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The support of global graph links

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Introduction.

In this paper we study global graph links i.e. graph links in three dimensional manifolds, and the main purpose is to answer Tamura's problem below for such links: In [14], [15], Tamura has studied *global knot theory*—the investigation of (codimension two) spheres embedded in manifolds. He called a knot *local* if it is contained in an embedded ball in the ambient manifold and raised in [14] the following

PROBLEM. Give criteria for a knot to be local.

In dimension three, the fundamental group of the exterior determines up to the Poincaré conjecture whether given knot is local or not, since Kneser's conjecture is true (see [21]). However, if one looks for a localness criterion in terms of the homotopy class of the knot, one encounters the following difficulty: Local knots are *inessential* i.e., null homotopic in the ambient manifold, but the converse does not hold. A counterexample in a solid torus is given in Figure 1. By the characterization of S^3 due to Bing [2], any three dimensional manifold admits a knot with irreducible exterior. Thus embedding the above example into a tubular neighborhood of such a knot, we have a counterexample in any manifold. (See Tamura [15] for counterexamples in higher dimensions.)

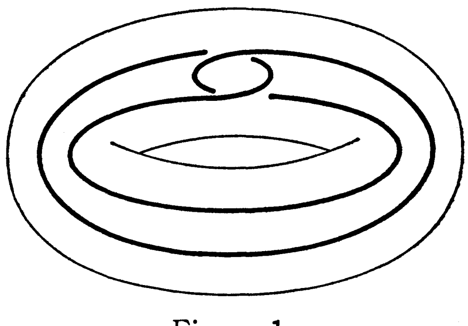

Figure 1.

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In contrast with this, if we restrict ourselves to graph knots and graph links, the converse does hold. For the precise argument, we need the notion of the support and the homotopy support of a link in an orientable three dimensional manifold. The *support* is the essential part of the ambient manifold, in which the link lies, and the *homotopy support* is that in the sense of homotopy (see section 1).

THEOREM. *Let M be an orientable graph manifold and suppose that M does not contain $S^1 \times S^2$ and manifolds with finite fundamental groups as prime factors. Then the support of a graph link in M coincides with its homotopy support.*

As a corollary to this, we have

LOCALNESS THEOREM FOR GRAPH LINKS. *A graph link in a manifold as in the above theorem is local if and only if it is inessential.*

In these results, the condition on the ambient manifold cannot be omitted. In fact, all graph manifolds with finite fundamental groups as well as $S^1 \times S^2$ admit graph knots which are not local but inessential (section 5). In case where the ambient manifold contains $S^1 \times S^2$ as a prime factor, we cannot define the homotopy support of links. We determine, however, the support of such a graph link if it is inessential and also that of a graph link consisting of local knots (section 5).

The graph manifold theory was first developed by Waldhausen [**19**] with his graph decompositions, and was then refined by Neumann [**12**] with the plumbing diagrams. In this paper we introduce a new method; $\mathfrak{R}$-decompositions of graph manifolds, which is a modification of round handle decompositions due to Asimov [**1**] (see also Morgan [**11**]). In section 2 we give the definition of this and prove fundamental results about it. Using this in section 3, we establish the following (see sections 1 and 3 for terminology).

CHARACTERIZATION OF GLOBAL GRAPH LINKS. *Every graph link is obtained from elementary ones by a finite number of operations of disjoint union, adding a cable, replacing by a cable, link connected sum and self link connected sum.*

This was known for graph knots in S^3 by Gordon [**4**] and Ue [**18**] (see also Soma [**13**] and Eisenbud-Neumann [**3**]). In section 4 we study how the support changes by the operations in this theorem and the main results are proved in section 5. Finally in section 6, we study Dehn surgeries along graph knots as an application of $\mathfrak{R}$-decompositions and prove

THEOREM. *All but two Dehn surgeries along a graph knot with irreducible exterior yield irreducible manifolds.*

A part of this paper was announced in [**20**].

The author would like to thank I. Tamura for leading him to this field and to thank M. Ue for helpful comments. Thanks are also due to T. Oshima, without whose excellent word processor, this paper could not appear.

1. Terminology.

We work in the smooth category, and manifolds are assumed to be connected, compact, oriented, and of dimension three unless otherwise stated. A manifold M is *irreducible* if every embedded S^2 bounds a ball in M and is *prime* if either M is irreducible and is not S^3 or M is $S^1 \times S^2$. It is known by Kneser [**9**] and Milnor [**10**] that every manifold is uniquely decomposed as a connected sum of prime factors. We say that a manifold is *prime to* $S^1 \times S^2$ if its prime decomposition does not contain $S^1 \times S^2$ as its factor. An embedded orientable surface in a manifold is *incompressible* if the homomorphism between the fundamental groups induced by the inclusion is injective, and *compressible* otherwise.

A *link* is a union of disjointly embedded oriented circles in the interior of a manifold and a *knot* is a link with a single component. For convenience, the empty set is also considered as a link, and is called an *empty link*. We identify two links if they coincide up to ambient isotopy. A link is *inessential* if each component of it is null homotopic in the ambient manifold, and *essential* otherwise. A link is *nonsplittable* if there does not exist an embedded S^2 which separates the ambient manifold into two nonempty sublinks.

Now we define the support and the homotopy support of a link L, which will be denoted by $\mathrm{supp}(L)$ and $\text{h-supp}(L)$, respectively. Although the support and the homotopy support are diffeomorphism classes of manifolds in the strict sense, by abuse of notation, we treat them as manifolds in the sequel.

Let $L=\bigcup k_i$ be a link in a manifold M. Let $N(L)=\bigcup N(k_i)$ be a tubular neighborhood and $E(L)$ the *exterior* $M-\mathrm{int}\, N(L)$.

DEFINITION. When L is nonsplittable, there exist a link L_0 in a manifold M_0 with irreducible exterior and another manifold M_1 such that the pair (M, L) is diffeomorphic to $(M_0 \# M_1, L_0)$. Then the *support* of L is the manifold M_0. In general, L is decomposed into nonsplittable sublinks $L=\bigcup L_j$, and then the *support* of L is the connected sum of the supports of L_j's. The *support* of an empty link is defined to be S^3.

In other words, the support of a link L is given as follows: Consider the prime decomposition of $E(L)$ and take prime factors which contain some component of $\partial N(L)$. Then the support of L is the connected sum of these factors with $N(L)$ glued back at $\partial N(L)$. A link is *local* if its support is S^3 and is *full* if the ambient manifold is its support.

Let M be a manifold prime to $S^1 \times S^2$ and fix a homeomorphism $\phi: M \simeq M_1 \# \cdots \# M_k$ which gives a prime decomposition. Then there is a natural isomorphism

$\phi_*:\pi_1(M)\to\pi_1(M_1)*\cdots*\pi_1(M_k)$. Let $p_j:\pi_1(M_1)*\cdots*\pi_1(M_k)\to\pi_1(M_j)$ be a homomorphism given by adding relations $\pi_1(M_i)=1$ for $i\neq j$.

DEFINITION. For a link $L=\bigcup k_i$ in M, the *homotopy support* of L is defined to be the connected sum of M_j's such that there is a k_i with $p_j(\phi_*([k_i]))\neq 1$ in $\pi_1(M_j)$. That is,

$$\text{h-supp}(L)=\mathop{\#}_{j \text{ such that } p_j(\phi_*([k_i]))\neq 1 \text{ for some } i} M_j .$$

This definition does not depend on the homeomorphism ϕ because of the lemma below, which is shown in Hempel [6] in the course of the proof of Milnor's uniqueness theorem for the prime decomposition. However, for links in manifolds not prime to $S^1\times S^2$, it cannot be a well defined notion for the lack of the property below (see Example 4.8).

