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Enumerating embeddings of homologically $(k-1)$-connected n-manifolds in Euclidean $(2n-k)$-space

Dedicated to Professor Nobuo Shimada on his 60th birthday

By Tsutomu YASUI

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§1. Introduction.

Throughout this paper, an n-manifold and an embedding mean a closed connected differentiable manifold of dimension n and a differentiable embedding, respectively. Let $[M\subset R^{m}]$ denote the set of isotopy classes of embeddings of M in Euclidean m-space R^{m}. In [**5**] (cf. [**6**]), Haefliger has proved the following theorem:

THEOREM (Haefliger). *If $k\leqq(n-4)/2$ and if M is an orientable homologically k-connected n-manifold, then $[M\subset R^{2n-k}]$ is equivalent to $H_{k+1}(M;Z)$ or $H_{k+1}(M;Z_2)$ according as $n-k$ is odd or even.*

Here a space X is called homologically k-connected if it satisfies the condition $\tilde{H}_i(M;Z)=0$ for $i\leqq k$. A k-connected path connected space is clearly homologically k-connected.

The purpose of this paper is to prove the following theorem, which is an extension of the above theorem:

MAIN THEOREM. *If $2\leqq k\leqq(n-4)/2$ and if M is a homologically $(k-1)$-connected n-manifold whose $(n-k)$-th normal Stiefel-Whitney class vanishes, then the set $[M\subset R^{2n-k}]$ is given as follows:*

(i) *if $k=2$ and M is not a spin manifold, then*

$$
\begin{aligned}
[M\subset R^{2n-2}] &= H^{n-3}(M;Z_2) && n\equiv 0\,(4),\\
&= H^{n-3}(M;Z_2)\times Z_2 && n\equiv 2\,(4),\\
&= H^{n-3}(M;Z)\times H^{n-2}(M;Z_2) && n\equiv 1\,(4),\ w_3\neq 0,\\
&= H^{n-3}(M;Z)\times H^{n-2}(M;Z_2)\times Z_2 && n\equiv 1\,(4),\ w_3=0,\ \text{ or } n\equiv 3\,(4);
\end{aligned}
$$

(ii) *if $k \geqq 3$ or M is a spin manifold, then*

$$
\begin{aligned}
[M\subset R^{2n-k}] &= H^{n-k-1}(M; Z_2) && n-k\equiv 0\,(4),\\
&= H^{n-k-1}(M; Z)\times H^{n-k}(M; Z_2) && \\
&\quad \times H^{n-k}(M; Z_2)/Sq^2\rho_2 H^{n-k-2}(M; Z) && n-k\equiv 1\,(4),\\
&= H^{n-k-1}(M; Z_2)\times H^{n-k}(M; Z_2)/Sq^2 H^{n-k-2}(M; Z_2) && n-k\equiv 2\,(4),\\
&= H^{n-k-1}(M; Z)\times H^{n-k}(M; Z_2) && \\
&\quad \times H^{n-k}(M; Z_2)/(Sq^1 H^{n-k-1}(M; Z_2)+Sq^2\rho_2 H^{n-k-2}(M; Z)) && n-k\equiv 3\,(4).
\end{aligned}
$$

In this theorem, the $(n-k)$-th normal Stiefel-Whitney class $\overline{W}_{n-k}$ of an orientable n-manifold M is defined by $\bar{w}_{n-k}$ or $\beta_2\bar{w}_{n-k-1}\in H^{n-k}(M; Z)$ according as $n-k$ is even or odd, where $\bar{w}_i$ is the i-th mod 2 normal Stiefel-Whitney class of M and β_2 is the Bockstein operator, and moreover $\overline{W}_{n-k}$ is the unique obstruction to embedding a homologically $(k-1)$-connected n-manifold in R^{2n-k} by the theorem in [**5**, § 1.3] (cf. [**6**, Theorem (2.3)]).

The remainder of this paper is organized as follows: In § 2, we shall state a method of computing $[M\subset R^{2n-k}]$ of a homologically $(k-1)$-connected n-manifold M (Theorem 2.5). In § 3, we state the cohomology group of the reduced symmetric product M^* $(=(M\times M-\varDelta M)/Z_2)$ of M (Theorem 3.3), postponing the proof till § 5, the last section. § 4 is devoted to proving the main theorem.

§ 2. The method of computing $[M\subset R^{2n-k}]$.

