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Scattering theory by Enss' method for operator valued matrices: Dirac operator in an electric field

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§1. Introduction.

The geometric method of Enss [1] for differential operators in $L^2(\boldsymbol{R}^n)$ is now well established. In this article we extend it to a class of differential operators in $[L^2(\boldsymbol{R}^n)]^m$, $m \geqq 2$. Our class includes the Dirac operator with an electric field in $[L^2(\boldsymbol{R}^3)]^4$; for details refer to example 2.2.

Spectral theory and scattering theory were considered for the operator $P^2/2+W_s$ on $L^2(\boldsymbol{R}^n)$ where W_s is a short range potential in [1, 2, 3, 4, 5]. For general operators of the form $h_0(P)+W_s$ on $L^2(\boldsymbol{R}^n)$ with $h_0(\infty)=\infty$ refer to [6, 7]. For a hint of developing the geometric method for opepators in $[L^2(\boldsymbol{R}^n)]^m$ refer to [6].

For the operator $P^2/2+W_s$ the boundedness of the eigenvalues is proved in [8].

For the operator $P^2/2+W_s+W_L(Q)$ where, now, W_L is a smooth long range local potential the theory is developed in [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. General operator of the form $h_0(P)+W_s(Q, P)+W_L(Q, P)$ with $h_0(\infty)=\infty$ is considered in [19]. For an account of all these results see [20, 21].

For a class of operators of the form $h_0(P)+W_s$ where h_0 need not have any limit at ∞ the geometric theory is developed in [22, 23].

Finally we sketch the contents of the article. In §2 we state the assumptions on the operator H and state the main theorem we intend to prove. In §3 we reproduce some technical theorems from [19]. These will be repeatedly used in §4 and §5. Existence of the wave operator is proved in §4 where as in §5 we prove asymptotic completeness.

§2. Statement of the result.

On the free and perturbed operators H_0 and H on the Hilbert space $[L^2(\boldsymbol{R}^n)]^m$, $n, m \geqq 1$ we make the following set of assumptions A1, A2, $\cdots$, A9.

A1. H_0: $\boldsymbol{R}^n \rightarrow \mathcal{M}_m(\boldsymbol{C})$, where $\mathcal{M}_m(\boldsymbol{C})$ is the space of all $m \times m$ matrices with entries from the complex numbers $\boldsymbol{C}$, is a C^∞ function and for each ξ in $\boldsymbol{R}^n$ the matrix $H_0(\xi)$ is Hermitian.

A2. The eigenvalues and eigenvectors of H_0 can be chosen to vary in a C^∞ manner: More precisely, for each ξ_0 in $\boldsymbol{R}^n$ there exists an open set $L(\xi_0)$ in $\boldsymbol{R}^n$ with ξ_0 in $L(\xi_0)$, m real valued C^∞ functions $h_1, \cdots, h_m : L(\xi_0) \to \boldsymbol{R}$ and m vector valued C^∞ functions $e_1, \cdots, e_m : L(\xi_0) \to \boldsymbol{C}^m$ such that

$$H_0(\xi)e_j(\xi) = h_j(\xi)e_j(\xi) \qquad j=1, \cdots, m, \quad \xi \text{ in } L(\xi_0).$$

In the last identity e_j are treated as column vectors and this convention we shall follow throughout.

From the theory of differentiable manifolds we borrow the term chart. A chart is (by our definition) a triple $(L, U, \boldsymbol{h})$ where L is an open subset of $\boldsymbol{R}^n$, $U : L \to \mathcal{M}_m(\boldsymbol{C})$ is a C^∞ function with $U(\xi)$ being unitary for each ξ, $\boldsymbol{h} : L \to \boldsymbol{R}^m$ where $\boldsymbol{h}=(h_1, \cdots, h_m)$ is a C^∞ function satisfying

$$H_0(\xi)U(\xi) = U(\xi)\operatorname{diag}(h_1(\xi), \cdots, h_m(\xi)) \qquad \text{for each } \xi \text{ in } L.$$

The columns of U will be denoted by $e_1, \cdots, e_m$ so that we have

$$H_0(\xi)e_j(\xi) = h_j(\xi)e_j(\xi).$$

For any chart $(L, U, \boldsymbol{h})$ we define the critical set and critical values by

$$C(L, U, \boldsymbol{h}) = \bigcup_{j=1}^{m} \{\xi \in L : h_j'(\xi)=0 \text{ or the matrix } h_j''(\xi) \text{ is singular}\},$$

$$C_v(L, U, \boldsymbol{h}) = \bigcup_{j=1}^{m} \{h_j(\xi) : \xi \in L,\ h_j'(\xi)=0 \text{ or the matrix } h_j'' \text{ is singular}\}.$$

Note that the usual definition of critical set or critical value does not depend on the second derivative. We impose

A3. $G = \bigcup \{L \setminus C(L, U, \boldsymbol{h}) : (L, U, \boldsymbol{h}) \text{ is a chart}\}$ is an open subset of $\boldsymbol{R}^n$ with $\boldsymbol{R}^n \setminus G$ having (Lebesgue) measure 0.

A4. The closure of $C_v(H_0)$ is a countable subset of $\boldsymbol{R}$ where $C_v(H_0) = \bigcup \{C_v(L, U, \boldsymbol{h}) : (L, U, \boldsymbol{h}) \text{ is a chart}\}$.

The vaguely elliptic property of H_0 is guaranteed by

A5. If $\{\lambda_1(\xi), \cdots, \lambda_m(\xi)\}$ is the set of eigenvalues of $H_0(\xi)$ then $\lim_{|\xi| \to \infty} \inf_j |\lambda_j(\xi)| = \infty$. Also there exists a polynomial p such that $\sum_j |\lambda_j(\xi)| \leqq p(\xi)$.

If the eigenvalues of $H_0(\xi)$ have atmost polynomial growth then, since $H_0(\xi)$ is Hermitian, it is clear that each entry of H_0 has at most polynomial growth.

Let Q, P denote the position and momentum operators on $L^2(\boldsymbol{R}^n)$ given by $Q=(Q_1, \cdots, Q_n)$, $P=(P_1, \cdots, P_n)$, $(Q_j f)(x) = x_j f(x)$, $(P_j f)(x) = -i(D_j f)(x)$, $D_j = \partial/\partial x_j$. The operator $H_0(P)$ will be denoted by H_0.

A6 (Condition on the long range). $W : \boldsymbol{R}^n \to \boldsymbol{R}$ is a C^∞ function and for some δ in $(0, 1]$ we have for each α

$$|(D^{\alpha}W)(x)| \leqq K_{\alpha}(1+|x|)^{-|\alpha|-\delta}$$

for suitable constants K_{α}. Here $\alpha=(\alpha_1, \cdots, \alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $D^{\alpha}=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$. K will always stand for a generic constant.

Note that if A6 holds for some δ then it also holds when δ is replaced by any δ_1 in $(0, \delta]$. So decreasing δ (if necessary) we (may and do) assume that $\delta \notin \{1, 1/2, 1/3, \cdots\}$.

By $W(Q)$ we mean the operator $W(Q)I$ where I is the identity operator on $[L^2(\boldsymbol{R}^n)]^m$: for $(f_1, \cdots, f_m)$ in $[L^2(\boldsymbol{R}^n)]^m$, $f_j \in L^2(\boldsymbol{R}^n)$ we have $W(Q)(f_1, \cdots, f_m) = (W(Q)f_1, \cdots, W(Q)f_m)$.

