\varepsilon<\varepsilon_0$, there is an $n_0=n_0(\varepsilon)\in \boldsymbol{Z}^+$ such that f_n is incompressible and ε-boundary incompressible for all $n>n_0$; more precisely, $f_n(\partial\Delta_n)$ consists of k mutually disjoint Jordan curves in $N(\partial M, \varepsilon)$ $(n>n_0)$.

We use the following notations throughout this section:

$$\hat{A} := \inf\{\liminf_{n\to\infty} A(f_n, \Delta_n);\ \{(f_n, \Delta_n)\}_{n=1}^{\infty} \text{ is an approximate sequence}\};$$

$$\hat{E} := \inf\{\liminf_{n\to\infty} E(f_n, \Delta_n);\ \{(f_n, \Delta_n)\}_{n=1}^{\infty} \text{ is an approximate sequence}\},$$

where the infima are taken with respect to all the approximate sequences.

We can verify $\hat{A}=\hat{E}$, since each f_n of any approximate sequence can be approximated by an immersion f_n of Δ_n into M in the C^2-topology, where f_n admits the isothermal parameters.

B. We take an approximate sequence $\{(f_n, \Delta_n)\}_{n=1}^{\infty}$ such that $E(f_n, \Delta_n)$ tends to $\hat{E}$ as $n\to\infty$, whose existence follows from the diagonal method. Then $A(f_n, \Delta_n)$ converges to $\hat{A}=\hat{E}$, since the area is always not greater than the energy. We may assume that there exists a positive constant C_0 satisfying $E(f_n, \Delta_n)\leqq C_0$ for any $n\in \boldsymbol{Z}^+$. Take a monotone decreasing sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers less than ε_0. Choosing a subsequence of $\{(f_n, \Delta_n)\}_{n=1}^{\infty}$, we may suppose that for all $n\in \boldsymbol{Z}^+$, $f_n(\partial\Delta_n)$ is contained in $N(\partial M, \varepsilon_n)$, and that f_n is ε_n-boundary incompressible. Note here that $f_n(\Delta_n)$ consists of k mutually disjoint Jordan curves. The collection of such k Jordan curves is denoted by Γ_n. For each n we consider Problem I in §2 for ε_n and Γ_n. Proposition 1 implies that there exists an area and energy minimizing solution in $\mathcal{F}_{\varepsilon_n}(\Gamma_n)$. Let this solution be denoted by (f_n^*, Δ_n^*). We pay attention to the sequence $\{(f_n^*, \Delta_n^*)\}_{n=1}^{\infty}$. It is an approximate sequence, and $A(f_n^*, \Delta_n^*)$ as well as $E(f_n^*, \Delta_n^*)$ converges to $\hat{A}=\hat{E}$ as $n\to\infty$. So we may adopt $\{(f_n^*, \Delta_n^*)\}_{n=1}^{\infty}$. We have the following lemma partly similar to Lemma 2.2; in fact we are led to a contradiction by the ε_n-boundary incompressibility of f_n instead of the previous disjointness of Jordan curves (cf. Case 2 in the proof of Lemma 2.2). Namely

LEMMA 3.1. $\{\Delta_n^*\}_{n=1}^{\infty}$ *contains a subsequence which converges to a normalized domain* Δ_∞^* *in the sense of Frèchet.*

Then we can choose diffeomorphisms $\chi_n^*: \Delta_\infty^*\to\Delta_n^*$ $(n=1, 2, \cdots)$ such that χ_n^* and $d\chi_n^*$ converge to the respective identities uniformly there. Note again that each (f_n^*, Δ_n^*) is an area minimizing solution in the fixed boundary setting: And so is $(f_n^* \circ \chi_n^*, \Delta_\infty^*)$ too, since χ_n^* is a diffeomorphism and accordingly $A(f_n^* \circ \chi_n^*, \Delta_\infty^*) = A(f_n^*, \Delta_n^*)$. Then by the argument of Meeks-Yau ([**10**], pp. 448-449), we have

LEMMA 3.2. *$\{f_n^* \circ \chi_n^*\}_{n=1}^\infty$ contains a subsequence which converges in $C^{p,\alpha}$-norm on any compact subset in the interior of Δ_∞^*, for any $p \in \boldsymbol{Z}^+$ and $\alpha \in (0, 1)$.*

C. In the following, $\Delta(\delta)$ denotes the compact set by removing the δ-boundary strip from Δ, i.e. $\{z \in \Delta;\ \mathrm{dist}(z, \partial\Delta) \geqq \delta\}$. Let f_∞^* be the limiting smooth map of $\{f_n^* \circ \chi_n^*\}_{n=1}^\infty$, which is guaranteed by Lemma 3.2. (Note that f_∞^* is defined only on $\mathrm{int}\Delta_\infty^*$.) Then we want to show that $(f_\infty^*, \Delta_\infty^*)$ belongs to the following class:

DEFINITION 5. Let (f, Δ) be a pair of map and domain. We say that it belongs to the admissible class $\tilde{\mathfrak{F}}$ if the following three conditions are satisfied:

1° Δ is a normalized domain,

2° f is a smooth map of $\mathrm{int}\Delta$ into M,

3° for any ε such that $0<\varepsilon<\varepsilon_0$, there is a $\delta_1=\delta_1(\varepsilon)>0$ such that $f|_{\Delta(\delta)}$ is incompressible and ε-boundary incompressible for all δ satisfying $0<\delta<\delta_0$; more precisely, $f(\mathrm{int}\Delta \setminus \Delta(\delta))$ is contained in $N(\partial M, \varepsilon)$ $(0<\delta<\delta_0)$.

For this purpose we require the following lemma:

LEMMA 3.3. *For any ε such that $0<\varepsilon<\varepsilon_0$, there exists a $\delta_1=\delta_1(\varepsilon)>0$ independent of n such that $f_n^* \circ \chi_n^*(\mathrm{int}\Delta_\infty(\delta) \setminus \Delta_\infty(\delta))$ is contained in $N(\partial M, \varepsilon)$ for all sufficiently large $n \in \boldsymbol{Z}^+$ and all δ satisfying $0<\delta<\delta_0$.*

PROOF. Suppose not. Then there exist a positive number $\varepsilon^*<\varepsilon_0$ and a point sequence $\{z_n\}_{n=1}^\infty$ in Δ_∞ satisfying the following two conditions:

$$(1) \qquad z_\infty = \lim_{n\to\infty} z_n\,,$$

$$(2) \qquad f_n^* \circ \chi_n^*(z_n) \notin N(\partial M, \varepsilon^*) \qquad (n \in \boldsymbol{Z}^+).$$