We begin this section by explaining notations. Let X^2 be the product $X\times X$ of a space X and let $\varDelta X$ be the diagonal in X^2. The cyclic group of order 2, Z_2, acts on X^2 via the map $t: X^2\rightarrow X^2$ defined by $t(x, y)=(y, x)$. Then $\varDelta X$ is the fixed point set of this action. The quotient space

$$X^*=(X^2-\varDelta X)/Z_2$$

is called the reduced symmetric product of X. Here the projection $p: X^2-\varDelta X\rightarrow X^*$ is a double covering, whose classifying map we denote by

$$\xi : X^* \longrightarrow P^\infty.$$

For a fibration $\pi: E\rightarrow B$ and a map $f: Y\rightarrow B$, let

$$Y\times_B E \longrightarrow Y \qquad \text{and} \qquad [Y, E; f]$$

be the pull-back of π along f and the homotopy set of liftings of f to E.

Notice that the sphere bundle $\pi: S^\infty\times_{Z_2}S^m\rightarrow P^\infty$ is homotopically equivalent to the natural inclusion $P^m\rightarrow P^\infty$ of the real projective m-space P^m. Hence we regard them as identical. Using the above notations, we deduce the following theorem from Haefliger's theorem [4, Théorème 1′] (cf. Yasui [18, § 1]):

THEOREM 2.1 (Haefliger). *For an n-manifold M, there is a bijection*

$$[M\subset R^{2n-k}]\cong[M^*, P^{2n-k-1}; \xi] \qquad \textit{if} \quad k\leqq(n-4)/2.$$

For any abelian group G and a homomorphism $\phi:\pi_1(P^\infty)=Z_2\to\mathrm{Aut}(G)$, let G_ϕ be the sheaf over P^∞, locally isomorphic to G, defined by ϕ, i.e., the local system associated with ϕ. This homomorphism ϕ gives an action of Z_2 on $(K(G, m), *)$. Hence we have a fibration

$$q : L_\phi(G, m)=S^\infty\times_{Z_2}K(G, m)\longrightarrow P^\infty$$

with fiber $K(G, m)$ and a canonical cross section s. It has been established (see, for example, G. W. Whitehead [17, Chap. VI, (6.13)]) that there exists a unique fundamental class $\iota\in H^m(L_\phi(G, m), P^\infty; q^*G_\phi)$, whose restriction to $K(G, m)$ is the ordinary one (ι is equal to $\bar{\delta}(sq, 1)$ up to sign in [17]), and that given $\hat{x}: X\to P^\infty$, the correspondence $f\to f^*\iota$ leads to a bijection

$$[X, L_\phi(G, m); \hat{x}]\cong H^m(X; \hat{x}^*G_\phi).$$

Further, if $\hat{x}$ has a lifting $\tilde{x}$ to P^{2n-k-1}, then there is a bijection

$$[X, P^{2n-k-1}\times_{P^\infty}L_\phi(G, m); \tilde{x}]\cong[X, L_\phi(G, m); \hat{x}]$$

by [8, Theorem 3.1] and hence we have a bijection

$$[X, P^{2n-k-1}\times_{P^\infty}L_\phi(G, m); \tilde{x}]\cong H^m(X; \hat{x}^*G_\phi). \tag{2.2}$$

Let

$$G_j=\pi_{2n-k-1+j}(S^{2n-k-1}).$$

Since the sphere bundle $\pi: P^{2n-k-1}\to P^\infty$ is the one associated with $(2n-k)\gamma$, γ being the universal real line bundle over P^∞, the action of $\pi_1(P^\infty)=Z_2$ on G_j is given by the homomorphism

$$\phi : Z_2(=\{1, a\})\longrightarrow \mathrm{Aut}(G_j)$$

defined by

$$\phi(a)(x)=(-1)^{2n-k}x \qquad \text{for} \quad x\in G_j$$

and moreover the sheaf $(G_j)_\phi$ is given by

$$(G_j)_\phi=\begin{cases} G_j & \text{if } k \text{ is even,}\\ G_j[u] & \text{if } k \text{ is odd,}\end{cases}$$

where $G_j[u]$ is the sheaf over P^∞, locally isomorphic to G_j, twisted by $u(\neq 0)\in H^1(P^\infty; Z_2)=Z_2$. For $\xi: M^*\to P^\infty$, let

$$\underset{\sim}{G}_j=\xi^*(G_j)_\phi=\begin{cases} G_j & \text{if } k \text{ is even,}\\ G_j[v]\ (v=\xi^*u) & \text{if } k \text{ is odd,}\end{cases} \tag{2.3}$$