A7 (Condition on the short range). The operator W_s on $[L^2(\boldsymbol{R}^n)]^m$ has $\mathrm{Dom}\, W_s \supset \mathrm{Dom}(1+P^2)^N$ for some $N>0$ and for some $\varepsilon_0>0$ the operator $W_s(1+P^2)^{-N}(1+Q^2)^{(1+\varepsilon_0)/2}$ defined on $\mathrm{Dom}(1+|Q|)^{1+\varepsilon_0}$ is bounded.

A8. The operator $H=H_0+W_s+W(Q)$ (defined as sum) on $[\mathcal{S}(\boldsymbol{R}^n)]^m$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing smooth functions, has a self adjoint extension denoted by the same letter H.

A9. $(H\pm i)^{-1}-(H_0\pm i)^{-1}$ is a compact operator.

With all these conditions we have

THEOREM 2.1. *Let* A1 *to* A9 *hold. Then*

(a) *There exist functions* $X(t, \cdot): \boldsymbol{R}^n \to \mathcal{M}_m(\boldsymbol{C})$ *taking values in the Hermitian matrices such that*

$$\Omega_{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} \exp[itH]\exp[-iX(t, P)] \qquad \textit{exists}.$$

(b) $\mathrm{Range}\, \Omega_{\pm} = \mathcal{H}_c(H)$, *the continuous subspace for* H.

(c) *Any eigenvalue of* H *in* $\boldsymbol{R}\setminus\overline{C_v(H_0)}$ *is of finite multiplicity and such eigenvalues can not accumulate in* $\boldsymbol{R}\setminus\overline{C_v(H_0)}$.

EXAMPLE 2.2 (Dirac operator in an electric field). Take $n=3$, $m=4$. For $\xi=(\xi_1, \xi_2, \xi_3)$ in $\boldsymbol{R}^3$ define $H_0(\xi)$ by

$$H_0(\xi)=\begin{pmatrix} 1 & 0 & \xi_3 & \xi_1-i\xi_2 \\ 0 & 1 & \xi_1+i\xi_2 & -\xi_3 \\ \xi_3 & \xi_1-i\xi_2 & -1 & 0 \\ \xi_1+i\xi_2 & -\xi_3 & 0 & -1 \end{pmatrix}.$$

Choose k in $(-1/2, 1/2)$, φ in $C_0^{\infty}(\boldsymbol{R}^3)$ with φ real valued, $\varphi=1$ for $|x|\leqq 1$ and 0 for $|x|\geqq 2$. Put $W_s(x)=k\varphi(x)|x|^{-1}$, $W(x)=k[1-\varphi(x)]|x|^{-1}$. Then $H=H_0(P)+W_s(Q)+W(Q)$ satisfies all the assumptions A1 to A9. In fact the eigenvalues are $h_1(\xi)=h_2(\xi)=-h_3(\xi)=-h_4(\xi)=(1+\xi^2)^{1/2}$ on the whole of $\boldsymbol{R}^3$ and the eigenvectors can be chosen in a C^{∞} manner on the whole of $\boldsymbol{R}^3$. A simple calculation shows that $h_j''(\xi)=((\partial^2 h_j/\partial\xi_k\partial\xi_m))$, $k, m=1, 2, 3$, is nonsingular at every point ξ of $\boldsymbol{R}^3$. For details refer to §6 of chapter 10 of [**24**]. With H_0 as above the self adjointness of $H_0+V(Q)$ for $V:\boldsymbol{R}^3\to\boldsymbol{R}$ is extensively studied. For a recent article refer to [**25**] and references therein. With H_0 as above and W satisfying A6, Theorem 2.1 (a) has been proved in [**26**] for $H=H_0+W_s+W$.

REMARK 2.3. In §3, 4, 5 we state the propositions for the positive time only. Once they are proved we implicitly assume that the corresponding propositions for negative times are stated and proved.

§3. Some technical results for operators on $L^2(\boldsymbol{R}^n)$.

Let G_0 be any open subset of $\boldsymbol{R}^n$ and $h:G_0\to\boldsymbol{R}$ any C^∞ function such that $|h'(\xi)|>0$ and $|\det h''(\xi)|>0$ for each ξ in G_0. Let $V:\boldsymbol{R}^n\to\boldsymbol{R}$ be any C^∞ function such that for some δ_0 in $(0, 1)\setminus\{1, 1/2, 1/3, \cdots\}$ the inequalities $|D^\alpha V(x)| \leqq K_\alpha(1+|x|)^{-|\alpha|-\delta_0}$ hold for all α. Now choose a positive integer m_0 such that $m_0\delta_0<1<(m_0+1)\delta_0$.

Let C be a fixed compact subset of G_0 such that for some $b>0$ we have

$$C_{3b}=\{p \text{ in } \boldsymbol{R}^n : \operatorname{dist}(p, C)\leqq 3b\}\subset G_0\,. \tag{3.1}$$

Let

$$3a=\inf\{|h'(\xi)| : \xi\in C_{3b}\}\,. \tag{3.2}$$

Assume further that C_{3b} satisfies

$$\sup\{|h'(\xi_1)-h'(\xi_2)| : |\xi_1-\xi_2|\leqq 2b,\ \xi_1, \xi_2\in C_{3b}\}\leqq a2^{-1/2}\,. \tag{3.3}$$

For the "momentum" ξ in G_0, time $t\geqq t_0\geqq 0$ and the (inductive) sequence $m=0, 1, 2, \cdots, m_0$ define the function Y by

$$Y(0, t_0, t, \xi)=0\,,$$

$$Y(m, t_0, t, \xi)=\int_{t_0}^{t} ds\, V(sh'(\xi)+Y_\xi'(m-1, t_0, s, \xi))\,.$$

Put

$$X(t_0, t, \xi)=th(\xi)+Y(m_0, t_0, t, \xi)$$

so that

$$\partial X(t_0, t, \xi)/\partial t=h(\xi)+V(th'(\xi)+Y_\xi'(m_0-1, t_0, t, \xi))\,.$$

Then we have

LEMMA 3.1. *Let C, b, X be as above. Then there exists $t_{-1}=t_{-1}(C_{3b})\geqq 0$ such that for each $t_0\geqq t_{-1}$, f in $\mathcal{S}(\boldsymbol{R}^n)$ and φ in $C_0^\infty(G_0)$ with* $\operatorname{supp}\varphi\subset\{\xi$ *in* $G_0:\operatorname{dist}(\xi, C)<b\}$ *the following hold:*

(i) $\int_{t_0}^{\infty} dt \|(1+|Q|)^{-1-\varepsilon} \exp[-iX(t_0, t, P)]\varphi(P)f\| < \infty$ *for each* $\varepsilon>0$,

(ii) $\int_{t_0}^{\infty} dt \|[V(Q)-V(th'(P)+Y'_P(m_0-1, t_0, t, P))] \exp[-iX(t_0, t, P)]\varphi(P)f\| < \infty$.

PROOF. Refer to Lemma 5.1 of [**19**]. Q.E.D.

We can improve the above Lemma to

THEOREM 3.2. *Let* $\varphi \in C_0^\infty(G_0)$ *and* $f \in \mathcal{S}(\boldsymbol{R}^n)$. *Then there exists* $t_{-1}=t_{-1}(\varphi)$ *such that* (i) *and* (ii) *of Lemma* 3.1 *is valid for our new* φ.