By the standard estimate, we can find, for each n, an r_n such that $1/n^2 \leqq r_n \leqq 1/n$ and that

$$\int \left\| \frac{\partial f_n}{\partial \theta}(z_n + r_n e^{i\theta}) \right\| d\theta < \sqrt{\frac{2\pi C}{|\log \eta|}}\,,$$

where the integral is over the set $\{\theta \in [0, 2\pi);\ z_\infty + r_n e^{i\theta} \in \Delta_\infty\}$. Hence

$$f_n^* \circ \chi_n^*(\partial(D(z_\infty, r_n) \cap \Delta_\infty)) \subset N(\partial M, \varepsilon^*/2)$$

for all sufficiently large n. Then in view of (2) we have

$$(3) \qquad \mathrm{dist}(f_n^* \circ \chi_n^*(z_n),\ f_n^* \circ \chi_n^*(\partial(D(z_\infty, r_n) \cap \Delta_\infty)) > \varepsilon^*/2$$

for all sufficiently large n. The condition (1) enables us to suppose $|z_n - z_\infty| < 1/n^2$. Hence noting $r_n > 1/n$, we obtain

$$(4) \qquad z_n \in D(z_\infty, r_n) \cap \Delta_\infty\,.$$

Then by Lemma 1 in [**10**] together with (2), (3) and (4), there exists a positive constant C_1 independent of n such that

$$A(f_n^* \circ \chi_n^*,\ D(z_\infty, r_n) \cap \Delta_\infty) > C_1,$$

accordingly

$$E(f_n^* \circ \chi_n^*,\ D(z_\infty, r_n) \cap \Delta_\infty) > C_1, \tag{5}$$

for all sufficiently large n.

Let Δ_n' denote the compact set $\Delta_\infty \setminus D(z_\infty, r_n)$. We may suppose that Δ_n' is k-ply connected (yet not normalized) for all n. Let ψ_n denote a conformal diffeomorphism of a normalized domain Δ_n'' onto Δ_n'. Then $\{(f_n^* \circ \chi_n^* \circ \psi_n, \Delta_n'')\}_{n=1}^{\infty}$ is an approximate sequence. Hence we have

$$\liminf_{n\to\infty} E(f_n^* \circ \chi_n^*,\ \Delta_n') = \liminf_{n\to\infty} E(f_n^* \circ \chi_n^* \circ \psi_n,\ \Delta_n'') \geqq \hat{E}. \tag{6}$$

Therefore we conclude from (5) and (6)

$$\hat{E} = \lim_{n\to\infty} E(f_n^*, \Delta_n^*) \geqq \liminf_{n\to\infty} E(f_n^* \circ \chi_n^*,\ \Delta_n') + C_1 \geqq \hat{E} + C_1.$$

This is a contradiction. q. e. d.

The above lemma proves the following:

LEMMA 3.4. *$(f_\infty^*, \Delta_\infty^*)$ minimizes the energy and the area in $\mathfrak{F}$: Both two minimum values are equal to each other.*

PROOF. We can see by Lemma 3.3 that $(f_\infty^*, \Delta_\infty^*)$ possesses the defining property 3° of $\tilde{\mathfrak{F}}$ (Definition 5). Obviously it also satisfies the other conditions 1°, 2°. Thus we conclude that $(f_\infty^*, \Delta_\infty^*)$ belongs to $\tilde{\mathfrak{F}}$. Hence we have

$$\tilde{E} \leqq E(f_\infty^*, \Delta_\infty^*) \leqq \liminf_{n\to\infty} E(f_n^* \circ \chi_n^*,\ \Delta_\infty^*) = \lim_{n\to\infty} E(f_n^*, \Delta_n^*) = \hat{E},$$

$$\tilde{A} \leqq A(f_\infty^*, \Delta_\infty^*) \leqq \liminf_{n\to\infty} A(f_n^* \circ \chi_n^*,\ \Delta_\infty^*) = \lim_{n\to\infty} A(f_n^*, \Delta_n^*) = \hat{A},$$

where

$$\tilde{E} := \inf\{E(f, \Delta);\ (f, \Delta) \in \tilde{\mathfrak{F}}\},$$

$$\tilde{A} := \inf\{A(f, \Delta);\ (f, \Delta) \in \tilde{\mathfrak{F}}\}.$$

(Note that $\lim_{n\to\infty} E(f_n^* \circ \chi_n^*, \Delta_\infty^*) = \lim_{n\to\infty} E(f_n^*, \Delta_n^*)$, since $d\chi_n$ converges to the identity uniformly on Δ_∞^*.)

Next we will show $\hat{E} \leqq \tilde{E}$ and $\hat{A} \leqq \tilde{A}$, which will complete the proof. (Note again $\hat{E} = \hat{A}$.) Let (f, Δ) be an arbitrary member of $\tilde{\mathfrak{F}}$. We define a sequence $\{(f_n, \Delta_n)\}_{n=1}^{\infty}$ by

$$\Delta_n := \Delta(1/n),$$

$$f_n := f|_{\Delta_n}.$$

Without loss of generality we may assume from the beginning that Δ_n is k-ply connected and that $f|_{\partial\Delta_n}$ consists of k mutually disjoint Jordan curves. Let ψ_n denote a conformal diffeomorphism of a normalized domain $\Delta_n^\natural$ onto Δ_n. Then we can verify that $\{(f_n\circ\psi_n, \Delta_n^\natural)\}_{n=1}^\infty$ is an approximate sequence. Hence we obtain

$$\hat{E}\leqq\liminf_{n\to\infty} E(f_n\circ\psi_n, \Delta_n^\natural).$$

On the other hand we have

$$E(f_n\circ\psi_n, \Delta_n^\natural)=E(f_n, \Delta_n)=E(f|_{\Delta_n}, \Delta_n)=E(f, \Delta_n)\leqq E(f, \Delta).$$

Therefore

$$\hat{E}\leqq\liminf_{n\to\infty} E(f_n\circ\psi_n, \Delta_n^\natural)\leqq E(f, \Delta).$$

Similarly

$$\hat{A}\leqq\liminf_{n\to\infty} A(f_n\circ\psi_n, \Delta_n^\natural)\leqq A(f, \Delta).$$

By the arbitrariness of $(f, \Delta)\in\mathfrak{F}$, we conclude $\hat{E}\leqq\tilde{E}$ and $\hat{A}\leqq\tilde{A}$. q. e. d.