LEMMA 1.1. *Let M_i and N_j be prime manifolds, not diffeomorphic to $S^1\times S^2$. If $\phi:M_1\#\cdots\# M_k\to N_1\#\cdots\# N_l$ is a homeomorphism, then $k=l$ and there is a permutation $\sigma:\{1,\cdots,k\}\to\{1,\cdots,k\}$ such that $\phi(\hat{M}_i)$ is isotopic into $\hat{N}_{\sigma(i)}$. Here $\hat{M}$ denotes the manifold obtained from M by deleting an embedded ball.*

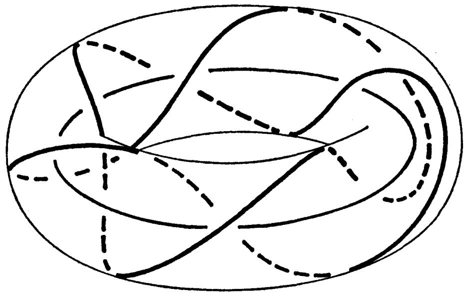
A manifold M is a *graph manifold* if there is a family of disjointly embedded tori in M such that each connected component of the manifold obtained by cutting M along these tori is the total space of an S^1-bundle over a surface. A link is called a *graph link* if its exterior is a graph manifold. Note that manifolds which admit graph links are also graph manifolds.

In the rest of this section we define operations for links, by which a characterization of graph links is given.

For two links $L_1\subset M_1$ and $L_2\subset M_2$, the *disjoint union* of L_1 and L_2 is the link $L_1\cup L_2$ in the manifold $M_1\# M_2$.

Let k be a knot in a manifold M and fix an identification $N(k)\simeq S^1\times D^2$, where S^1 and D^2 are oriented. Then there is a natural basis $\{\mu,\lambda\}$ for $H_1(\partial N(k);\boldsymbol{Z})\simeq\boldsymbol{Z}+\boldsymbol{Z}$, defined by $\mu=[\{*\}\times\partial D^2]$ and $\lambda=[S^1\times\{*\}]$. For two integers m and n, the *(m,n)-cable knot* of k is a knot on $\partial N(k)$ which represents $m\mu+n\lambda$ in $H_1(\partial N(k);\boldsymbol{Z})$, where $m=0$, $n=\pm1$ or m and n are coprime (Figure 2). Note that this definition allows $(m,\pm1)$-cable knots, which are excluded by the usual definition.

Let $L_1\subset M_1$, $L_2\subset M_2$ be links and $k_1\subset L_1$, $k_2\subset L_2$ be components. Take a small ball $B_i\subset N(k_i)$ which intersects k_i by an arc, for $i=1, 2$. Glue $M_1-\operatorname{int}B_1$ and $M_2-\operatorname{int}B_2$ to make $(k_1-\operatorname{int}B_1)\cup(k_2-\operatorname{int}B_2)$ again a circle and the orientation compatible. Then the resulting link in $M_1\# M_2$ (Figure 3) is called the *link connected sum* of L_1 and L_2 by k_1 and k_2 and is denoted by $L_1\# L_2(k_1,k_2)$. For simplicity, we rather write $L_1\# L_2$ instead of $L_1\# L_2(k_1,k_2)$ and call any one of $L_1\# L_2$'s a link connected sum of L_1 and L_2. If both L_1 and L_2 are



(5, 2)-cable knot

Figure 2.

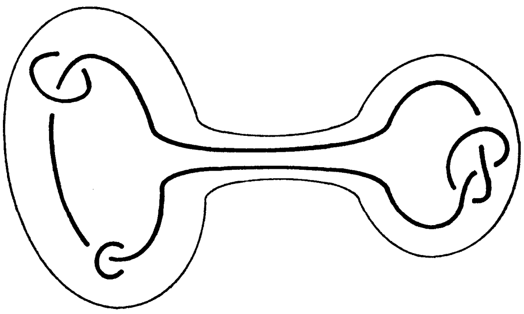


Figure 3.

knots, $L_1 \# L_2$ is called the *knot connected sum* of L_1 and L_2.

Let L be a link in a manifold M with more than one component, and let k_1, k_2 be two distinct components of L. Let B_1 and B_2 be balls chosen as above and glue ∂B_1 and ∂B_2 in $M - \operatorname{int} B_1 - \operatorname{int} B_2$ to make $(k_1 - \operatorname{int} B_1) \cup (k_2 - \operatorname{int} B_2)$ a circle and the orientation compatible. Then the resulting link in $M \# S^1 \times S^2$ (Figure 4) is called the *self link connected sum* of L by k_1 and k_2 and is denoted by ${}^{\#}L(k_1, k_2)$. For simplicity, we rather write ${}^{\#}L$ instead of ${}^{\#}L(k_1, k_2)$ and call any one of ${}^{\#}L$'s a self link connected sum of L.

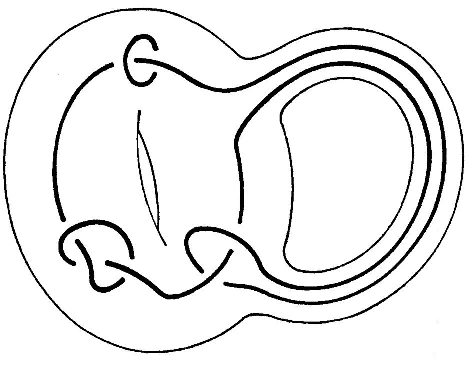


Figure 4.

2. $\mathscr{R}$-decompositions of graph manifolds.

To deal with graph manifolds, we introduce in this section the notion of $\mathscr{R}$-decompositions, which is a modification of that of round handle decompositions (compare with graph decompositions by Waldhausen [**19**] and plumbing diagrams by Neumann [**12**]). For round handle decompositions, we refer to Asimov [**1**] and Morgan [**11**].

DEFINITION. A decomposition $\mathscr{D}=\{S_1, \cdots, S_m; H_1, \cdots, H_n\}$ of a manifold M is called an *$\mathscr{R}$-decomposition* if

(i) there is a family of disjointly embedded tori such that $\mathscr{D}$ is the set of connected components of the manifold obtained by cutting M along these tori, and

(ii) $S_i \simeq S^1 \times D^2 \quad i=1, \cdots, m,$ and

$$H_j \simeq S^1 \times (\text{two punctured disk}) \qquad j=1, \cdots, n.$$

We write S instead of S_i and H instead of H_j except when confusion might occur.

Since the twisted S^1-bundle over the Möbius band has a Seifert fibered structure over D^2, we have

LEMMA 2.1. *A manifold admits an $\mathscr{R}$-decomposition if and only if it is a graph manifold.*

Let $\mathscr{D}$ be an $\mathscr{R}$-decomposition of a manifold M. We say that an element A of $\mathscr{D}$ is a *neighbor* of another element B if A and B have a common boundary component in M. If A and B are neighbors with the common boundary component T, we write $A \underset{T}{\sim} B$ or simply $A \sim B$. We set $h(\mathscr{D})$ to be the number n of elements of $\mathscr{D}$ diffeomorphic to $S^1 \times$(two punctured disk).