PROOF. Follows from Lemma 3.1 by using the techniques of the partition of unity [**27**]. Q.E.D.

LEMMA 3.3.

(i) $\lim_{t\to\infty} \{Y(m_0, 0, t, \xi)-Y(m_0, t_0, t, \xi)\}$ *exists on* G_0 *for each* $t_0 \geqq 0$,

(ii) $\lim_{t\to\infty} \{Y(m_0, 0, t+s, \xi)-Y(m_0, 0, t, \xi)\}=0$ *in* G_0 *for all* s.

PROOF. Follows from Lemma 4.2 of [**19**]. Q.E.D.

The above results will be used for proving the existence of wave operators in §4 while for completeness in §5 we need the following results.

For "the position" x in $\boldsymbol{R}^n$ and "momentum" ξ in G_0 and time $t \geqq t_0 \geqq 0$ define

$$Y(0, t_0, t, x, \xi)=0,$$

$$Y(m, t_0, t, x, \xi)=\int_{t_0}^{t} ds\, V(x+sh'(\xi)+Y'_\xi(m-1, t_0, s, x, \xi)) \quad \text{for} \quad m=1, 2, \cdots, m_0,$$

$$X(t_0, t, x, \xi)=x\cdot\xi+(t-t_0)h(\xi)+Y(m_0, t_0, t, x, \xi)$$

so that

$$X(t_0, t_0, x, \xi)=x\cdot\xi,$$

$$\partial X(t_0, t, x, \xi)/\partial t=h(\xi)+V(x+th'(\xi)+Y'_\xi(m_0-1, t_0, t, x, \xi)).$$

Note that

$$Y(m, t_0, t, 0, \xi)=Y(m, t_0, t, \xi)$$

but

$$X(t_0, t, 0, \xi)=X(t_0, t, \xi)-t_0h(\xi).$$

We introduce now a positive operator valued measure on the Borel subsets of $\boldsymbol{R}^n \times \boldsymbol{R}^n$ due to [**4, 28**].

Let $b^*>0$ be given. Choose η in $\mathcal{S}(\boldsymbol{R}^n)$ such that $\hat{\eta}$, the Fourier transform of η given by

$$\hat{\eta}(k)=(2\pi)^{-n/2}\int dx \exp[-ik\cdot x]\eta(x)$$

has

(3.4) $$\operatorname{supp}\hat{\eta}\subset\left\{k \text{ in } \boldsymbol{R}^n : |k|\leqq\frac{1}{8}b^*\right\}$$

and

$$\|\eta\|^2=\int dx|\eta(x)|^2=1.$$

Define for (x, k) in $\boldsymbol{R}^n\times\boldsymbol{R}^n$ the function η_{xk} by

$$\eta_{xk}(y)=\eta(y-x)\exp[ik\cdot(y-x)]$$

so that

$$\hat{\eta}_{xk}(p)=\hat{\eta}(p-k)\exp[-ix\cdot p].$$

For any Borel subset M of $\boldsymbol{R}^n\times\boldsymbol{R}^n$ define an operator $T(M)$ by the weak integral

$$T(M)=(2\pi)^{-n/2}\int_M dx\,dk\langle\ ,\eta_{xk}\rangle\eta_{xk}.$$

For various properties of the positive operator valued measure T refer to [**4**, **8**, **14**, **15**, **16**, **18**, **19**, **20**, **22**, **23**, **28**].

For $M\subset G_0$ and $r>0$ define the subset $E(M, \pm, r)$ of $\boldsymbol{R}^n\times G_0$ by

$$E(M, \pm, r)=\{(x, k) : k\in M,\ x\cdot h'(k)\gtrless 0,\ |x|\geqq r\}.$$

For any subset L of G_0 with $\{\xi : \operatorname{dist}(\xi, L)\leqq b^*/8\}\subset G_0$ and φ in $C_0^\infty(G_0)$ define two operators $A(t_0, t, L, \varphi, +, r)$ and $B(t_0, t, L, \varphi, +, r)$ by

$$[A(t_0, t, L, \varphi, +, r)f](q)$$
$$=(1+|q|)^{-1-\varepsilon_0}\int_{E(L,+,r)}dx\,dk\langle f, \eta_{xk}\rangle\int d\xi\,\hat{\eta}(\xi-k)\varphi(\xi)\exp(i[q\cdot\xi-X(t_0, t, x, \xi)]),$$

$$[B(t_0, t, L, \varphi, +, r)f](q)$$
$$=\int_{E(L,+,r)}dx\,dk\langle f, \eta_{xk}\rangle\int d\xi[V(q)-V(x+th'(\xi)+Y'_\xi(m_0-1, t_0, t, x, \xi))]\varphi(\xi)$$
$$\cdot\hat{\eta}(\xi-k)\exp(i[q\cdot\xi-X(t_0, t, x, \xi)]).$$

For the evolutions A and B we state

LEMMA 3.4. *Let C_{3b} be as in* (3.1), (3.2), (3.3) *and b^* of* (3.4) *be 'any element' in* $(0, b]$ *and φ be in $C_0^\infty(G_0)$. If the diameter of C_{3b} is small enough then there exists $t_{-1}=t_{-1}(C_{3b})\geqq 0$ such that for all $t_0\geqq t_{-1}$ we have*

(i) $$\lim_{r\to\infty}\int_{t_0}^\infty dt\|A(t_0, t, C, \varphi, +, r)\|=0,$$

(ii) $$\lim_{r\to\infty}\sup_{t\geq t_0}\|A(t_0, t, C, \varphi, +, r)\|=0,$$

(iii) $$\lim_{r\to\infty}\int_{t_0}^{\infty}dt\|B(t_0, t, C, \varphi, +, r)\|=0.$$

The corresponding statements for the negative time hold with the *same* η.

PROOF. Refer to Lemma 6.1 of [**19**]. Q.E.D.

THEOREM 3.5. *Let* $\varphi\in C_0^\infty(G_0)$. *Then there exist* η *of* (3.4) $\eta=\eta(\operatorname{supp}\varphi)$ *and* $t_{-1}=t_{-1}(\operatorname{supp}\varphi)\geqq 0$ *such that for all* $t_0\geqq t_{-1}$ *we get*

(i) $$\lim_{r\to\infty}\int_{t_0}^{\infty}dt\|A(t_0, t, G_0, \varphi, +, r)\|=0,$$

(ii) $$\lim_{r\to\infty}\sup_{t\geq t_0}\|A(t_0, t, G_0, \varphi, +, r)\|=0,$$

(iii) $$\lim_{r\to\infty}\int_{t_0}^{\infty}dt\|B(t_0, t, G_0, \varphi, +, r)\|=0.$$

The corresponding statements for the negative time hold with the *same* η.

PROOF. We prove (i) only. For (ii), (iii) it is similar. Denote the open {closed} ball of centre x and radius r by $S(x, r)$ $\{S[x, r]\}$ so that $S(x, r)=\{y:|y-x|<r\}$ and $S[x, r]=\{y:|y-x|\leqq r\}$. For each x in G_0 we can find $b=b(x)>0$ so that for $C=S[x, b]$, Lemma 3.4 holds. If $\lambda>0$ is any preassigned number by taking $\min\{b(x), \lambda\}$ if necessary we can assume that $b(x)\leqq\lambda$.