The argument of Jäger [7] proves the following lemma:

LEMMA 3.5. *Let (f, Δ) be a member of $\tilde{\mathfrak{F}}$ which minimizes the energy in this class. (Note that f is a branched minimal immersion of the interior.) Then f has a smooth extension up to the boundary of Δ.*

Lemma 3.4 and Lemma 3.5 enable us to suppose that the limit map f_∞^* is smoothly defined over Δ_∞^*. Then $(f_\infty^*, \Delta_\infty^*)$ belongs to the class $\mathfrak{F}$. Hence we conclude that it minimizes the energy and the area in $\mathfrak{F}$, since $\mathfrak{F}$ is contained in $\tilde{\mathfrak{F}}$. After all we have

PROPOSITION 2. *If $\mathfrak{F}$ is not empty, there exists a solution (f^*, Δ^*) of Problem II: (f^*, Δ^*) minimizes the energy and the area in this class. f^* is a branched minimal immersion.*

§ 4. Embeddedness and non-embeddedness.

In this section we investigate whether the solution map f^* of Problem II is an embedding or not. First in case $m=3$, f^* has no interior (true and false) branch points (Gulliver-Osserman-Royden [5]), and no boundary branch points (Meeks-Yau [12]) because of the convexity of ∂M. On the contrary in case $m\geqq 4$, f^* admits branch points in general. Henceforth we assume $m=3$ fundamentally. Our main purpose here is to prove the following proposition:

PROPOSITION 3. *Let $m=3$ and $k=2$. Suppose that M is orientable and that ∂M is incompressible. Then any solution map f^* of Problem II is either an embedding, or a double covering map onto an embedded Möbius strip.*

Note that in case $k\geqq 3$, similar results to the above-mentioned do not hold in general (Example 1).

REMARK 1. The boundary ∂M is said to be incompressible if the inclusion map $i:\partial M\to M$ is incompressible, i.e. $i_{\#}:\pi_1(\partial M)\to\pi_1(M)$ is injective. If M is not incompressible, we can take a finite number of mutually disjoint embedded minimal disks whose boundary circles generate $\mathrm{Ker}\, i_{\#}$ (*General Geometric Loop Theorem* of Meeks-Yau [**10**]). Cut M along them and we obtain a new manifold M' which has a piecewise smooth incompressible boundary with non-negative mean curvature. Using the approximation method of Meeks-Yau [**9**] (see also Meeks-Yau [**12**], § 1), we can verify that Proposition 3 still holds for this manifold M'. Therefore the assumption of this proposition is sufficiently general.

The proof of Proposition 3 follows from subsequent three Lemmas 4.4, 4.5, 4.6 with three prerequisite Lemmas 4.1, 4.2, 4.3. We will carry out our argument as follows:

Let a_1 and a_2 denote the boundary components of the annulus Δ^*. First in case that $f^*(a_1)$ and $f^*(a_2)$ do not lie on the same component of ∂M, we can conclude that f^* is an embedding (Lemma 4.2). Therefore we have only to consider the case that they lie on a single component C of ∂M. We take a covering $p:\tilde{M}\to M$ corresponding to a subgroup conjugate to $j_{\#}\pi_1(C)$ in $\pi_1(M)$, where $j:C\to M$ is an inclusion map. Then:

(a) Any lift of f^* to $\tilde{M}$ has some area-minimality, and hence it is an embedding (Lemma 4.4).

(b) For any two lifts $\tilde{f}_1, \tilde{f}_2:\Delta^*\to\tilde{M}$, we have either $\tilde{f}_1(\Delta^*)\cap\tilde{f}_2(\Delta^*)=\emptyset$ or $\tilde{f}_1(\Delta^*)=\tilde{f}_2(\Delta^*)$ (Lemma 4.5).

Let $\hat{\Delta}^*$, $\hat{M}$ denote the universal covering spaces of Δ^*, M respectively. Let $\hat{f}_1, \hat{f}_2:\hat{\Delta}^*\to\hat{M}$ be any two embeddings lifted of f^*. We can take each $\hat{f}_i$ down to an embedding $\tilde{f}_i:\Delta^*\to\tilde{M}_i$ lifted of f^*, where $\tilde{M}_i$ denotes a covering space corresponding to a subgroup H_i conjugate to $j_{\#}\pi_1(C)$ in $\pi_1(M)$. In case that we can choose the covering $\tilde{M}_i$ such that $H_1=H_2$, the above facts provide us with all things to prove for the pair $\hat{f}_1$ and $\hat{f}_2$. In the other case we will show $\hat{f}_1(\hat{\Delta}^*)\cap\hat{f}_2(\hat{\Delta}^*)=\emptyset$ by the cutting-gluing argument in a finite covering of $\tilde{M}_i$ (Lemma 4.6).

Now we begin the proof of Proposition 3 along the above line. Hereafter we suppose $k=2$, i.e. Δ^* is doubly connected.

LEMMA 4.1. *Let N be a (possibly non-compact) 3-manifold with incompressible boundary. Let $\alpha:S^1\to\Delta^*$ denote one of the boundary Jordan curves of Δ^*. For any continuous map $f:(\Delta^*, \partial\Delta^*)\to(N, \partial N)$, the following two conditions are equivalent:*

$1°$ *f is incompressible.*

$2°$ *$f\circ\alpha$ is not zero-homotopic in ∂N.*

PROOF. Since $1°\Rightarrow 2°$ is obvious, we have only to show $2°\Rightarrow 1°$. Suppose that f is not incompressible. Then there exists an $n\in\boldsymbol{Z}^+$ such that $n[f\circ\alpha]=f_{\#}(n[\alpha])$ is trivial as an element of $\pi_1(N)$. Therefore it is also trivial as an element of $\pi_1(\partial N)$, since ∂N is incompressible. On the other hand any non-trivial element of $\pi_1(\partial N)$ is torsion-free, since ∂N consists of mutually disjoint orientable surfaces. Hence we conclude that $f\circ\alpha$ is trivial as an element of $\pi_1(\partial N)$. This is a contradiction. q. e. d.

Let a_1, a_2 denote the boundary components of Δ^*.

LEMMA 4.2. *Let N be a (possibly non-compact) orientable Riemannian 3-manifold with convex incompressible boundary, which has at least two boundary components B_1, B_2. Let $\mathcal{Q}$ denote a class of maps f satisfying the following three conditions:*

1° $f:(\Delta^*, \partial\Delta^*)\to(N, \partial N)$ *is a smooth map,*

2° f *is incompressible,*

3° $f(a_i)\subset B_i$ $(i=1, 2)$.

If an immersion $f^\dagger$ minimizes the area in $\mathcal{Q}$, then $f^\dagger$ is an embedding.