When $h(\mathscr{D}) \geqq 1$, every element S of $\mathscr{D}$, diffeomorphic to $S^1 \times D^2$, has a unique neighbor $H \in \mathscr{D}$, and thus ∂S has the structure of an S^1-bundle induced by the inclusion $\partial S \to H$. Then the *winding number* of S is defined by $w(S)=[\text{fibre}] \in H_1(S; \boldsymbol{Z}) \simeq \boldsymbol{Z}$. In this case we define the *induced decomposition* $\hat{\mathscr{D}}=\{\hat{H}_j\}$ of M by $\hat{H}_j=H_j \cup \bigcup_{S_i \sim H_j} S_i$ for each $H_j \in \mathscr{D}$. Here the union is taken in M, for example, if two boundary components of H_j have the same image under the inclusion $H_j \to M$, then they are identified in $\hat{H}_j$.

For an $\mathscr{R}$-decomposition $\mathscr{D}$ of M, we define the *diagram* of $\mathscr{D}$ as follows: We write $\bigcirc$ for each $S \in \mathscr{D}$, $\triangle$ for each $H \in \mathscr{D}$ and $\bullet$ for each connected component of ∂M. Here each edge of $\triangle$ indicates a boundary component of the corresponding H. Two of $\bigcirc$'s, edges of $\triangle$'s and $\bullet$'s are connected by—if the corresponding boundary components have the same image under the inclusion into M. (See Figure 5 for examples.)

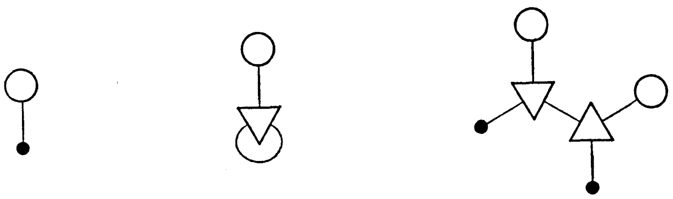


Figure 5.

An $\mathcal{R}$-decomposition $\mathcal{D}$ of M is called *minimal* unless some union of more than one elements of $\mathcal{D}$ in M is again diffeomorphic either to $S^1 \times D^2$ or to $S^1 \times$(two punctured disk). The arguments below are based on those of Morgan's in [**11**], while he used irreducible minimal round handle decompositions.

LEMMA 2.2. *Suppose that $\mathcal{D}$ is a minimal $\mathcal{R}$-decomposition of a manifold M and that there is an $S \in \mathcal{D}$ with $w(S) = \pm 1$. Then M is diffeomorphic either to $T^2 \times I$ or to a T^2-bundle over S^1.*

PROOF. Let $S \underset{T}{\sim} H$. Then $S \cup_T H \simeq T^2 \times I$; thus either $\{S, H\}$ can be eliminated from $\mathcal{D}$ or $\mathcal{D} = \{S, H\}$. The former case does not occur since $\mathcal{D}$ is minimal, and in the latter case, M is diffeomorphic either to $T^2 \times I$ or to a T^2-bundle over S^1. $\square$

LEMMA 2.3. *Suppose that $\mathcal{D}$ is a minimal $\mathcal{R}$-decomposition of an irreducible manifold. Then every $S \in \mathcal{D}$ satisfies $w(S) \neq 0$.*

PROOF. If there is an S such that $w(S) = 0$ with $S \underset{T}{\sim} H$, then $S \cup_T H \simeq S^1 \times D^2 \# S^1 \times D^2$. This contradicts the minimality of $\mathcal{D}$ or the irreducibility of M. $\square$

Thus all minimal $\mathcal{R}$-decompositions of most of irreducible manifolds have no S's with $w(S) = 0$ or ± 1, and the following proposition asserts the converse.

PROPOSITION 2.4. *Let $\mathcal{D}$ be an $\mathcal{R}$-decomposition of a manifold M. If every $S \in \mathcal{D}$ satisfies $w(S) \neq 0$, ± 1, then $\mathcal{D}$ is minimal and M is either irreducible or diffeomorphic to $S^1 \times S^2$.*

For the proof, we need the following two lemmas. The proof of Lemma 2.5 is obvious and is omitted.

LEMMA 2.5. *If a manifold M admits an $\mathcal{R}$-decomposition $\mathcal{D}$ with $h(\mathcal{D}) = 0$, then M is diffeomorphic to one of S^3, $S^1 \times S^2$, lens spaces and $S^1 \times D^2$.*

LEMMA 2.6. *If a manifold M admits an $\mathcal{R}$-decomposition $\mathcal{D}$ such that $h(\mathcal{D}) = 1$ and each $S \in \mathcal{D}$ satisfies $w(S) \neq 0$, ± 1, then $\mathcal{D}$ is minimal and M is diffeomorphic to one of the following:*

(1) *$S^1 \times$(two punctured disk),*

(2) *a manifold obtained by gluing two boundary components of $S^1 \times$(two punc-*

tured disk),

(3) *a Seifert fibred space over $S^1 \times D^1$ with one exceptional fibre,*

(4) *a manifold obtained by gluing the two boundary components of a manifold in* (3),

(5) *a Seifert fibred space over D^2 with two exceptional fibres, and*

(6) *a Seifert fibred space over S^2 with three exceptional fibres.*

PROOF. The diagram of $\mathscr{D}$ is one of those in Figure 6 and thus the result follows. □

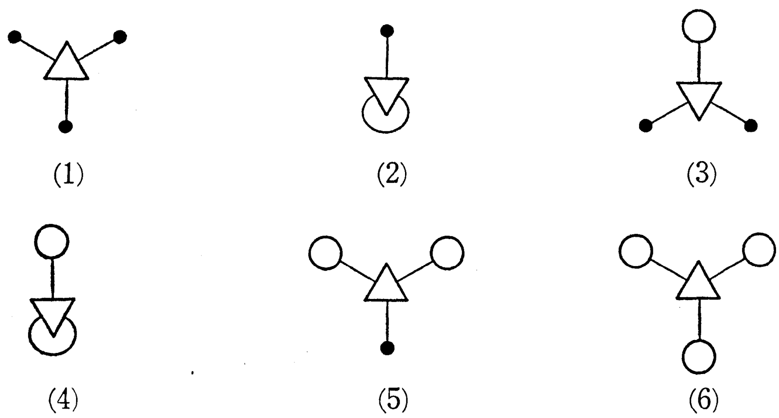


Figure 6.

PROOF OF PROPOSITION 2.4. In case of $h(\mathscr{D}) \leqq 1$, Lemmas 2.5 and 2.6 imply the result. When $h(\mathscr{D}) \geqq 2$, each element $\hat{H}_j$ of the induced decomposition $\hat{\mathscr{D}}$ is diffeomorphic to one of (1), (2), (3) and (5) in Lemma 2.6. Therefore every $\hat{H}_j$ is irreducible with incompressible boundary, and thus M is irreducible and each $\partial \hat{H}_j$ consists of incompressible tori in M. The minimality of $\mathscr{D}$ follows from the fact that any incompressible torus in $S^1 \times$(two punctured disk) is boundary parallel. □

COROLLARY 2.7 (Waldhausen [**19**]). *A manifold is a graph manifold if and only if so is each prime factor.*

PROOF. "If" part follows from the fact that the connected sum of two solid tori is a graph manifold.