Let L be any compact subset of G_0 and $\lambda>0$. Since L is a compact subset of G_0 we can choose points $x_1, \cdots, x_q$ in G_0 so that $L\subset\bigcup_{j=1}^{q}S(x_j, b_j/8)$ where $b_j=b(x_j)$ are as above and $b_j\leqq\lambda$. Let $b=\min\{b_1, \cdots, b_q\}/8$. Choose η so that b^* of η in (3.4) satisfies $0<b^*\leqq b$.

Clearly $\{\xi:\operatorname{dist}(\xi, L)\leqq b\}\subset\bigcup_j S(x_j, b_j/4)$. Choose φ_j in $C_0^\infty(S(x_j, b_j/4))$ so that $\varphi_1+\cdots+\varphi_q=1$ on $\{\xi:\operatorname{dist}(\xi, L)\leqq b\}$. Then for any Ψ in $C_0^\infty(G_0)$ we see that

$$\begin{aligned}A(t_0, t, L, \Psi, +, r)&=\sum_j A(t_0, t, L, \Psi\varphi_j, +, r)\\&=\sum_j A(t_0, t, S[x_j, b_j], \Psi\varphi_j, +, r)\end{aligned}\tag{3.5}$$

and a similar expression for $B(t_0, t, L, \Psi, +, r)$. By Lemma 3.4 and (3.5) we get that

$$\lim_{r\to\infty}\int_{t_0}^{\infty}dt\|A(t_0, t, L, \Psi, +, r)\|=0.\tag{3.6}$$

Now let $\varphi\in C_0^\infty(G_0)$. Choose $\lambda>0$ so that $\{\xi:\operatorname{dist}(\xi, \operatorname{supp}\varphi)\leqq 4\lambda\}\subset G_0$ and take $L=\{\xi:\operatorname{dist}(\xi, \operatorname{supp}\varphi)\leqq\lambda\}$. For this L and λ choose η so that (3.6) holds. Now it is easy to see that $A(t_0, t, G_0, \varphi, +, r)$ makes sense and $A(t_0, t, G_0, \varphi, +, r)=A(t_0, t, L, \varphi, +, r)$. This completes the proof of the theorem. Q.E.D.

We can slightly generalise this theorem for proving Lemma 5.4 (i). If $\varphi_1, \cdots, \varphi_k$ are in $C_0^\infty(G_0)$, we can take $\lambda>0$ so that $\{\xi : \mathrm{dist}(\xi, \bigcup_j \mathrm{supp}\, \varphi_j) \leqq 4\lambda\} \subset G_0$. Now take $L=\{\xi : \mathrm{dist}(\xi, \bigcup_j \mathrm{supp}\, \varphi_j) \leqq \lambda\}$. For this L and λ choose η as before. Then the theorem holds with the *same* η when φ is replaced by any of $\varphi_1, \cdots, \varphi_k$.

LEMMA 3.6. *For φ in $C_0^\infty(G_0)$ and A as in Theorem 3.5 define $J=(1+|Q|)^{1+\varepsilon_0}A$. Then*

(i) $[J(t_0, t, \varphi, +, r)f](q)$

$$=\int_{E(G_0,+,r)} dx\, dk \langle f, \eta_{xk}\rangle \int d\xi\, \varphi(\xi)\hat{\eta}(\xi-k) \exp(i[q\cdot\xi - X(t_0, t, x, \xi)]),$$

(ii) $\operatorname*{s-lim}_{t\to\infty} \exp(i[(t-t_0)h(P)+Y(m_0, o, t-t_0, P)])J(t_0, t, \varphi, +, r)$ *exists for each r.*

The corresponding statements for the negative time hold with the *same* η.

PROOF. (i) Obvious. (ii) Similar to the proof of Lemma 6.2 (iii) of [**19**].

Q.E.D.

§4. Existence of the wave operator.

First we construct a modified free evolution. Since $\delta \notin \{1, 1/2, \cdots\}$ by the assumption A6 we can choose a positive integer m_0 such that

$$m_0\delta < 1 < (m_0+1)\delta. \tag{4.1}$$

Let $(L, U, \boldsymbol{h})$ be a chart. For ξ in L and time $t \geqq t_0 \geqq 0$ define $X_1(t_0, t, \xi), \cdots, X_m(t_0, t, \xi)$ by

$$\begin{cases} X_j(t_0, t, \xi) = th_j(\xi) + Y_j(m_0, t_0, t, \xi), \\ Y_j(0, t_0, t, \xi) = 0, \\ Y_j(p, t_0, t, \xi) = \int_{t_0}^t ds\, W(sh_j'(\xi) + \nabla_\xi Y_j(p-1, t_0, s, \xi)) \quad \text{for} \quad p=1, \cdots, m_0. \end{cases} \tag{4.2}$$

Now define $Z(L, U, \boldsymbol{h}; t_0, t, \xi)$ for ξ in L by

$$\begin{aligned} &Z(L, U, \boldsymbol{h}\ ;\ t_0, t, \xi) \\ &= U(\xi)\, \mathrm{diag}(\exp[-iX_1(t_0, t, \xi)], \cdots, \exp[-iX_m(t_0, t, \xi)])U^*(\xi). \end{aligned} \tag{4.3}$$

If $(L, U, \boldsymbol{h})$ and $(M, V, \boldsymbol{g})$ are two charts then we show that $Z(L, U, \boldsymbol{h}; t_0, t, \cdot) = Z(M, V, \boldsymbol{g}; t_0, t, \cdot)$ on $L \cap M$ in Lemma 4.3. So we (can and do) define $Z(t_0, t, \xi) = Z(L, U, \boldsymbol{h}; t_0, t, \xi)$ for ξ in L and show in Theorem 4.5 that $\Omega_+ = \text{s-lim}_{t\to\infty} \exp[itH]Z^*(0, t, P)$ exists.

LEMMA 4.1. *Let G_0 be any open subset of $\boldsymbol{R}^n$ and $a: G_0\to\boldsymbol{C}$, $p, q: G_0\to\boldsymbol{R}$ be C^∞ functions such that $a(\xi)p(\xi)=a(\xi)q(\xi)$ for all ξ in G_0. Let $A:\boldsymbol{R}^\Theta\to\boldsymbol{R}$ be any C^∞ function. Define A_p, A_q on G_0 by $A_p(\xi)=A(p(\xi), \partial p/\partial\xi_1, \cdots, \partial p/\partial\xi_n, \partial^2 p/\partial\xi_1^2, \partial^2 p/\partial\xi_1\partial\xi_2, \cdots)$. Then*

(i) $$a(\xi)\{-1+\exp(\pm i[A_p(\xi)-A_q(\xi)]\}=0 \qquad \text{for } \xi \text{ in } G_0,$$

(ii) $$a\exp[\pm iA_p]=a\exp[\pm iA_q] \qquad \text{on } G_0.$$

PROOF. (i) Let $N=\{\xi\in G_0: a(\xi)=0\}$. For ξ in N the conclusion is clear. On the open set $G_0\setminus N$ we have $p=q$ by the assumption. So again the result is obvious on $G\setminus N_0$. (ii) Easily follows from (i). Q.E.D.