PROOF. The proof is due to the tower construction of Meeks-Yau, for which we refer to [**9**]. It suffices to argue in the real analytic category, by the approximation method of Meeks-Yau. Ordinarily the real analyticity simplifies the argument in the cutting-gluing process. We construct the tower for $f^\dagger$ in the usual way: Let $p_i:N_i\to N_{i-1}$ denote the constituent of the tower and let $f_i:\Delta^*\to N_i$ be the lift of $f^\dagger$ to N_i for $i=1, 2, \cdots, n$, where n is the height of the tower. By a standard consideration involving the argument about the first Betti number of N_n, we can verify that ∂N_n consists of a single torus component T and some spheres, and that $f_n(a_i)$ lies on T and represents a non-trivial homology class of $H_1(T; \boldsymbol{Z}_2)$ $(i=1, 2)$. Then by a topological argument of curves on a torus, we may conclude that there exist two mutually disjoint annular regions Ω_1 and Ω_2 on T satisfying the following three conditions: For $i=1, 2$;

(i) one of the two boundary components of Ω_i is a Jordan curve contained in $f_n(a_1)$, which is denoted by $\gamma_i: S^1\to\Omega_i$,

(ii) another component is a Jordan curve contained in $f_n(a_2)$,

(iii) $p_1\circ\cdots\circ p_n\circ\gamma_i$ is not zero-homotopic in ∂N_n.

Let $R: N_n\to f_n(\Delta^*)$ be a simplicial retraction such that $R|_{N_n}$ covers $f_n(\Delta^*)$ twice and is locally one-to-one except $f_n(\partial\Delta^*)$. Then we have

$$A(R, \Omega_1)+A(R, \Omega_2)\leqq A(R, T)\leqq 2A(f_n, T).$$

Note that by Lemma 4.1, $p_1\circ\cdots\circ p_n\circ R|_{\Omega_i}$ is incompressible $(i=1, 2)$. If f_n is not an embedding, we have a contradiction by the above inequality and the folding curve argument. Thus we conclude that f_n is an embedding. Next we see that f_{n-1} is an embedding, since otherwise the standard cutting-gluing argument leads us to a contradiction, except that we must pay a special attention to the defining property 3° of $\mathcal{Q}$ and the incompressibility together with Lemma 4.1. Similarly descending the tower step by step, we can verify that each f_i is an embedding

$(i=1, \cdots, n)$. q.e.d.

We note here that $\pi_1(\varDelta^*, \partial\varDelta^*)$ consists of a trivial element and a single non-trivial one, since $k=2$. Let $\beta:(I, \partial I)\to(\varDelta^*, \partial\varDelta^*)$ be a continuous path which meets a_1 and a_2 on its endpoints. β is unique up to relative homotopy, and represents the non-trivial element of $\pi_1(\varDelta^*, \partial\varDelta^*)$. Note also that a continuous map $f:(\varDelta^*, \partial\varDelta^*)\to(M, \partial M)$ is boundary incompressible if and only if $f\circ\beta$ represents a non-trivial element of $\pi_1(M, \partial M)$.

We will go on with our proof of Proposition 3. We distinguish the following two cases:

Case 1. $f^*(a_1)$ and $f^*(a_2)$ do not lie on the same component of ∂M.

Case 2. $f^*(a_1)$ and $f^*(a_2)$ lie on a single component C of ∂M.

As for Case 1, applying Lemma 4.2 for $N=M$, we conclude that f^* is an embedding. (Note that the defining property 3° of Ω in Lemma 4.2 implies the boundary incompressibility of f.) Therefore we have only to consider Case 2. We fix our eyes upon the covering $p:\tilde{M}\to M$ corresponding to a subgroup conjugate to $j_{\#}\pi_1(C)$ in $\pi_1(M)$, where $j:C\to M$ is an inclusion map. This covering plays an important role as mentioned in the introduction. $\tilde{M}$ is endowed with the pullback metric via the projection map p and has convex incompressible boundary. For any continuous map $f:(\varDelta^*, \partial\varDelta^*)\to(M, C)$, we can take a lift $\tilde{f}:(\varDelta^*, \partial\varDelta^*)\to(\tilde{M}, \partial\tilde{M})$ of f to $\tilde{M}$. The following property suggests that this covering is nice for our purpose.

LEMMA 4.3. *Under the above situation, the following two conditions are equivalent:*

1° *f is boundary incompressible.*

2° *$\tilde{f}(a_1)$ and $\tilde{f}(a_2)$ do not lie on the same component of $\partial\tilde{M}$.*

PROOF. 1°⇒2°: Suppose on the contrary that $\tilde{f}(a_1)$ and $\tilde{f}(a_2)$ lie on a single component $\tilde{C}$ of $\tilde{M}$. Note that $\tilde{C}$ is homeomorphic to C by $p|_{\tilde{C}}$. Then we have the following diagram:

$$\begin{tikzcd}
\pi_1(\tilde{C}) \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi_1(\tilde{M}) \arrow[r] \arrow[d, "\cong"] & \pi_1(\tilde{M}, \tilde{C}) \arrow[r] & \pi_0(\tilde{C}) \arrow[r, "\text{inj.}"'] & \pi_0(\tilde{M}) \\
\pi_1(C) \arrow[r, "\cong"'] & j_{\#}\pi_1(C)=p_{\#}\pi_1(M)<\pi_1(M).
\end{tikzcd}$$

The diagram-chasing shows $\pi_1(\tilde{M}, \tilde{C})=0$. It follows that $\tilde{f}\circ\beta$ represents a trivial element of $\pi_1(\tilde{M}, \partial\tilde{M})$. Hence $f\circ\beta=p\circ\tilde{f}\circ\beta$ is trivial as an element of $\pi_1(M, \partial M)$. This contradicts 1°.

2°⇒1°: Suppose that f is not boundary incompressible. Then $f\circ\beta$ represents a trivial element of $\pi_1(M, \partial M)$, i.e. there exists a homotopy

$$H:(I, \partial I) \longrightarrow (M, \partial M)$$

such that $H(\ ,0)=f\circ\beta$ and $H(\ ,1)=$a point of ∂M. We can take a lift of H

$$\tilde{H}:(I, \partial I) \longrightarrow (\tilde{M}, \partial\tilde{M})$$

satisfying $\tilde{H}(\ ,0)=\tilde{f}\circ\beta$. Thus $\tilde{f}\circ\beta$ represents a trivial element of $\pi_1(\tilde{M}, \partial\tilde{M})$. Hence both the endpoints of $\tilde{f}\circ\beta$, accordingly $\tilde{f}(a_1)$ and $\tilde{f}(a_2)$ lie on a single boundary component of $\tilde{M}$. q. e. d.