Suppose that M is a graph manifold and thus admits an $\mathscr{R}$-decomposition. Let $\mathscr{D}$ be a minimal one. If there are no S's in $\mathscr{D}$ with $w(S)=0$, then Lemma 2.2 and Proposition 2.4 imply that M is irreducible or is diffeomorphic to $S^1 \times S^2$. Suppose now that there is an $S \in \mathscr{D}$ such that $w(S)=0$ with $S \underset{T}{\sim} H$. Then $S \cup_T H \simeq S^1 \times D^2 \# S^1 \times D^2$. Replacing $S \cup_T H$ by two disjoint solid tori, we have a new manifold M', which may be disconnected, and an $\mathscr{R}$-decomposition $\mathscr{D}'$ of M' with $h(\mathscr{D}') < h(\mathscr{D})$. If M' is disconnected, then M is the connected sum of the two components of M', and if M' is connected, then M is the connected sum of M' and $S^1 \times S^2$. Thus the result is proved by induction on $h(\mathcal{D})$. □

3. Characterization of graph links.

In this section we give a constructive characterization of global graph links.

For an $\mathcal{R}$-decomposition $\mathcal{D}=\{S_1, \cdots, S_m; H_1, \cdots, H_n\}$ of a manifold M with $h(\mathcal{D})\geqq 1$, we let $\partial\mathcal{D}$ denote the family of disjointly embedded tori {connected components of $\partial\hat{H}_j\}_{j=1,\cdots,n}$, where $\hat{H}_j$'s are elements of the induced decomposition $\hat{\mathcal{D}}$. Note that if $\mathcal{D}$ is minimal, each element T of $\partial\mathcal{D}$ is an incompressible torus in M (see the proof of Proposition 2.4). We say that a link $L=\bigcup_{i=1}^{l} k_i$ in M *lies on* $T\in\partial\mathcal{D}$, if there is a collar $T\times[0, 1]$ of T in M and k_i is a knot on $T\times\{i/l\}$ not null homotopic in $T\times\{i/l\}$ (Figure 7).

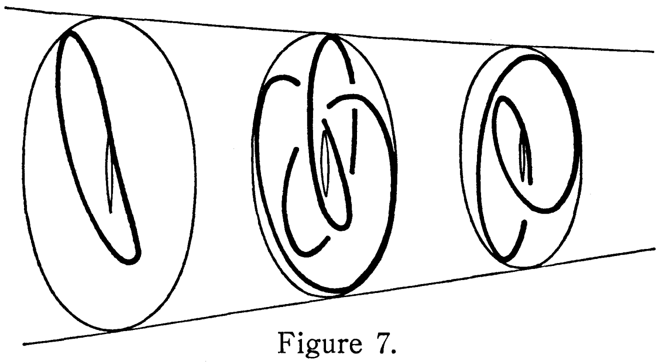


Figure 7.

DEFINITION. A graph link L in a manifold M is *elementary* if the pair (M, L) satisfies one of the following:

(i) M is a graph manifold and L is empty,

(ii) M is diffeomorphic to one of S^3, $S^1\times S^2$, lens spaces and $S^1\times D^2$, and L is the knot given by the core of a solid torus element of some minimal $\mathcal{R}$-decomposition $\mathcal{D}$ of M with $h(\mathcal{D})=0$, or

(iii) M is an irreducible graph manifold not diffeomorphic to any of S^3, lens spaces and $S^1\times D^2$, and $\mathcal{D}=\{S_1, \cdots, S_m; H_1, \cdots, H_n\}$ is some minimal $\mathcal{R}$-decomposition of M with $\partial\mathcal{D}=\{T_1, \cdots, T_l\}$. For subsets I' of $\{1, \cdots, m\}$ and J' of $\{1, \cdots, l\}$, the link L is given by $L=\bigcup_{j'\in J'} L_{j'}\cup\bigcup_{i'\in I'} k_{i'}$, where the sublink $L_{j'}$ lies on $T_{j'}\in\partial\mathcal{D}$ and the knot $k_{i'}$ is the core of $S_{i'}\in\mathcal{D}$.

Note that each component of an elementary graph link is essential unless the ambient manifold is S^3.

Now the main result of this section is stated as follows:

THEOREM 3.1 (*Characterization of global graph links*). *Every graph link is* *obtained from elementary ones by a finite number of operations of*

(1) *disjoint union,*
(2) *adding a cable knot of some component,*
(3) *replacing one component by its cable knot,*
(4) *link connected sum, and*
(5) *self link connected sum.*

COROLLARY 3.2. *Every graph link in S^3 is obtained from the unknot by a finite number of operations of disjoint union, adding a cable of some component, replacing one component by its cable, and link connected sum.*

COROLLARY 3.3. *Every full graph knot in a manifold prime to $S^1\times S^2$ is obtained from elementary graph knots by a finite number of operations of replacing by a cable and knot connected sum.*

Corollaries 3.2 and 3.3 are generalizations of a characterization of graph knots in S^3 due to Gordon [**4**] and Ue [**18**] (see also Soma [**13**]), and the former is obtained also by Eisenbud-Neumann [**3**].

PROOF OF THEOREM 3.1. (Compare with the proof of Corollary 2.7.) Let L be a graph link in a manifold M. Since each prime factor of a graph manifold is also a graph manifold, we can assume without loss of generality that L is nonsplittable, full and nonempty. This implies that the exterior $E(L)$ is irreducible. We set $h(L)$ to be $\min h(\mathcal{D})$, where $\mathcal{D}$ runs over all minimal $\mathcal{R}$-decompositions of $E(L)$, and we prove the theorem by induction on $h(L)$. If $h(L)=0$, then $E(L)$ is a solid torus by Lemma 2.3, and thus the link L is an elementary graph knot corresponding to (ii) of the definition. Suppose now that $h(L)\geqq 1$ and that the theorem is verified for every graph link L' with $h(L')<h(L)$. Let $\mathcal{D}=\{S_1, \cdots, S_m; H_1, \cdots, H_n\}$ be a minimal $\mathcal{R}$-decomposition of $E(L)$ with $h(\mathcal{D})=h(L)$. Note that there are no S's in $\mathcal{D}$ such that $w(S)=0$, since $E(L)$ is irreducible. By adding $N(L)$ to $\mathcal{D}$, we have an $\mathcal{R}$-decomposition $\bar{\mathcal{D}}=\{S_1, \cdots, S_m, \bar{S}_1, \cdots, \bar{S}_l; H_1, \cdots, H_n\}$ of M where $\bar{S}_1, \cdots, \bar{S}_l$ denote components of $N(L)$. The rest of the proof is divided into six cases.

Case 1. *There are no S's and $\bar{S}$'s such that $w(S)=0, \pm 1$ and $w(\bar{S})=0, \pm 1$.* By Proposition 2.4, the $\mathcal{R}$-decomposition $\bar{\mathcal{D}}$ of M is minimal, and the link L consists of the cores of some solid tori of $\bar{\mathcal{D}}$. Since $h(\bar{\mathcal{D}})=h(\mathcal{D})\geqq 1$, we see that L is an elementary graph link corresponding to (iii) of the definition.

Case 2. *There is an $\bar{S}\in\bar{\mathcal{D}}$ with $w(\bar{S})=0$.* Let $\bar{S}\underset{T}{\sim}H$. Then $\bar{S}\cup_T H\simeq S^1\times D^2 \# S^1\times D^2$ and the core of $\bar{S}$ is as in Figure 8. Replacing ($\bar{S}\cup_T H$, the core of $\bar{S}$) by the disjoint union ($S^1\times D^2$, the core)$\cup$($S^1\times D^2$, the core), we get a graph link L' in a manifold M', which may be disconnected, such that $h(L')<h(L)$. If M' is disconnected, L is a link connected sum of two components of (M', L'), and if M' is connected, L is a self link connected sum of L'.