LEMMA 4.2. *Let G_0, A be as above. Let $B: G_0\to\mathcal{M}_m(\boldsymbol{C})$ be C^∞ and for each ξ the matrix $B(\xi)$ be unitary. Let $p_1, \cdots, p_m, q_1, \cdots, q_m$ be C^∞ real valued functions on G_0 satisfying*

$$B(\xi)\operatorname{diag}(p_1(\xi), \cdots, p_m(\xi))=\operatorname{diag}(q_1(\xi), \cdots, q_m(\xi))B(\xi) \qquad \text{for } \xi \text{ in } G_0.$$

Then we get on G_0

$$B\operatorname{diag}(e^{\pm iA_{p_1}}, \cdots, e^{\pm iA_{p_m}})=\operatorname{diag}(e^{\pm iA_{q_1}}, \cdots, e^{\pm iA_{q_m}})B.$$

PROOF. Let

$$B=\begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{m1} & \cdots & b_{mm}\end{pmatrix}.$$

Then a simple calculation shows that $b_{jk}(p_k-q_j)=0$ on G_0 for all k, j. Now by Lemma 4.1 we get $b_{jk}\exp[\pm iA_{p_k}]=\exp[\pm iA_{qj}]b_{jk}$ on G_0 for all j, k. Now the result is clear. Q.E.D.

LEMMA 4.3. *For any two charts $(L, U, \boldsymbol{h})$, $(M, V, \boldsymbol{g})$ we have $Z(L, U, \boldsymbol{h}; t_0, t, \cdot)=Z(M, V, \boldsymbol{g}; t_0, t, \cdot)$ on $L\cap M$.*

PROOF. By definition of chart we have for ξ in $L\cap M$

$$H_0(\xi)=U(\xi)\operatorname{diag}(h_1(\xi), \cdots, h_m(\xi))U^*(\xi)$$
$$=V(\xi)\operatorname{diag}(g_1(\xi), \cdots, g_m(\xi))V^*(\xi)$$

so that

$$V^*(\xi)U(\xi)\operatorname{diag}(h_1(\xi), \cdots, h_m(\xi))=\operatorname{diag}(g_1(\xi), \cdots, g_m(\xi))V^*(\xi)U(\xi).$$

Now the result follows from Lemma 4.2. Q.E.D.

Now define $Z(t_0, t, \cdot)$ on $\boldsymbol{R}^n$ by

(4.4) $\quad Z(t_0, t, \xi)=Z(L, U, \boldsymbol{h}; t_0, t, \xi) \quad$ if $\xi\in L$ and $(L, U, \boldsymbol{h})$ is a chart.

Set

(4.5) $$Z_t(\xi)=Z(0,\ t,\ \xi) \quad \text{for} \quad t\geqq 0\,,$$

(4.6) $$V_t=\exp[-itH] \quad \text{for all real } t\,.$$

For any vector $\boldsymbol{f}=(f_1, \cdots, f_m)'$ [i.e. $\boldsymbol{f}$ is written as a column vector] with the functions f_j in $L^2(\boldsymbol{R}^n)$ define

(4.7) $$\operatorname{supp}\boldsymbol{f}=\bigcup_j \operatorname{supp} f_j\,.$$

For any chart $(L, U, \boldsymbol{h})$ define $L_0\subset L$ by

(4.8) $$L_0=\bigcap_j \{\xi\in L\ :\ h_j'(\xi)\neq 0 \text{ and } \det h_j''(\xi)\neq 0\}\,.$$

LEMMA 4.4. *Let $(L, U, \boldsymbol{h})$ be a chart and $\boldsymbol{f}\in[\mathcal{S}(\boldsymbol{R}^n)]^m$ be such that* $\operatorname{supp}\hat{\boldsymbol{f}}$ *is a compact subset of L_0. Then there exists $t_{-1}=t_{-1}(\boldsymbol{f})\geqq 0$ such that for all $t_0\geqq t_{-1}$,* $\text{s-lim}_{t\to\infty}V_t^*Z(t_0, t, P)\boldsymbol{f}$ *exists.*

PROOF. For each $t_0\geqq 0$ for the vector $Z(t_0, t, P)\boldsymbol{f}$ we have $\operatorname{supp}[Z(t_0, t, P)\boldsymbol{f}]^\wedge \subset\operatorname{supp}\hat{\boldsymbol{f}}$ and $[Z(t_0, t, P)\boldsymbol{f}]^\wedge(\xi)$ is a C^∞ function of ξ. So $Z(t_0, t, P)\boldsymbol{f}\in[\mathcal{S}(\boldsymbol{R}^n)]^m \subset\operatorname{Dom} H$. Put $g=U^*(P)\boldsymbol{f}$ so that $\boldsymbol{g}\in[\mathcal{S}(\boldsymbol{R}^n)]^m$ and $\operatorname{supp}\hat{\boldsymbol{g}}\subset\operatorname{supp}\hat{\boldsymbol{f}}$. A simple calculation shows that

$$\begin{aligned}
&-iV_{t-t_0}\frac{d}{dt}V_{t-t_0}^*Z(t_0, t, P)\boldsymbol{f}\\
&=W_s(1+P^2)^{-N}(1+Q^2)^{(1+\varepsilon_0)/2}(1+Q^2)^{-(1+\varepsilon_0)/2}\sum_{j=1}^m(1+P^2)^N\\
&\qquad\cdot\exp[-iX_j(t_0, t, P)]e_j(P)g_j\\
&\quad+\sum_{j=1}^m[W(Q)-W(th_j'(P)+\nabla_P Y_j(m_0-1, t_0, t, P))]\\
&\qquad\cdot\exp[-iX_j(t_0, t, P)]e_j(P)g_j\,.
\end{aligned}$$

From the above identity, using Theorem 3.2 we infer that for some $t_{-1}=t_{-1}(\boldsymbol{f})\geqq 0$ and for all $t_0\geqq t_{-1}$, we get

$$\int_{t_0}^\infty dt\left\|\frac{d}{dt}V_{t-t_0}^*Z(t_0, t, P)\boldsymbol{f}\right\|<\infty\,.$$

Now the result is clear. Q.E.D.

THEOREM 4.5.

(i) $\Omega_+=\text{s-lim}_{t\to\infty}V_t^*Z_t$ *exists where* $Z_t=Z(0, t, P)$,

(ii) Ω_+ *is an isometry,*

(iii) $V_s\Omega_+=\Omega_+U_s$ *for all real s where* $U_s=\exp[-isH_0]$,

(iv) $\operatorname{Range}\Omega_+\subset\mathcal{H}_{ac}(H)$, *the absolutely continuous subspace for H.*

PROOF. (i) For a given chart $(L, U, \boldsymbol{h})$ let L_0 be as in (4.8). Then by Lemma 3.3 (i) we see that $\boldsymbol{g}(t_0)=\text{s-lim}_{t\to\infty}Z^*(t_0, t, P)Z(0, t, P)\varphi(P)\boldsymbol{f}$ exists for each φ in

$C_0^\infty(L_0)$ and $\boldsymbol{f}$ in $[\mathcal{S}(\boldsymbol{R}^n)]^m$ and that supp $\hat{\boldsymbol{g}}(t_0)$ is compact in L_0. By using Lemma 4.4 we deduce that s-$\lim_{t\to\infty} V_t^* Z_t \varphi(P)$ exists for each φ in $C_0^\infty(L_0)$. Now by the techniques of the partition of unity [27] we see that s-$\lim_{t\to\infty} V_t^* Z_t \varphi(P)$ exists for φ in $C_0^\infty(\bigcup\{L_0 : (L, U, \boldsymbol{h})$ is a chart$\})$. Now the result is clear by the assumption A3. (ii) Obvious. (iii) Let $(L, U, \boldsymbol{h})$ be a chart and $\varphi \in C_0^\infty(L_0)$. Then by Lemma 3.3 (ii) we get $V_s \Omega_+ \varphi(P) = \Omega_+ U_s \varphi(P)$. Now the result follows as in (i). (iv) By the assumption A3 the operator H_0 has only absolutely continuous spectrum. Now the result is standard [24, 29]. Q.E.D.