LEMMA 4.4. *Any lift of f^* to $\tilde{M}$ is an embedding.*

PROOF. Let $\tilde{f}^*:(\Delta^*, \partial\Delta^*)\to(\tilde{M}, \partial\tilde{M})$ be an arbitrary lift of f^* to $\tilde{M}$. Let $\tilde{C}_i$ denote the boundary component on which $\tilde{f}^*(a_i)$ lies, for $i=1, 2$. Lemma 4.3 implies $\tilde{C}_1\neq\tilde{C}_2$, since f^* is boundary incompressible. In view of Lemma 4.2, it is sufficient to show that $\tilde{f}^*$ minimizes the area in the class $\mathcal{G}$ for $N=\tilde{M}$ and $B_i=\tilde{C}_i$. Suppose not. Then there exists a map g belonging to $\mathcal{G}$ such that $A(g, \Delta^*)<A(f^*, \Delta^*)$. Since $\tilde{M}$ is endowed with the pullback metric via the projection map p, we have $A(p\circ g, \Delta^*)<A(f^*, \Delta^*)$. Obviously $p\circ g$ is incompressible, since so is g. By Lemma 4.3, the defining property 3° of $\mathcal{G}$ implies that $p\circ g$ is boundary incompressible. This contradicts the area-minimality of f^*. q. e. d.

LEMMA 4.5. *For any two distinct lifts $\tilde{f}_1, \tilde{f}_2$ of f^* to $\tilde{M}$, we have either $\tilde{f}_1(\Delta^*)\cap\tilde{f}_2(\Delta^*)=\emptyset$ or $\tilde{f}_1(\Delta^*)=\tilde{f}_2(\Delta^*)$.*

PROOF. It suffices to argue in the real analytic category, by the approximation method of Meeks-Yau. Suppose $\tilde{f}_1(\Delta^*)\cap\tilde{f}_2(\Delta^*)\neq\emptyset$ and $\tilde{f}_1(\Delta^*)\neq\tilde{f}_2(\Delta^*)$. Then either of the following two cases occur:

Case 1. There exists a Jordan curve $\gamma: S^1\to\tilde{f}_1(\Delta^*)\cap\tilde{f}_2(\Delta^*)$.

Case 2. There exists a Jordan arc $\gamma_1:(I, \partial I)\to(\tilde{f}_1(\Delta^*)\cap\tilde{f}_2(\Delta^*), \tilde{f}_1(\partial\Delta^*)\cap\tilde{f}_2(\partial\Delta^*))$.

We first consider Case 1. Suppose that γ bounds a disk in either of $\tilde{f}_1(\Delta^*)$ or $\tilde{f}_2(\Delta^*)$, say $\tilde{f}_1(\Delta^*)$. It follows from the incompressibility of $\tilde{f}_2$ that γ also bounds a disk in $\tilde{f}_2(\Delta^*)$. Then the simple cutting-gluing argument leads us to a contradiction. Thus we conclude that γ bounds a disk neither in $\tilde{f}_1(\Delta^*)$ nor in $\tilde{f}_2(\Delta^*)$. Therefore the Jordan curve $\tilde{f}_i^{-1}\circ\gamma$ divides Δ^* into a couple of annular domains $D_i^{(1)}$ and $D_i^{(2)}$ $(i=1, 2)$. We may assume that $A(\tilde{f}_1, D_1^{(1)})$ is the minimum among $A(\tilde{f}_i, D_i^{(j)})$ $(i, j=1, 2)$. Then $A(\tilde{f}_1, D_1^{(2)})$ is the maximum. Choosing a diffeomorphism $\phi^{(j)}: D_1^{(2)}\to D_2^{(j)}$ satisfying $\phi^{(j)}(\tilde{f}_1^{-1}\circ\gamma(t))=\tilde{f}_2^{-1}\circ\gamma(t)$ $(t\in S^1)$, we define a piecewise smooth continuous map $g^{(j)}:(\Delta^*, \partial\Delta^*)\to(M, \partial M)$ by

$$g^{(j)}(z):=\begin{cases} p\circ\tilde{f}_1(z) & (z\in D_1^{(1)}), \\ p\circ\tilde{f}_2\circ\phi^{(j)}(z) & (z\in D_1^{(2)}). \end{cases}$$

Then either $g^{(1)}$ or $g^{(2)}$ is boundary incompressible, since f^* is so and $(g^{(1)}\circ\beta)^{inv.}\circledcirc(g^{(2)}\circ\beta)=f^*\circ\beta$, where $(g^{(1)}\circ\beta)^{inv.}$ denotes the inverse path of $g^{(1)}\circ\beta$, and ⊚ denotes the product of paths. Obviously both $g^{(1)}$ and $g^{(2)}$ are incompressible. Furthermore $A(g^{(j)}, \Delta^*) \leqq A(f^*, \Delta^*)$, and $g^{(j)}$ has folding curves. This contradicts the area-minimality of f^*.

We next consider Case 2. We will find another Jordan arc γ_2 corresponding to γ_1. We pay attention to the maps $\tilde{f}_2 \circ \tilde{f}_1^{-1} \circ \gamma_1$ and $\tilde{f}_1 \circ \tilde{f}_2^{-1} \circ \gamma_1$ which are well-defined, since $\gamma_1(I) \subset \tilde{f}_1(\Delta^*) \cap \tilde{f}_2(\Delta^*)$ and $\tilde{f}_i$ is an embedding $(i=1, 2)$. Obviously

$$(7) \qquad p(\tilde{f}_2 \circ \tilde{f}_1^{-1} \circ \gamma_1(t)) = p(\gamma_1(t)) = p(\tilde{f}_1 \circ \tilde{f}_2^{-1} \circ \gamma_1(t)) \qquad (t \in I),$$

especially

$$p(\tilde{f}_2 \circ \tilde{f}_1^{-1} \circ \gamma_1(0)) = p(\tilde{f}_1 \circ \tilde{f}_2^{-1} \circ \gamma_1(0)) .$$

Thus we have

$$(8) \qquad \tilde{f}_2 \circ \tilde{f}_1^{-1} \circ \gamma_1(0) = \tilde{f}_1 \circ \tilde{f}_2^{-1} \circ \gamma_1(0) ,$$

since they lie on the same boundary component of $\tilde{M}$, and the projection p is one-to-one on each boundary component of $\tilde{M}$. Therefore by the uniqueness of lift, the above equalities (7), (8) show

$$\tilde{f}_2 \circ \tilde{f}_1^{-1} \circ \gamma_1(t) = \tilde{f}_1 \circ \tilde{f}_2^{-1} \circ \gamma_1(t) \qquad (t \in I).$$

So we define a Jordan arc $\gamma_2 := \tilde{f}_1 \circ \tilde{f}_2^{-1} \circ \gamma_1 = \tilde{f}_2 \circ \tilde{f}_1^{-1} \circ \gamma_1$.