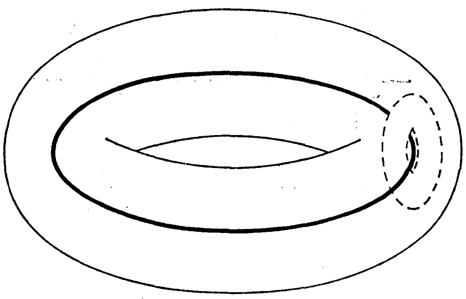


Figure 8.

Case 3. *There is an* $S\in\mathfrak{D}$ *with* $w(S)=\pm 1$. By Lemma 2.2, $E(L)$ is diffeomorphic to $T^2\times I$ and thus the number of components of L is one or two. If L has only one component, then it is an elementary graph knot in $S^1\times D^2$. If L has two components, then it is a link in one of S^3, $S^1\times S^2$ and lens spaces, consisting of the cores of two solid tori of some minimal $\mathfrak{R}$-decomposition. Hence L is obtained from an elementary graph knot which is in one of S^3, $S^1\times S^2$ and lens spaces, by adding a cable knot.

Case 4. *There are* $\bar{S}$, S *and* H *in* $\mathfrak{D}$ *such that* $\bar{S}\sim H$, $S\sim H$ *and* $w(\bar{S})=\pm 1$. In this case $\bar{S}\cup H\cup S\simeq S^1\times D^2$ and since $w(S)\neq 0$, the knot given by the core of $\bar{S}$ is a cable knot of the core of this $S^1\times D^2$. Thus the link L is obtained from a graph link L' with $h(L')<h(L)$ by replacing one component by its cable.

Case 5. *There are* $\bar{S}$, $\bar{S}'$ *and* H *in* $\mathfrak{D}$ *such that* $\bar{S}\sim H$, $\bar{S}'\sim H$ *and* $w(\bar{S})=\pm 1$. We can assume $w(\bar{S}')\neq 0$. Then $\bar{S}\cup H\cup\bar{S}'\simeq S^1\times D^2$, and the link given by the cores of $\bar{S}$ and $\bar{S}'$ consists of the core of this $S^1\times D^2$ and its cable. Thus L is obtained from a graph link L' with $h(L')<h(L)$ by adding a cable knot of one component.

Case 6. We assume that Cases 1-5 do not occur and rearrange $\bar{S}$'s and H's such that $\bar{S}_j\sim H_j$ and $w(\bar{S}_j)=1$ for $j=1, 2, \cdots, i_l-1$ and $H_j\sim H_{j+1}$ for $j=i_0, i_0+1, \cdots, i_1-2, i_1, i_1+1, \cdots, i_2-2, \cdots, i_{l-1}, i_{l-1}+1, \cdots, i_l-2$, where $1=i_0<i_1<\cdots<i_l$ and $l\geqq 0$, and each $\bar{S}$ with $w(\bar{S})=1$ is one of $\bar{S}_j$'s above (Figure 9). For $k=0, \cdots, l-1$, the union $\bigcup_{i_k\leqq j\leqq i_{k+1}-1}\bar{S}_j\cup H_j$ in M is diffeomorphic to $T^2\times I$ or to a T^2-bundle over S^1. Therefore, in the case where $\mathfrak{D}\neq\{\bar{S}_j, H_{j'}\}_{1\leqq j, j'\leqq i_l-1}$, we can get a new $\mathfrak{R}$-decomposition $\mathfrak{D}'$ of M by deleting $\{\bar{S}_j, H_{j'}\}_{1\leqq j, j'\leqq i_l-1}$ from $\mathfrak{D}$. Since none of H_j, $j=i_0, i_1-1, i_1, i_2-1, \cdots, i_{l-1}, i_l-1$ is a neighbor of S or $\bar{S}\in\mathfrak{D}$ other than $\bar{S}_j$'s above, we see that $\mathfrak{D}'$ contains no S's with $w(S)=0$, 1 or $\bar{S}$'s with $w(\bar{S})=0$, 1 and that there exists an $H\in\mathfrak{D}$ other than H_j's above. This implies that $\mathfrak{D}$ is minimal and $h(\mathfrak{D}')\geqq 1$. Since L consists of some of the cores of solid tori in $\mathfrak{D}'$ and sublinks each of which lies on some element of $\partial\mathfrak{D}'$, we see that L is an elementary graph link corresponding to (iii) of the definition. In the case when $\mathfrak{D}=\{\bar{S}_j, H_{j'}\}_{1\leqq j, j'\leqq i_l-1}$, a similar argument shows that L is an

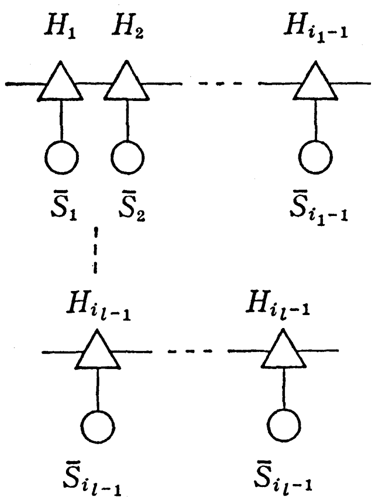


Figure 9.

elementary graph link in $T^2 \times I$ or in a T^2-bundle over S^1.

Thus we have checked all the cases and proved the theorem by induction. □

4. Properties of support.

In this section we study how the support changes by operations in Theorem 3.1. We begin with an easy lemma.

LEMMA 4.1. *Let L be a link obtained from another link L_1 by adding a cable knot of one component. Then* $\text{supp}(L)=\text{supp}(L_1)$, *and when the ambient manifold is prime to* $S^1 \times S^2$, $\text{h-supp}(L)=\text{h-supp}(L_1)$.

In the following lemmas, links L_1 and L_2 are assumed to be nonempty, nonsplittable and full. Note that then $E(L_i)$ is irreducible and that, by the loop theorem (see Hempel [6], for example), either $E(L_i)$ is diffeomorphic to $S^1 \times D^2$ or every component of $\partial N(L_i)$ is incompressible in $E(L_i)$. In proving our lemmas, by abuse of notation, we use the winding number $w(S)$ of a solid toral component S of any decomposition of a manifold, if S is a neighbor of a component diffeomorphic to $S^1 \times$(two punctured disk). (See section 2 for the definition.)

Let k_1 be an elementary graph knot in a lens space $L(p, q)$. Then the exterior $E(k_1)$ is a solid torus and, for a fixed identification $N(k_1) \simeq S^1 \times D^2$, the meridian curve of $E(k_1)$ is the (m_0, p)-cable knot of k_1 for some m_0.

LEMMA 4.2. *Let k_1 and m_0 be as above and let k denote the (m, n)-cable knot of k_1. Then*

(i) $\text{supp}(k)=S^3$ *i.e., k is local if* $(m, n)=\pm(m_0, p)$, *and* $\text{supp}(k)=L(p, q)$ *otherwise, and*

(ii) $\text{h-supp}(k)=S^3$ *if n is divisible by p, and* $\text{h-supp}(k)=L(p, q)$, *otherwise.*

PROOF. Since (ii) is trivial, we only prove (i). There is an $\mathcal{R}$-decomposition $\mathcal{D}=\{S_1, S_2, H\}$ of $E(k)$ such that S_1 corresponds to $N(k_1)$ and S_2 to $E(k_1)$. (See Figure 10 for the diagram.) Since $w(S_1)=\pm n$ and $w(S_2)=\pm(pm-m_0n)$, it follows that $E(k)$ is irreducible if and only if $(m, n)\neq\pm(m_0, p)$. □

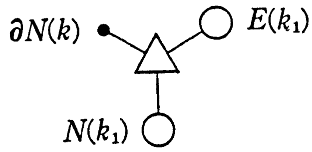


Figure 10.