§5. Proof of asymptotic completeness.

Let $(L, U, \boldsymbol{h})$ be a chart and L_0 be as in (4.8). For x in $\boldsymbol{R}^n$, ξ in L_0, $t \geqq t_0 \geqq 0$ define $X_1(t_0, t, x, \xi), \cdots, X_m(t_0, t, x, \xi)$ by, with m_0 as in (4.1),

$$
(5.1)\quad \begin{cases} X_j(t_0, t, x, \xi) = x\cdot\xi + (t-t_0)h_j(\xi) + Y_j(m_0, t_0, t, x, \xi), \\ Y_j(0, t_0, t, x, \xi) = 0, \\ Y_j(p, t_0, t, x, \xi) = \displaystyle\int_{t_0}^{t} ds W(x + s h_j'(\xi) + \nabla_\xi Y_j(p-1, t_0, s, x, \xi)) \\ \qquad\qquad\qquad\qquad \text{for } p = 1, 2, \cdots, m_0. \end{cases}
$$

Note that

$$
(5.2)\quad \begin{cases} X_j(t_0, t, x, \xi) = x\cdot\xi, \\ \partial X_j(t_0, t, x, \xi)/\partial t = h_j(\xi) + W(x + t h_j'(\xi) + \nabla_\xi Y_j(m_0 - 1, t_0, t, x, \xi)). \end{cases}
$$

For b^* in $(0, 1)$ [to be chosen properly later] let η be as in (3.4). Let $\varphi \in C_0^\infty(L_0)$ be real valued. Define $I(t_0, t, \varphi, +, r)$ for $t \geqq t_0 \geqq 0$, $r > 0$ by

$$
(5.3)\quad \begin{aligned} &[I(t_0, t, \varphi, +, r)\boldsymbol{f}](q) \\ &= \sum_j \int_{E_j(L_0, +, r)} dx_j dk_j \langle f_j, \eta_{x_j k_j}\rangle \int d\xi\, \varphi(\xi)\hat{\eta}(\xi - k_j) \\ &\quad \cdot \exp(i[q\cdot\xi - X_j(t_0, t, x_j, \xi)]) e_j(\xi) \end{aligned}
$$

where

$$
(5.4)\quad E_j(L_0, \pm, r) = \{(x, k) : k \in L_0,\ x\cdot\nabla h_j(k) \gtrless 0,\ |x| \geqq r\}.
$$

A simple calculation shows that

$$
(5.5)\quad I(t_0, t_0, \varphi, +, r) = U(P)\varphi(P)\,\mathrm{diag}(T(E_1(L_0, +, r)), \cdots, T(E_m(L_0, +, r))).
$$

LEMMA 5.1. *Given φ, L_0 as above there exists some $d = d(\varphi) > 0$ and $t_{-1} = t_{-1}(\varphi) \geqq 0$ so that for any η of (3.4) with $b^* \leqq d$ and $t_0 \geqq t_{-1}$ we get*

$$
\text{(i)}\quad \lim_{r\to\infty} \int_{t_0}^{\infty} dt \left\| \frac{d}{dt} V_{t-t_0}^* I(t_0, t, \varphi, +, r) \right\| = 0,
$$

(ii) $$\lim_{r\to\infty}\sup_{t\geq t_0}\|V_{t-t_0}I(t_0, t_0, \varphi, +, r)-I(t_0, t, \varphi, +, r)\|=0\,,$$

(iii) $$\lim_{r\to\infty}\sup_{t\geq t_0}\|(1+|Q|)^{-1-\varepsilon_0}I(t_0, t, \varphi, +, r)\|=0\,.$$

The corresponding statements for the negative time hold with the *same* η.

PROOF. (i) It is easy to see that for $\boldsymbol{f}$ in $[L^2(\boldsymbol{R}^n)]^m$ we get

$$\left[-iV_{t-t_0}\frac{d}{dt}\{V_{t-t_0}^*I(t_0, t, \varphi, +, r)\}\boldsymbol{f}\right](q)$$

$$= \{W_s(1+P^2)^{-N}(1+|Q|)^{1+\varepsilon_0}(1+|Q|)^{-1-\varepsilon_0}(1+P^2)^N I(t_0, t, \varphi, +, r)\boldsymbol{f}\}(q) \tag{5.6}$$

$$+\sum_j\int_{E_j(L_0,+,r)} dx_j dk_j\langle f_j, \eta_{x_jk_j}\rangle$$

$$\cdot\int d\xi\,[W(q)-W(x+th_j'(\xi)+\nabla_\xi Y_j(m_0-1, t_0, t, x, \xi))]$$

$$\cdot\varphi(\xi)\hat{\eta}(\xi-k_j)\exp(i[q\cdot\xi-X_j(t_0, t, x_j, \xi)])e_j(\xi)$$

where

$$\{(1+|Q|)^{-1-\varepsilon_0}(1+P^2)^N I(t_0, t, \varphi, +, r)\boldsymbol{f}\}(q)$$

$$=\sum_j(1+|Q|)^{-1-\varepsilon_0}\int_{E_j(L_0,+,r)} dx_j dk_j\langle f_j, \eta_{x_jk_j}\rangle \tag{5.7}$$

$$\cdot\int d\xi\,\varphi(\xi)(1+\xi^2)^N\hat{\eta}(\xi-k_j)\exp(i[q\cdot\xi-X_j(t_0, t, x_j, \xi)])e_j(\xi)\,.$$

Now the result follows from (5.6), (5.7), Theorem 3.5 and the assumption A7. (ii) Follows from (i). (iii) Put $N=0$ in (5.7) to get an expression for the operator $(1+|Q|)^{-1-\varepsilon_0}I(t_0, t, \varphi, +, r)$ and apply Theorem 3.5. Q.E.D.

For any measurable function $f:[0, \infty)\to[0, \infty)$ define $\mathcal{E}(f)$ by

$$\mathcal{E}(f)=\limsup_{s\to\infty} s^{-1}\int_0^s dt\, f(t)\,. \tag{5.8}$$

$\mathcal{E}$ stands for the ergodic average. $\mathcal{E}(f)$ shall also be denoted by $\mathcal{E}(f(t))$ in the sequel.