We assert $\gamma_1(I) \neq \gamma_2(I)$. Suppose otherwise. Then $\gamma_2 = \tilde{f}_2 \circ \tilde{f}_1^{-1} \circ \gamma_1$ has a fixed point P_0, since it is a continuous map over $\gamma_1(I) = \gamma_2(I)$. Hence $\tilde{f}_2(\tilde{f}_1^{-1}(P_0)) = P_0 = \tilde{f}_1(\tilde{f}_1^{-1}(P_0))$. Thus by the uniqueness of lift, we conclude $\tilde{f}_1 = \tilde{f}_2$, which is a contradiction. We next claim that the end-points of the Jordan arc γ_j do not lie on the same boundary component $(j=1, 2)$, since otherwise the simple cutting-gluing argument together with the boundary incompressibility leads us to a contradiction.

We may assume $\gamma_1(I) \cap \gamma_2(I) = \emptyset$, since otherwise we can find either a Jordan curve γ as in Case 1, or a Jordan arc γ_1 as in Case 2 such that its end-points lie on the same boundary component of M. Then the two disjoint Jordan arcs $\tilde{f}_i^{-1} \circ \gamma_1$ and $\tilde{f}_i^{-1} \circ \gamma_2$ divide Δ^* into a couple of simply connected domains $D_i^{(1)}$ and $D_i^{(2)}$ $(i=1, 2)$. It loses no generality to suppose that $A(\tilde{f}_1, D_1^{(1)})$ is the smallest among $A(\tilde{f}_i, D_i^{(j)})$ $(i, j=1, 2)$. Choosing a diffeomorphism $\phi^{(j)} : D_1^{(2)} \to D_2^{(j)}$ satisfying $\phi^{(j)}(\tilde{f}_1^{-1} \circ \gamma_i(t)) = \tilde{f}_2^{-1} \circ \gamma_i(t)$ $(i=1, 2 : t \in I)$, we define a piecewise smooth continuous map $g^{(j)} : (\Delta^*, \partial\Delta^*) \to (M, \partial M)$ by the same relation as (1). Then either $g^{(1)}$ or $g^{(2)}$ are incompressible, since f^* is so and $[f^* \circ \alpha] = [g^{(1)} \circ \alpha] \cdot [g^{(2)} \circ \alpha]$ in $\pi_1(M)$ (note also Lemma 4.1). Obviously both $g^{(1)}$ and $g^{(2)}$ are boundary incompressible. Moreover we can verify $A(g^{(j)}, \Delta^*) \leqq A(f^*, \Delta^*)$ as well as the existence of folding curves of $g^{(j)}$. This is a contradiction. Thus we conclude either $\tilde{f}_1(\Delta^*) \cap \tilde{f}_2(\Delta^*) = \emptyset$ or $\tilde{f}_1(\Delta^*) = \tilde{f}_2(\Delta^*)$. q. e. d.

Let $\hat{f}_1, \hat{f}_2 : \hat{\Delta}^* \to \hat{M}$ be arbitrary two distinct embeddings lifted of f^*, i. e.

$$\begin{tikzcd} \hat{\Delta}^* \arrow[r, "\hat{f}_i"] \arrow[d, "\text{proj.}"'] & \hat{M} \arrow[d, "\text{proj.}"] \\ \Delta^* \arrow[r, "f^*"'] & M, \end{tikzcd}$$

where $\hat{\Delta}^*$ and $\hat{M}$ respectively denote the universal covering spaces of Δ^* and M. $\hat{f}_i$ can be taken down to a map $\tilde{f}_i$ which is a lift of f^* to a covering $\tilde{M}_i$ corresponding to a subgroup H_i conjugate to $j_{\#}\pi_1(C)$ in $\pi_1(M)$ $(i=1, 2)$:

$$\begin{tikzcd} \hat{\Delta}^* \arrow[rr, "\hat{f}_i"] \arrow[dd] & & \hat{M} \arrow[d] \\ & & \tilde{M}_i \arrow[d] \\ \Delta^* \arrow[urr, "\tilde{f}_i"] \arrow[rr, "f^*"'] & & M\,. \end{tikzcd}$$

In case that we can choose the covering $\tilde{M}_i$ such that $H_1=H_2$, Lemmas 4.4 and 4.5 give us all the things to prove for the pair $\hat{f}_1$ and $\hat{f}_2$.

LEMMA 4.6. *If we cannot take the covering $\tilde{M}_i$ satisfying $H_1=H_2$, then $\hat{f}_1(\hat{\Delta}^*)\cap\hat{f}_2(\hat{\Delta}^*)=\emptyset$.*

PROOF. We first claim $\hat{f}_1(\hat{\Delta}^*)\neq\hat{f}_2(\hat{\Delta}^*)$. Suppose otherwise. Then $\hat{f}_1(\hat{\Delta}^*)$ and $\hat{f}_2(\hat{\Delta}^*)$ touch, on their boundaries, the same boundary component of $\hat{M}$, and accordingly we can choose the covering $\tilde{M}_i$ such that $H_1=H_2$. This is a contradiction.

Now we suppose the contrary to our assertion: $\hat{f}_1(\Delta^*)\cap\hat{f}_2(\Delta^*)\neq\emptyset$. Note that $\hat{f}_i(\hat{\Delta}^*)$ is an embedded infinite strip. Then we have the following four cases to be considered in general:

Case 1. There exists a Jordan curve $\gamma: S^1\to\hat{f}_1(\hat{\Delta}^*)\cap\hat{f}_2(\hat{\Delta}^*)$.

Case 2. There exists a Jordan arc $\iota: (I, \partial I)\to(\hat{f}_1(\hat{\Delta}^*)\cap\hat{f}_2(\hat{\Delta}^*), \hat{f}_1(\partial\hat{\Delta}^*)\cap\hat{f}_2(\partial\hat{\Delta}^*))$.

Case 3. There exists a simple curvilinear ray $\kappa: [0, \infty)\to\hat{f}_1(\hat{\Delta}^*)\cap\hat{f}_2(\hat{\Delta}^*)$ such that $\kappa(0)\in\hat{f}_1(\partial\hat{\Delta}^*)\cap\hat{f}_2(\partial\hat{\Delta}^*)$.

Case 4. There exists a simple infinite arc $\lambda: (-\infty, \infty)\to\hat{f}_1(\hat{\Delta}^*)\cap\hat{f}_2(\hat{\Delta}^*)$.

We can rule out Cases 2 and 3, since if either of these cases occurred, we could take the covering $\tilde{M}_i$ satisfying $H_1=H_2$ by the same reason as at the beginning of the proof. As for the other two cases we will be led to a contradiction by the cutting-gluing argument: Since $\hat{f}_i(\hat{\Delta}^*)$ is non-compact, we need get down to an appropriate intermediate covering.