LEMMA 4.3. *Let L be the link obtained from L_1 by replacing a component k_1 by its cable k. Then* $\mathrm{supp}(L)=\mathrm{supp}(L_1)$ *unless L_1 is an elementary graph knot in a lens space.*

PROOF. Let k be an (m, n)-cable knot of k_1. Then, for a decomposition $E(L)=E(L_1)\cup S^1\times$(two punctured disk)$\cup N(k_1)$, the winding number $w(N(k_1))$ is $\pm n$ and thus $S^1\times$(two punctured disk)$\cup N(k_1)$ is irreducible with incompressible boundary. This implies that if $\partial N(k_1)$ is incompressible in $E(L_1)$, then $E(L)$ is irreducible. In this case L is full and $\mathrm{supp}(L)=\mathrm{supp}(L_1)$. If not, $E(L_1)$ is a solid torus and thus L_1 is an elementary graph knot in S^3, $S^1\times S^2$ or a lens space. If L_1 is in S^3, this lemma is trivial and if L_1 is in $S^1\times S^2$, every cable of it is essential and thus is full in $S^1\times S^2$. □

LEMMA 4.4. *Let k_1 be an elementary graph knot in $S^1\times S^2$. Then, for any link L_2 and any component k_2 of L_2, we have* $\mathrm{supp}(k_1\#L_2(k_1, k_2))=S^1\times S^2\#\mathrm{supp}(L_2-k_2)$, *where L_2-k_2 denotes the sublink of L_2 consisting of all the components of L_2 but k_2.*

PROOF. There is an embedded S^2 in the ambient manifold of $k_1\#L_2$ which separates $k_1\#k_2$ and L_2-k_2 (Figure 11). □

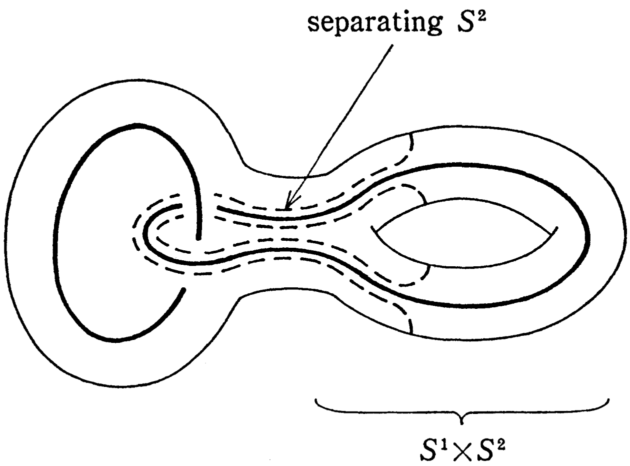


Figure 11.

LEMMA 4.5. *Suppose that neither of L_1 and L_2 is an elementary graph knot in $S^1\times S^2$ and let L be a link connected sum of L_1 and L_2. Then* $\mathrm{supp}(L)=\mathrm{supp}(L_1)\,\#\,\mathrm{supp}(L_2)$, *and when the ambient manifold is prime to $S^1\times S^2$,* $\text{h-supp}(L)=\text{h-supp}(L_1)\,\#\,\text{h-supp}(L_2)$.

PROOF. Since the statement for the homotopy support is trivial, we only deal with the support. Let $L=L_1\,\#\,L_2(k_1, k_2)$. Then $E(L)=E(L_1)\cup_{\partial N(k_1)}S^1\times$(two punctured disk)$\cup_{\partial N(k_2)}E(L_2)$. There are four cases corresponding to whether each $\partial N(k_i)$ is incompressible or not. Suppose, for example, that $\partial N(k_1)$ is compressible and $\partial N(k_2)$ is incompressible. Then $E(L_1)$ is a solid torus and, since L_1 is not an elementary graph knot in $S^1\times S^2$, the winding number $w(E(L_1))$ for the decomposition above is not zero. It follows that $E(L_1)\cup S^1\times$(two punctured disk) is irreducible, and $\partial E(L_1)$ is incompressible in $E(L_1)$. Hence $E(L)$ is irreducible. Similar proof can be applied for the other cases. □

LEMMA 4.6. *The support of a self link connected sum $^{\#}L_1$ of L_1 is* $\mathrm{supp}\,(L_1)\,\#\,S^1\times S^2$.

PROOF. Let $L={}^{\#}L_1(k_1, k_2)$. Then $E(L)=E(L_1)\cup_{\partial N(k_1)\cup\partial N(k_2)}S^1\times$(two punctured disk). Since $E(L_1)$ is not a solid torus, both $\partial N(k_1)$ and $\partial N(k_2)$ are incompressible in $E(L_1)$; thus the result follows. □

LEMMA 4.7. *Let L be the disjoint union $L_1\cup L_2$, and let $k_i\subset L_i$ be a component of L_i for $i=1, 2$. Then* $\mathrm{supp}({}^{\#}L(k_1, k_2))=\mathrm{supp}(L_1\,\#\,L_2(k_1, k_2))$.

Finally we give an example that indicates the difficulty for defining the homotopical position of links in manifolds not prime to $S^1\times S^2$.

EXAMPLE 4.8. Let k_0 be an elementary graph knot in $S^1\times S^2$ and k_1 be its cable. Let k be any knot in any manifold. Then by Lemmas 4.4 and 4.5, $\mathrm{supp}(k\,\#\,k_0)=\mathrm{supp}(k_1)=S^1\times S^2$ but $\mathrm{supp}(k\,\#\,k_0\cup k_1)=\mathrm{supp}(k)\,\#\,S^1\times S^2$.

5. Support of global graph links.

In this section we study the support of graph links. First we prove the following, which is the main result of this paper.

THEOREM 5.1. *The support of a graph link in a manifold prime to $S^1\times S^2$ is a connected sum of its homotopy support and graph manifolds with finite fundamental groups.*

It is easy to see that this theorem implies the results stated in the introduction which are concerned with the support of graph links.

PROOF OF THEOREM 5.1. We prove this by induction on the number of operations in Theorem 3.1. If L is an elementary graph link, then either $\mathrm{supp}(L)=S^3$ or each component of L is essential. Thus this theorem holds for such links. The operations (1) and (4) in Theorem 3.1 make no difference between the support and the homotopy support by the definition and Lemma 4.5, and the operation (2) does not change them by Lemma 4.1. Since the operation (5) does not appear in this case, we only need to treat with the operation (3). This operation reduces the homotopy support only if the component represents a torsion element in the fundamental group and the difference between the support and the homotopy support is produced only if this reduction occurs. It is known that if the fundamental group of an irreducible manifold has a non-trivial torsion, then the group is of finite order (see, for example, Hempel [**6**]). Thus this reduction takes place only by cutting off a manifold with finite fundamental group. This completes the induction and we get the theorem. □

Although the homotopy support is not defined for links in manifolds not prime to $S^1\times S^2$, for a class of such graph links containing inessential ones, we have

THEOREM 5.2. *Suppose that L is a graph link in a manifold M and that each component of L represents a torsion element in the fundamental group of M. Then the support of L is a connected sum of $S^1\times S^2$'s and graph manifolds with finite fundamental groups.*

While the proof of this theorem needs Lemmas 4.6 and 4.7, it is done by induction similar to that of Theorem 5.1 and we omit it.