and let

$$\bar{\rho}_2 : H^i(M^*; \underline{Z}) \longrightarrow H^i(M^*; Z_2),$$
$$\bar{\beta}_2 : H^{i-1}(M^*; Z_2) \longrightarrow H^i(M^*; \underline{Z})$$

be the ordinary reduction mod 2 and Bockstein operator or the ones twisted by v according as k is even or odd. Then

$$\bar{\rho}_2\bar{\beta}_2 = \begin{cases} Sq^1 & \text{if } k \text{ is even,} \\ Sq^1+v & \text{if } k \text{ is odd,} \end{cases} \tag{2.4}$$

by [2], [14]. With the above notations, we shall prove

THEOREM 2.5. *Let* $2 \leqq k \leqq (n-4)/2$ *and let* M *be a homologically* $(k-1)$-*connected* n-*manifold. If* M *can be embedded in* R^{2n-k}, *then there exists a bijection*

$$[M \subset R^{2n-k}] = H^{2n-k-1}(M^*; \underline{Z}) \times \mathrm{Coker}\,\Theta$$

where

$$\Theta = \left(Sq^2 + \binom{2n-k}{2} v^2\right)\bar{\rho}_2 : \quad H^{2n-k-2}(M^*; \underline{Z}) \longrightarrow H^{2n-k}(M^*; Z_2).$$

In order to prove this, it is sufficient, by Theorem 2.1, to show that

$$[M^*, P^{2n-k-1}; \xi] = H^{2n-k-1}(M^*; \underline{Z}) \times \mathrm{Coker}\,\Theta.$$

Let $P = P^{2n-k-1}$ and let $\pi' : P' \to P$ be the pull-back of $\pi : P \to P^\infty$ along π. If M can be embedded in R^{2n-k}, then ξ has a lifting $\xi' : M^* \to P$ by the first half of [4, Théorème 1′] and so

$$[M^*, P'; \xi'] \cong [M^*, P; \xi] \tag{2.6}$$

by [8, Theorem 3.1]. Since $\pi : P \to P^\infty$ is the sphere bundle associated with $(2n-k)\gamma$, the Postnikov tower of $\pi' : P' \to P$ is given as follows:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & E_{j+1} \arrow[d, "p_j"'] & \\
P' \arrow[r, "h_j"] & E_j \arrow[r, "k_j"] \arrow[d, dotted] & P \times_{P^\infty} L_\varphi(G_j, 2n-k+j) \\
 & E_2 \arrow[r, "k_2"] \arrow[d, "p_1"'] & P \times K(Z_2, 2n-k+2) \\
P \times_{P^\infty} L_\varphi(Z, 2n-k-1) = & E_1 \arrow[r, "k_1"] \arrow[d] & P \times K(Z_2, 2n-k+1) \\
 & P^{2n-k-1} = P &
\end{tikzcd}$$

where h_j is a $(2n-k-1+j)$-equivalence, $p_j: E_{j+1} \to E_j$ is a P-principal fibration with classifying map k_j in the category TP of P-sectioned spaces and maps. By [**10**, Part IV, Theorem 1], for $\xi': M^* \to P$, $p_j: E_{j+1} \to E_j$ induces an exact sequence

$$\cdots \xrightarrow{(\Omega_P k_j)_\#} [M^*, P \times_{P^\infty} L_\phi(G_j, 2n-k-1+j);\ \xi'] \longrightarrow [M^*, E_{j+1};\ \xi']$$
$$\xrightarrow{(p_j)_\#} [M^*, E_j;\ \xi'] \xrightarrow{(k_j)_\#} [M^*, P \times_{P^\infty} L_\phi(G_j, 2n-k+j);\ \xi'] \qquad (j \geqq 1),$$

where $\Omega_P k_j$ is the map of loops associated with k_j in TP. With the help of (2.2), (2.3), this is converted into the exact sequence

$$\cdots \xrightarrow{(\Omega_P k_j)_\#} H^{2n-k-1+j}(M^*;\ \underline{G}_j) \longrightarrow [M^*, E_{j+1};\ \xi'] \xrightarrow{(p_j)_\#} [M^*; E_j;\ \xi']$$
$$\xrightarrow{(k_j)_\#} H^{2n-k+j}(M^*;\ \underline{G}_j) \quad (j \geqq 1),$$

where

$$[M^*, E_1;\ \xi'] = H^{2n-k-1}(M^*;\ \underline{Z}).$$

Now it has been shown by Haefliger and Hirsch [**7**, p. 237] that if M is a homologically $(k-1)$-connected n-manifold $(k \geqq 2)$, then

$$H^{2n-k-1+j}(M^*;\ \underline{G}_j) = H^{2n-k-1+j}(M^*;\ \underline{G}_{j-1}) = 0 \qquad \text{for} \quad j \geqq 2.$$