We consider Case 1. Note that the Jordan curve $\hat{f}_i^{-1}\circ\gamma$ bounds a closed disk D_i in Δ^* $(i=1, 2)$. Then we can verify that there exists a diagram as follows:

$$\begin{tikzcd}
\hat{\Delta}^* \arrow[r, "\hat{f}_i"] \arrow[d, "q_i"'] & \hat{M} \arrow[d, "p_i"] \\
\bar{\Delta}^* \arrow[r, "\bar{f}_i"] \arrow[d, "\text{finite covering}"'] & \bar{M}_i \arrow[d, "\text{finite covering}"] \\
\Delta^* \arrow[r, "\tilde{f}_i"] & \tilde{M}_i
\end{tikzcd}$$

such that p_i projects the compact set $\hat{f}_1(D_1)\cup\hat{f}_2(D_2)$ into $\bar{M}_i$ injectively, where $\bar{\Delta}^*$ is a cover of Δ^* corresponding to the cover $\bar{M}_i$ of $\tilde{M}_i$. (Note that since $\pi_1(\tilde{M}_i)\cong j_*\pi_1(C)\cong\pi_1(C)$ is a surface group, it is a *locally extended residually finite* one. See Freedman-Hass-Scott [**3**].) Choosing a diffeomorphism $\phi: D_1\to D_2$ satisfying $\hat{f}_1\circ\phi=\hat{f}_2$ on ∂D_1, we define a piecewise smooth continuous map $g_i:\Delta^*\to\bar{M}_i$ $(i=1, 2)$ by

$$g_1(x):=\begin{cases} \bar{f}_1(x) & (x\in\Delta^*\setminus q_1(D_1)), \\ p_1\circ\hat{f}_2\circ\phi\circ q_1^{-1}(x) & (x\in q_1(D_1)), \end{cases}$$

$$g_2(x):=\begin{cases} \bar{f}_2(x) & (x\in\Delta^*\setminus q_2(D_2)), \\ p_2\circ\hat{f}_1\circ\phi\circ q_2^{-1}(x) & (x\in q_2(D_2)). \end{cases}$$

Then we have

$$A(\bar{f}_1, \Delta^*)+A(\bar{f}_2, \Delta^*)=A(g_1, \Delta^*)+A(g_2, \Delta^*)\,,$$

and g_i has folding curves $(i=1, 2)$. This is a contradiction.

Finally we consider Case 4. We can find the following diagram to be valid: For i, $j=1, 2$;

$$\begin{tikzcd}
\hat{\Delta}^* \arrow[r, "\hat{f}_j"] \arrow[d] & \hat{M} \arrow[d, "r"] \\
\Delta_0^* \arrow[r, "f'_j"] \arrow[d] & M' \arrow[d, "s_i"] \\
\Delta_i^* \arrow[r, "f''_{ij}"] \arrow[d, "\text{finite covering}"'] & M''_i \arrow[d, "\text{finite covering}"] \\
\Delta^* & \tilde{M}_i
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta_i^* \arrow[r, "f''_{ii}"] \arrow[d] & M''_i \arrow[d] \\
\Delta^* \arrow[r, "\tilde{f}_i"] & \tilde{M}_i
\end{tikzcd}$$

satisfying the following three conditions:

1° Δ_0^* and Δ_i^* are covers of Δ^* respectively corresponding to the cover M' and M''_i of $\tilde{M}_i$: f'_i and f''_{ii} are embeddings,

2° the map r projects λ onto a Jordan curve in $f'_1(\Delta_0^*)\cap f'_2(\Delta_0^*)$,

3° the map s_i projects this Jordan curve injectively into M''_i.

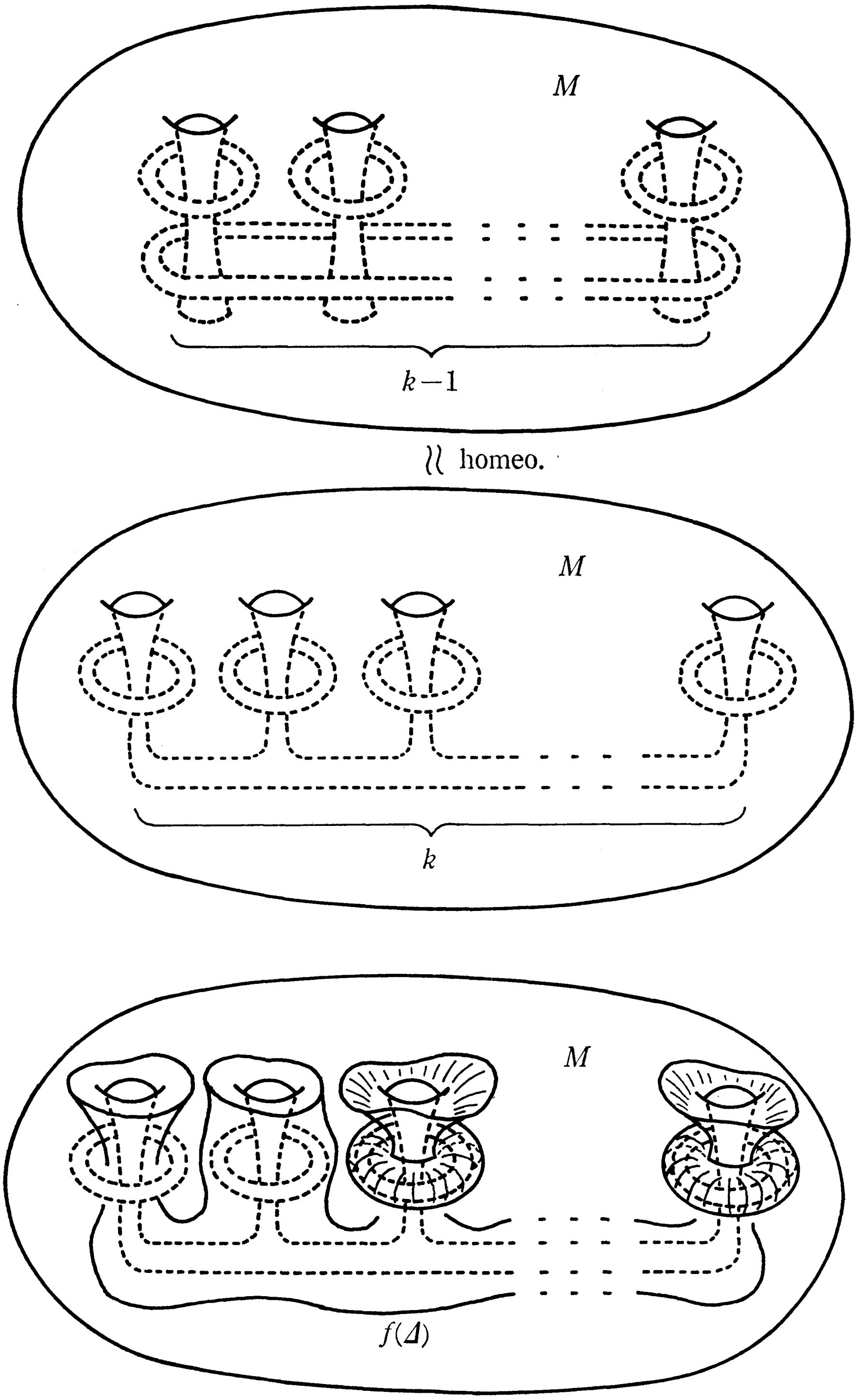

Figure 1.