LEMMA 5.2. *Let φ, L_0, $I(t_0, t, \varphi, +, r)$ be as in Lemma 5.1. Put $T_\pm=$ diag$(T(E_1(L_0, \pm, 0)), \cdots, T(E_m(L_0, \pm, 0)))$. Then for each f in $\mathcal{H}_c(H)$*

(i) $$\mathcal{E}[\|T_+\varphi(P)U^*(P)V_t^*f\|]=0\,,$$

(ii) $$\mathcal{E}[\|T_-\varphi(P)U^*(P)V_tf\|]=0\,.$$

PROOF. (i) By Lemma 5.1 (ii), (iii) and (5.5) we get

$$\lim_{r\to\infty}\sup_{t\geq t_0}\|\mathrm{diag}(T(E_1(L_0, +, r)), \cdots, T(E_m(L_0, +, r)))$$

$$\cdot\varphi(P)U^*(P)V_{t-t_0}^*(1+|Q|)^{-1-\varepsilon_0}\|=0\,.$$

So by using density of Range$(1+|Q|)^{-1-\varepsilon_0}$ we deduce that for each g in $[L^2(\boldsymbol{R}^n)]^m$

$$\lim_{r\to\infty}\sup_{t\geq t_0}\|\operatorname{diag}(T(E_1(L_0, +, r))\varphi(P), \cdots, T(E_m(L_0, +, r))\varphi(P))U^*(P)V_t^*g\|=0.$$

Now the result follows by the compactness of

$$T\{(x_j, k_j) : k_j\in L_0, x_j\cdot h_j'(k_j)\geqq 0, |x_j|\leqq r\}\varphi(P)$$

for each $j=1, \cdots, m$, each $r>0$ and RAGE Theorem [**29**]. (ii) Similar to (i).

Q.E.D.

LEMMA 5.3. *Let* φ, L_0, $I(t_0, t, \varphi, +, r)$ *be as in Lemma* 5.1. *Then*

(i) $\omega_1(t_0, \varphi, +, r)=\text{s-}\lim_{t\to\infty} Z_{t-t_0}^* I(t_0, t, \varphi, +, r)$ *exists.*

(ii) $\Omega_1(t_0, \varphi, +, r)=\text{s-}\lim_{t\to\infty} V_{t-t_0}^* I(t_0, t, \varphi, +, r)$ *exists.*

(iii) $\Omega_1(t_0, \varphi, +, r)=\Omega_+\omega_1(t_0, \varphi, +, r)$.

(iv) $\lim_{r\to\infty}\|(1-\Omega_+\Omega_+^*)U(P)\varphi(P)\operatorname{diag}(T(E_1(L_0, +, r)), \cdots, T(E_m(L_0, +, r)))\|=0$.

(v) $(1-\Omega_+\Omega_+^*)U(P)\varphi(P)T_+$ *is compact.*

The corresponding statements for the negative time hold with the *same* η.

PROOF. (i) For any ψ in $C_0^\infty(L_0)$ with $\psi\varphi=\varphi$ it is easily seen that $\psi(P)I(t_0, t, \varphi, +, r)=I(t_0, t, \varphi, +, r)$. With the above observation the result follows by using Lemma 3.6 (ii). (ii) Follows from Lemma 5.1 (i). (iii) Follows from (i) and (ii). (iv) Similar to the proof of Lemma 6.2 (vii) of [**19**]. (v) Follows from (iv) by the compactness of

$$\varphi(P)T\{(x_j, k_j) : |x_j|\leqq r, x_j\cdot h_j'(k_j)\geqq 0, k_j\in L_0\}$$

for each j.

Q.E.D.

LEMMA 5.4. (i) *Let* $(L, U, \boldsymbol{h})$ *be a chart and* L_0 *as in* (4.8). *Then for* φ *in* $C_0^\infty(L_0)$ *and* f *in* $\mathcal{H}_c(H)$ *one has* $\mathcal{E}[\|(1-\Omega_+\Omega_+^*)\varphi(P)V_tf\|]=0$.

(ii) *Let* G *be as in assumption* A.3 *and* $\varphi\in C_0^\infty(G)$. *Then for* f *in* $\mathcal{H}_c(H)$ *we get* $\mathcal{E}[\|(1-\Omega_+\Omega_+^*)\varphi(P)V_tf\|]=0$.

(iii) $\mathcal{H}_c(H)\ominus\text{Range}\,\Omega_+=\{f\in\mathcal{H}_c(H) : \mathcal{E}[\|\varphi(P)V_tf\|]=0$ *for each* φ *in* $C_0^\infty(G)\}$.

(iv) $\mathcal{H}_c(H)\ominus\text{Range}\,\Omega_+=\{f\in\mathcal{H}_c(H) : \mathcal{E}[\|\boldsymbol{\varphi}(P)V_tf\|]=0$ *for each* $\boldsymbol{\varphi}$ *in* $C_0^\infty(G)\}$. *Here* $\boldsymbol{\varphi}$ *is a matrix* $((\varphi_{jk}))_{j,k=1,\cdots,m}$. $\boldsymbol{\varphi}$ *is in* $C_0^\infty(G)$ *means each* φ_{jk} *is in* $C_0^\infty(G)$.

PROOF. (i) Clearly we can assume φ to be real valued. Given such φ choose a real valued ψ in $C_0^\infty(L_0)$ such that $\psi\varphi=\varphi$. Let $f\in\mathcal{H}_c(H)$. Now choose b^* in (3.4) so that we have Lemma 5.3 (v) holds for φ *and* Lemma 5.2 (ii) holds for ψ i.e. $(1-\Omega_+\Omega_+^*)U(P)\varphi(P)T_+$ is compact and $\mathcal{E}[\|T_-\psi(P)U^*(P)V_tf\|]=0$. Now by RAGE Theorem and boundedness of $U^*(P)\psi(P)$ we have

(5.9) $$\mathcal{E}[\|(1-\Omega_+\Omega_+^*)U(P)\varphi(P)T_+U^*(P)\psi(P)V_tf\|]=0\,.$$

Again, using the boundedness of $(1-\Omega_+\Omega_+^*)U(P)\varphi(P)$ we trivially have

(5.10) $$\mathcal{E}[\|(1-\Omega_+\Omega_+^*)U(P)\varphi(P)T_-U^*(P)\psi(P)V_tf\|]=0\,.$$

Now add (5.9) and (5.10) and use $T_++T_-=1$, $\varphi\psi=\varphi$ to get the result. (ii) Follows from (i) by the techniques of partition of unity [27]. (iii) Let $f\in$ L. H. S. and $\varphi\in C_0^\infty(G)$. Put $\psi=\varphi\bar{\varphi}$ so that $\psi\in C_0^\infty(G)$. Then

$$\|\varphi(P)V_tf\|^2=|\langle(1-\Omega_+\Omega_+^*)\psi(P)V_tf,\ V_tf\rangle|\leqq\|(1-\Omega_+\Omega_+^*)\psi(P)V_tf\|\|f\|\,.$$

By (ii) we see that $f\in$ R. H. S. Thus L. H. S. $\subset$ R. H. S.

Let $f\in$ R. H. S. and $g\in$ Range Ω_+. Put $g=\Omega_+h$. Then for any real valued φ in $C_0^\infty(G)$ we have

$$\langle f,\ g\rangle=\langle V_tf,\ V_tg-Z_th\rangle+\langle\varphi(P)V_tf,\ Z_th\rangle+\langle V_tf,\ Z_t[1-\varphi(P)]h\rangle\,.$$

Now using the hypothesis on f and $g=\Omega_+h$ we conclude

$$|\langle f,\ g\rangle|\leqq\|[1-\varphi(P)]h\|\|f\|$$

for each real valued φ in $C_0^\infty(G)$. Since $\boldsymbol{R}^n\setminus G$ has measure zero we get $\langle f,\ g\rangle=0$. Thus R. H. S. $\subset$ L. H. S.