COROLLARY 5.3. *The support of an inessential graph link is a connected sum of $S^1\times S^2$'s and graph manifolds with finite fundamental groups.*

We remark here that the conclusion of Corollary 5.3 is best possible; more precisely, we have

PROPOSITION 5.4. *Let M be a connected sum of $S^1\times S^2$'s and graph manifolds with finite fundamental groups. Then there exists a graph knot which is inessential and full in M.*

PROOF. For $M=S^1\times S^2$, the self link connected sum of the Hopf link in S^3 gives an example. For a graph manifold M with finite fundamental group, some cable of an elementary graph knot is a required example by Lemmas 4.2 and 4.3. In general, one can take a knot connected sum of these. □

We give a figure of lifts to S^3 of a counterexample in $L(3, 1)$ (Figure 12).

It is natural to ask: if one assumes the localness of each component instead of inessentiality, what is the support of a link? In general, the characterization of S^3 due to Bing [**2**] implies the existence of a full link consisting of local knots as well as that of an inessential full knot in any manifold as in the introduction. For graph links, however, we have the following

THEOREM 5.5. *Suppose that L is a graph link and that the support of each component of L is a connected sum of lens spaces. Then the support of L is a*

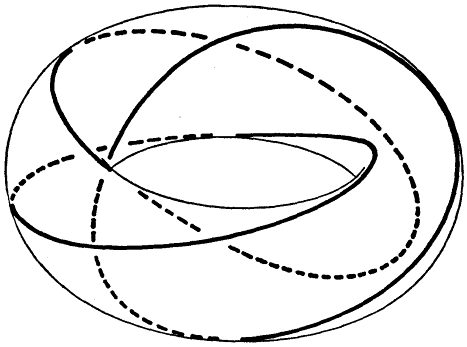


Figure 12.

connected sum of $S^1 \times S^2$*'s and lens spaces.*

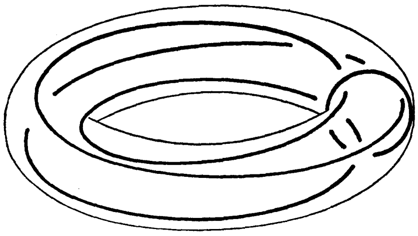
PROOF. We prove this by induction on the number of operations in Theorem 3.1. For elementary graph links, this theorem obviously holds. The operations in Theorem 3.1 add new factors to the support of the link only in the case that the operation (5) adds $S^1 \times S^2$. They reduce the support of some component only in the following two cases: either the operation (2) reduces it by a lens space or the operation (4) reduces it by any manifold when the connected sum is taken by an elementary graph knot in $S^1 \times S^2$. In the latter case, however, there remains a component the support of which is $S^1 \times S^2$. This completes the induction and we get the theorem. □

Note that by Example 4.8, there exists a graph link L such that the support of each component of L is $S^1 \times S^2$ but the support of L is not a connected sum of $S^1 \times S^2$'s and lens spaces.

COROLLARY 5.6. *The support of a graph link consisting of local knots is a connected sum of* $S^1 \times S^2$*'s and lens spaces.*

The conclusion of Corollary 5.6 is best possible. More precisely, we have

PROPOSITION 5.7. *Let* M *be a connected sum of* $S^1 \times S^2$*'s and lens spaces. Then there exists a full graph link in* M *consisting of local knots.*



In a solid torus of a minimal $\mathfrak{R}$-decomposition of $L(3, 1)$

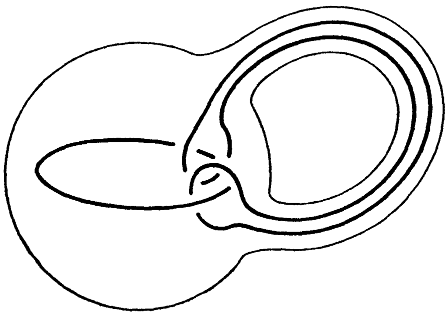


Figure 13.

We give here figures of such links in $L(3, 1)$ and $S^1\times S^2$ instead of the proof (Figure 13).

6. Dehn surgery along full graph knots.

As an application of $\mathcal{R}$-decompositions, we study Dehn surgeries along full graph knots in this section.

A knot k is called *framed* if we are given an identification $S^1\times D^2\simeq N(k)$, where S^1 and D^2 are oriented. Then the basis $\{\lambda, \mu\}$ for $H_1(\partial N(k);\boldsymbol{Z})$ is given by $\mu=[\{*\}\times\partial D^2]$ and $\lambda=[S^1\times\{*\}]$ under this identification. For a framed knot k in a manifold M and a number $r\in\boldsymbol{Q}\cup\{\infty\}$, we let $\chi_M(k, r)$ or simply $\chi(k, r)$ denote the manifold obtained from M by Dehn surgery of type r on k. That is,

$$\chi_M(k, r)=(M-N(k))\cup_{\varphi_r}S^1\times D^2,$$

where $\varphi_r:\partial N(k)\to\partial(S^1\times D^2)$ satisfies $\varphi_{r*}(p\mu+q\lambda)=0$ in $H_1(S^1\times D^2;\boldsymbol{Z})$ for coprime integers p and q with $r=p/q$.

The main result of this section is the following, and a motivation of proving this is Thurston's theory of hyperbolic Dehn surgery ([**16**], §5). (See Ue [**18**] for Dehn surgery along graph knots in S^3.)

THEOREM 6.1. *Let k be a framed full graph knot in a manifold M. Then there are at most two values of r such that $\chi_M(k, r)$ is not irreducible. Moreover, if there exist two such values, then one of the resulting manifolds is the connected sum of a lens space and an irreducible manifold.*

For the proof of this theorem, we need some lemmas. Let $H=S^1\times(S^2-\mathrm{int}D_A^2-\mathrm{int}D_B^2-\mathrm{int}D_C^2)=S^1\times(\text{two punctured disk})$, where S^1 and $S^2-\mathrm{int}D_A^2-\mathrm{int}D_B^2-\mathrm{int}D_C^2$ are oriented, and let $T_A=S^1\times\partial D_A^2$, $T_B=S^1\times\partial D_B^2$, $T_C=S^1\times\partial D_C^2$ be components of ∂H. Let α, β, γ and δ denote the homology classes of H given by $[\{*\}\times\partial D_A^2]$, $[\{*\}\times\partial D_B^2]$, $[\{*\}\times\partial D_C^2]$ and $[S^1\times\{*\}]$, respectively. Fix pairs (b_1, b_2) and (c_1, c_2) of coprime integers with $b_1\geqq 2$ and $c_1\geqq 2$, and let $\psi_B:T_B\to\partial(S^1\times D^2)$ and $\psi_C:T_C\to\partial(S^1\times D^2)$ be diffeomorphisms which satisfy $\psi_{B*}(b_1\beta+b_2\delta)=0$ and $\psi_{C*}(c_1\gamma+c_2\delta)=0$ in $H_1(S^1\times D^2;\boldsymbol{Z})$, respectively. For coprime integers x and y, we let $\varphi:T_A\to\partial(S^1\times D^2)$ be a diffeomorphism which satisfies $\varphi_*(x\alpha+y\delta)=0$ in $H_1(S^1\times D^2;\boldsymbol{Z})$.