Then the cutting-gluing argument for f''_{i1} and f''_{i2} leads us to a contradiction (cf. Case 2 in the proof of Lemma 4.5). q. e. d.

Thus we have proved Proposition 3 aimed at.

EXAMPLE 1. As mentioned in the introduction, Proposition 3 does not hold in general in case $k \geqq 3$. We give here a simple example indicating it. Let M be a 3-manifold obtained by removing k tori from a solid handle body of genus $k-1$, as illustrated in Figure 1, with a metric such that ∂M is convex. M admits incompressible and boundary incompressible maps f of $(\Delta, \partial\Delta)$ into $(M, \partial M)$; however there does not exist any such embedding, nor double covering map.

§ 5. Main result.

In this final section we will prove our main theorem. Under "annulus" we understand an orientable surface of genus 0 with two boundary components, which shall be denoted by A. We recall here the following definition:

DEFINITION 6 (Waldhausen [**15**], Jaco [**6**]). (i) A continuous map $f:(A, \partial A) \to (M, \partial M)$ is said to be *essential* (in M) if it is incompressible and boundary incompressible. (Note that Jaco [**6**] uses the same term in different sense.)

(ii) A Jordan arc $\gamma:(I, \partial I) \to (A, \partial A)$ is said to be *essential* (in A) if $\gamma_{\#}: \pi_1(I, \partial I) \to \pi_1(A, \partial A)$ is injective.

We restate our main theorem:

THEOREM (Geometric Annulus Theorem). *Let M be a compact orientable Riemannian 3-manifold with convex incompressible boundary and let A be a smooth annulus. Suppose that there is an essential smooth map $f:(A, \partial A) \to (M, \partial M)$. Then:*

(1) *There exists an essential smooth immersion $f^*:(A, \partial A) \to (M, \partial M)$ which has least area among all such essential smooth map.*

(2) *Any such immersion of least area is either an embedding, or a double covering map onto an embedded Möbius strip.*

(3) *The images of any two such extremal maps either are disjoint, or are identical or intersect each other along a single essential arc. Furthermore the distinct images of the double covering maps, which happen to appear, are all mutually disjoint.*

REMARK 2. The above theorem holds good also for a smooth 3-*manifold pair* (M, F) in place of $(M, \partial M)$, where (M, F) is said to be a smooth 3-manifold pair if F is a smooth incompressible surface contained in ∂M (cf. Jaco [**7**], p. 154, viii. 13). The proof is carried out similarly with slight modifications.

As a direct consequence of our theorem, we have the smooth version of the ordinary annulus theorem:

COROLLARY. *Let M be a compact orientable smooth 3-manifold with boundary and let A be a smooth annulus. Suppose that there is an essential smooth map $f:(A, \partial A)\to(M, \partial M)$. Then there exists an essential smooth embedding $f^*:(A, \partial A)\to(M, \partial M)$.*

PROOF. M admits such a metric as ∂M is convex. Furthermore by Remark 1, we may assume that ∂M is incompressible. Then our theorem provides a smooth map $f^*:(A, \partial A)\to(M, \partial M)$, which is either an embedding or a double covering map. As for the latter, we can perturb the mapping slightly to obtain a new essential smooth embedding. q. e. d.

REMARK 3. The above theorem implies equivariant versions of the Annulus Theorem (cf. Meeks-Yau [**11**]). The third case of the first half in the conclusion (3) indicates the obstruction to the general equivariant assertion. To these circumstances, we have some versions (for example, Kobayashi [8]).

PROOF OF THEOREM. The conclusion (1) is a direct consequence of Proposition 2. The conclusion (2) follows from Proposition 3. Hence we have only to show the conclusion (3). It suffices as usual to carry out in the real analytic category. Let f_1 and f_2 be any two such least area immersions as in Theorem. Suppose $f_1(\Delta^*)\neq f_2(\Delta^*)$ and $f_1(\Delta^*)\cap f_2(\Delta^*)\neq\emptyset$. Then either of the following two cases may arise:

Case 1. There exists a Jordan curve $\gamma:S^1\to f_1(\Delta^*)\cap f_2(\Delta^*)$.

Case 2. There exists a Jordan arc $\gamma_1:(I, \partial I)\to(f_1(\Delta^*)\cap f_2(\Delta^*), f_1(\partial\Delta^*)\cap f_2(\partial\Delta^*))$.

As for Case 1, using a lift $\sigma_i:S^1\to\Delta^*$ of γ for the one-to-one or two-to-one covering $f_i:\Delta^*\to f_i(\Delta^*)$ $(i=1, 2)$, we can obtain a contradiction by the cutting-gluing argument. (See the proof of Lemma 4.5. In case f_i is an embedding, $\sigma_i=f_i^{-1}\circ\gamma$.)

We next consider Case 2. If γ_1 were not essential, the simple cutting-gluing argument leads us to a contradiction. $f_1(\Delta^*)\cap f_2(\Delta^*)$ consists of mutually disjoint essential Jordan arcs, since otherwise we can find either an unessential Jordan arc or a Jordan curve, whose image is contained in it. If there is another Jordan arc $\gamma_2:(I, \partial I)\to(f_1(\Delta^*)\cap f_2(\Delta^*), f_1(\partial\Delta^*)\cap f_2(\partial\Delta^*))$, we get a contradiction by the cutting-gluing argument, together with the use of lifts $\tau_i^j:(I, \partial I)\to(\Delta^*, \partial\Delta^*)$ of γ_j $(j=1, 2)$ for the covering $f_i:\Delta^*\to f_i(\Delta^*)$. Thus we conclude the first half of (3).

Finally we show the latter half. The above argument proves that $f_1(\Delta^*)\cap f_2(\Delta^*)$ consists of at most one essential Jordan arc. If there were such essential Jordan arc γ_1, we can similarly obtain a contradiction, by adopting two distinct lifts $\tau_i, \tau_i':(I, \partial I)\to(\Delta^*, \partial\Delta^*)$ of γ_1, for the double covering $f_i:\Delta^*\to f_i(\Delta^*)$, in place of τ_i^1, τ_i^2. This completes the proof. q. e. d.

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