(iv) Let E_{jk} be the matrix units i.e. 1 at the jkth place and 0 everywhere else. Then (iv) follows from (iii) by noting $\boldsymbol{\varphi}=((\varphi_{jk}))=\sum_{j,k}\varphi_{jk}E_{jk}$ and $\varphi=\sum_j\varphi E_{jj}$. Q. E. D.

THEOREM 5.5. Range $\Omega_+=\mathcal{H}_c(H)=$ Range Ω_-.

PROOF (For the positive sign only). Let $f\in\mathcal{H}_c(H)\ominus$ Range Ω_+. Let $C_v(H_0)$ be as in assumption A4. Choose $\psi\in C_0^\infty(\boldsymbol{R}\setminus\bar{C}_v(H_0))$. Put $\boldsymbol{\varphi}(P)=\psi(H_0)$. Then $\boldsymbol{\varphi}\in C_0^\infty(G)$. So by Lemma 5.4 (iv)

(5.11) $$\mathcal{E}[\|\psi(H_0)V_tf\|]=0\,.$$

By assumption A9 and Stone Weierstrass theorem the operator $\psi(H)-\psi(H_0)$ is compact. So by RAGE Theorem

(5.12) $$\mathcal{E}[\|[\psi(H)-\psi(H_0)]V_tf\|]=0\,.$$

From (5.11) and (5.12) we get $0=\psi(H)f$ for each ψ in $C_0^\infty(\boldsymbol{R}\setminus\bar{C}_v(H_0))$. Since $\bar{C}_v(H_0)$ is countable and $f\in\mathcal{H}_c(H)$ we conclude $f=0$. This completes the proof. Q. E. D.

THEOREM 5.6. *Any eigenvalue of H not in $\bar{C}_v(H_0)$ is of finite multiplicity. Such eigenvalues can accumulate only at the points of $\bar{C}_v(H_0)$.*

PROOF. Let E be the orthogonal projection onto the point subspace for H.

Let $(L, U, \boldsymbol{h})$ be a chart and $\varphi \in C_0^\infty(L_0)$ be real valued. Then by Lemma 5.3 (v) we can choose η of (3.4) such that $(1-\Omega_\pm \Omega_\pm^*)U(P)\varphi(P)T_\pm$ is compact. Since $1-\Omega_\pm \Omega_\pm^* = E$ by Theorem 5.5 we get $EU(P)\varphi(P)$ is compact. So $E\varphi(P)$ is compact. Clearly now $E\boldsymbol{\varphi}$ is compact for each $\boldsymbol{\varphi}$ in $C_0^\infty(G)$. So $E\psi(H_0)$ is compact for each ψ in $C_0^\infty(\boldsymbol{R}\setminus \bar{C}_v(H_0))$. Since $\psi(H)-\psi(H_0)$ is compact we get $E\psi(H)$ is compact for each ψ in $C_0^\infty(\boldsymbol{R}\setminus \bar{C}_v(H_0))$. Now the result is clear. Q.E.D.

By using the same techniques we can prove a general theorem when $W(Q)$ is replaced by a general pseudo differential operator $W(Q, P)$ with a smooth symbol $W(x, \xi)$. More precisely we have

THEOREM 5.7. *Define* A*6, A*8 *and* A*9 *by*

A*6 (*condition on the long range*). $W: \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ *is a* C^∞ *function. There exist a polynomial* $q: \boldsymbol{R}^n \to \boldsymbol{R}$ *and* δ *in* $(0, 1]$ *such that*

$$|W(x, \xi)| \leqq (1+|x|)^{-\delta} q(\xi)$$

for all x, ξ. *Also for each compact subset* B *of* $\boldsymbol{R}^n$ *and multi-indices* α, β

$$|D_\xi^\alpha D_x^\beta W(x, \xi)| \leqq K(B, \alpha, \beta)(1+|x|)^{-|\beta|-\delta} \qquad \text{for } (x, \xi) \text{ in } \boldsymbol{R}^n \times B$$

holds for suitable constants $K(B, \alpha, \beta)$. *In such a case define* $W(Q, P)$ *on* $\mathcal{S}(\boldsymbol{R}^n)$ *by*

$$[W(Q, P)f](q) = (2\pi)^{-n/2} \int d\xi \hat{f}(\xi) W(q, \xi) \exp(iq \cdot \xi).$$

(*Assume that*) $W(Q, P)$ *maps* $\mathcal{S}(\boldsymbol{R}^n)$ *into* $L^2(\boldsymbol{R}^n)$.

A*8. *Same as* A8 *with* $W(Q)$ *replaced by* $W(Q, P)$.

A*9. *Same as* A9 *for new* H.

Let A1, $\cdots$, A5, A*6, A7, A*8, A*9 *hold. Then Theorem* 2.1 *is true for* (*the new*) H.

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Note added after submission.

In the assumption A4 the definition of critical values involves the second derivatives. Let us for simplicity take $m=1$, $n=2$ i.e. $H_0 = h_0(P)$ where $h_0: \boldsymbol{R}^2 \to \boldsymbol{R}$ is C^∞. Then $\bar{C}_v$ is countable, where

$$C_v = \{h_0(\xi) : h_0'(\xi) = 0 \text{ or } \det h_0''(\xi) = 0\}$$

for $h_0(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^r$, $r > 0$ or $h_0(\xi_1, \xi_2) = \xi_1^4 + \xi_2^4 + a(\xi_1^2 + \xi_2^2)$, $a > 0$ or $h_0(\xi_1, \xi_2) =$

$\xi_1^4+\xi_2^6+a(\xi_1^2+\xi_2^2)$, $a>0$ but not for the elliptic case $h_0(\xi_1, \xi_2)=\xi_1^4+\xi_2^4$. The aim of this note is to overcome this highly unsatisfactory state of affairs.

Let G_0 be any open set of $\boldsymbol{R}^n$ and $h_0: G_0\to\boldsymbol{R}$ any C^∞ function such that $|h_0'(\xi)|>0$ for each ξ in G_0. (Note that we have removed the condition $|\det h_0''(\xi)|>0$ on G_0 which was imposed in § 3). Then using the techniques of [**30**] we can prove Theorem 3.2. For example refer to the proof of Lemma 3.5 in [**31**].

With G_0 as above we can prove, using the techniques of [**30**], Theorem 3.5. A similar result is proved as Theorem 5.5 in [**32**].

Now proceeding exactly as in § 4 and § 5 we see that the assumptions A3 and A4 can be improved to the assumptions A′3 and A′4: for any chart $(L, U, \boldsymbol{h})$ we define the critical set C and critical values C_v by

$$C(L, U, \boldsymbol{h})=\bigcup_{j=1}^{m}\{\xi\in L : h_j'(\xi)=0\} ,$$

$$C_v(L, U, \boldsymbol{h})=\bigcup_{j=1}^{m}\{h_j(\xi) : \xi\in L,\ h_j'(\xi)=0\} .$$

A′3. $G=\bigcup\{L\setminus C(L, U, \boldsymbol{h}) : (L, U, \boldsymbol{h}) \text{ is a chart}\}$ is an open subset of $\boldsymbol{R}^n$ with $\boldsymbol{R}^n\setminus G$ having (Lebesgue) measure zero.

A′4. The closure of $C_v(H_0)$ is countable where

$$C_v(H_0)=\bigcup\{C_v(L, U, \boldsymbol{h}) : (L, U, \boldsymbol{h}) \text{ is a chart}\} .$$

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