LEMMA 6.2. *Let $N=H\cup_\varphi S^1\times D^2$. Then*
(1) *$N\simeq S^1\times D^2\,\#\,S^1\times D^2$ if $(x, y)=(0, \pm 1)$, and*
(2) *N is irreducible with incompressible boundary, otherwise.*

LEMMA 6.3. *Let $N=(H\cup_{\psi_B}S^1\times D^2)\cup_\varphi S^1\times D^2$. Then*
(1) *$N\simeq L(b_1, b_2)\,\#\,S^1\times D^2$ if $(x, y)=(0, \pm 1)$,*
(2) *$N\simeq S^1\times D^2$ if $(x, y)=\pm(1, n)$, and*

(3) *N is irreducible with incompressible boundary, otherwise.*
In case of (1), *the meridian curve of the prime factor $S^1\times D^2$ of N represents $\pm\delta$ in $H_1(T_C;\boldsymbol{Z})$ and in case of* (2), *the meridian curve of $N\simeq S^1\times D^2$ represents $\pm(b_1\gamma+(b_2+nb_1)\delta)$ in $H_1(T_C;\boldsymbol{Z})$. In particular, no pair of such homology classes in $H_1(T_C;\boldsymbol{Z})$ have intersection number ± 1.*

LEMMA 6.4. *Let $N=(H\cup_{\phi_B}S^1\times D^2\cup_{\phi_C}S^1\times D^2)\cup_{\varphi}S^1\times D^2$. Then*

(1) *$N\simeq L(b_1, b_2)\,\#\,L(c_1, c_2)$ if $(x, y)=(0, \pm 1)$,*

(2) *$N\simeq S^1\times S^2$ if there exists an integer n such that $(c_1, c_2)=(b_1, b_2+nb_1)$ and $(x, y)=\pm(1, n)$ for this n, and*

(3) *N is irreducible, otherwise.*

The proof of Lemmas 6.2-6.4 are immediate and are omitted.

PROOF OF THEOREM 6.1. Let $\mathcal{D}$ be a minimal $\mathcal{R}$-decomposition of $E(k)$. If $h(\mathcal{D})=0$, then this theorem is obvious. Suppose that $h(\mathcal{D})\geqq 1$. Then there is a unique $H\in\mathcal{D}$ such that $H\supset\partial N(k)$. We divide the proof into three cases.

Case 1. *There are no S's of $\mathcal{D}$ such that $H\sim S$.* Let $M'=N(k)\cup H$. Then, since $\partial M'$ is incompressible in $M-\operatorname{int}M'$, we see by Lemma 6.2 that $\chi(k, r)$ is irreducible for all but one value of r.

Case 2. *There are S and S' of $\mathcal{D}$ such that $H\sim S$ and $H\sim S'$.* Since $M=N(k)\cup H\cup S\cup S'$, the result follows from Lemma 6.4.

Case 3. *There is a unique S of $\mathcal{D}$ such that $H\sim S$.* Rearrange S's and H's in $\mathcal{D}$ such that the diagram of $\mathcal{D}$ is as in Figure 14. Here l is taken to be maximal, i.e., either $H_l\supset B$ for some boundary component B of M or $H_l\sim H_{l+1}$ with H_{l+1} as in Case 1 or 2. Note that there exists a unique r_1 such that $w(S^1\times D^2)=0$ with respect to $S^1\times D^2\sim H_1$, where $S^1\times D^2$ is the attaching solid torus of this Dehn surgery. Let (M_m, k_m) denote the pair $(N(k)\cup\bigcup_{n=1}^{m}(H_n\cup S_n), k)$. Then for $r\neq r_1$, $\chi(k, r)$ is not irreducible only if there exists an m such that $\chi(k_n, r)\simeq S^1\times D^2$ for all $n\leqq m$ and $w(\chi(k_m, r))=0$ with respect to $\chi(k_m, r)\sim H_{m+1}$. Let r_2 be such a value of r with the number m minimal. Then for every r with $r\neq r_1, r_2$, Lemma 6.3 implies that either $\chi(k_m, r)$ is irreducible with incompressible boundary or $\chi(k_m, r)\simeq S^1\times D^2$ and also that, in the latter case, $w(\chi(k_m, r))\neq 0, \pm 1$ with respect to $\chi(k_m, r)\sim H_{m+1}$ even if there is an H_{m+1} with $H_m\sim H_{m+1}$. Thus in both cases, $\chi(k, r)$ is irreducible.

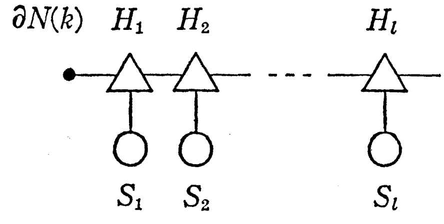


Figure 14.

Now suppose that there are two such values and let r_1 and r_2 be as above. Then $\chi(k_1, r_2) \simeq S^1 \times D^2$ and $w(\chi(k_1, r_2))=0$ or ± 1 with respect to $\chi(k_1, r_2) \sim H_2$. Therefore Lemma 6.3 implies that $\chi(k_1, r_1) \simeq$(a lens space) $\# S^1 \times D^2$ and $w(S^1 \times D^2) \neq 0, \pm 1$ with respect to $S^1 \times D^2 \sim H_2$ for the prime factor $S^1 \times D^2$ of $\chi(k_1, r_1)$ above. This completes the proof of Theorem 6.1. □

This, together with Thurston's uniformization theorem [**17**], implies the following

COROLLARY 6.5. *Let k be a framed full knot in a closed manifold M. Then $\chi(k, r)$ is irreducible for all but finitely many r.*

This is proved by Gordon [**4**] for knots in S^3, and the proof we give here is similar to that in [**4**]. We remark that this corollary also follows from a result of Hatcher [**5**].

PROOF. If $\partial E(k)$ is compressible in $E(k)$, then $E(k)$ is a solid torus and the result follows easily. Otherwise, $E(k)$ is irreducible and $\partial E(k)$ is an incompressible torus. Then, by the torus decomposition theorem due to Jaco-Shalen [**7**] and Johannson [**8**] and by Thurston's uniformization theorem, we see that there exists a family of disjointly embedded incompressible tori in $E(k)$ such that each connected component of the manifold obtained by cutting $E(k)$ along these tori is either a graph manifold or "hyperbolic" in the sense of Thurston [**16**]. Let P denote the component of this decomposition which contains $\partial E(k)$ and let $(\bar{P}, \bar{k})=(P \cup N(k), k)$. There are two cases.

Case 1. *P is a graph manifold.* If ∂P consists only of $\partial E(k)$, then k is a graph knot and the result follows from Theorem 6.1. Suppose that ∂P consists of more than two components. Then the proof of Theorem 6.1 says that, for all but two values of r, the manifold $\chi_{\bar{P}}(\bar{k}, r)$ is irreducible with incompressible boundary and thus $\chi(k, r)=\chi_{\bar{P}}(\bar{k}, r) \cup (M-\operatorname{int} P)$ is irreducible. Suppose that ∂P consists of two components. Let $\partial P=\partial E(k) \cup \partial_0 P$ and let P' denote the other component of the decomposition above which contains $\partial_0 P$. We can assume that P' is hyperbolic. Now the proof of Theorem 6.1 says that for all but two values of r, the manifold $\chi_{\bar{P}}(\bar{k}, r)$ is either irreducible with incompressible boundary or diffeomorphic to $S^1 \times D^2$. In the latter case, however, the meridian curves represent different homology classes of $\partial_0 P$ for different values of r, and thus the result follows from Thurston's theory of hyperbolic Dehn surgery ([**16**], §5), since P' is hyperbolic.

Case 2. *P is hyperbolic.* In this case, the result follows directly from Thurston's theory of hyperbolic Dehn surgery. □

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