We know, on the other hand, that $\Omega_P k_1$ induces an operation $\left(Sq^2 + \binom{2n-k}{2} v^2\right)\bar{\rho}_2$, i. e.

$$\begin{array}{ccccc} (\Omega_P k_1)_\# : & [M^*, \Omega_P E_1;\ \xi'] & \longrightarrow & [M^*, \Omega_P(P \times K(Z_2, 2n-k+1));\ \xi'] \\ \| & \| & & \| \\ \Theta = \left(Sq^2 + \binom{2n-k}{2} v^2\right)\bar{\rho}_2 : & H^{2n-k-2}(M^*;\ \underline{Z}) & \longrightarrow & H^{2n-k}(M^*;\ Z_2), \end{array}$$

because $(k_1)_\#$ corresponds to $(Sq^2 + w_2((2n-k)\gamma))\bar{\rho}_2$. From the above argument, it is clear that there exists a short exact sequence

$$0 \longrightarrow H^{2n-k}(M^*;\ Z_2)/\mathrm{Im}\,\Theta \longrightarrow [M^*, P';\ \xi'] \longrightarrow H^{2n-k-1}(M^*;\ \underline{Z}) \longrightarrow 0.$$

This, together with Theorem 2.1 and (2.6), completes the proof of Theorem 2.5.

§ 3. The cohomology of M^*.

The mod 2 cohomology of M^* has been studied by Bausum [**1**], Haefliger [**3**], Thomas [**16**], Yasui [**19**], Yo [**21**] and others. The notations used here are the same as those explained in [**19**] (most of them are the same as in [**16**, § 2]). Let $M \in H^n(M;\ Z_2)$ be the generator, i. e.

$$H^n(M;Z_2)=Z_2\langle M\rangle,$$

and let

$$\sigma=1+t^*:H^*(M^2;Z_2)\longrightarrow H^*(M^2;Z_2).$$

LEMMA 3.1. *Assume that M is a homologically $(k-1)$-connected n-manifold $(k\geqq 2)$. Then*

(i) $H^i(M^*;Z_2)=0$ *if* $i>2n-k$,

(ii) $H^{2n-k}(M^*;Z_2)=\{\rho\sigma(M\otimes x)\mid x\in H^{n-k}(M;Z_2)\}$ $(\cong H^{n-k}(M;Z_2))$,

(iii) $H^{2n-k-1}(M^*;Z_2)$

$$=\{\rho(u^{k-1}\otimes x^2)\mid x\in H^{n-k}(M;Z_2)\}\ (\cong H^{n-k}(M;Z_2))$$
$$+\{\rho(u^{k+1}\otimes x^2)\mid x\in H^{n-k-1}(M;Z_2)\}\ (\cong H^{n-k-1}(M;Z_2)),$$

(iv) $H^{2n-k-2}(M^*;Z_2)$

$$=\{\rho(u^k\otimes x^2)\mid x\in H^{n-k-1}(M;Z_2)\}\ (\cong H^{n-k-1}(M;Z_2))$$
$$+\{\rho(u^{k-2}\otimes x^2)\mid x\in H^{n-k}(M;Z_2)\}\ (\cong H^{n-k}(M;Z_2))$$
$$+\{\rho(u^{k+2}\otimes x^2)\mid x\in H^{n-k-2}(M;Z_2)\}\ (\cong H^{n-k-2}(M;Z_2))$$
$$+[\{\rho\sigma(x\otimes y)\mid x,y\in H^{n-2}(M;Z_2),\ x\neq y\}]$$

where the term in the square brackets [ ] *is present only when* $k=2$.

PROOF. (i), (ii) are given by Thomas [**16**, Proposition 2.9]. By [**19**, Proposition 2.6], there are two relations:

$$\rho(u^{k+1}\otimes x^2)=\rho(U(1\otimes x)+u^{k-1}\otimes(Sq^1x)^2)\qquad \text{if } x\in H^{n-k-1}(M;Z_2),$$
$$\rho(u^{k+2}\otimes x^2)=\rho(U(1\otimes x)+u^k\otimes(Sq^1x)^2+u^{k-2}\otimes((Sq^2+w_2)x)^2)$$
$$\text{if } x\in H^{n-k-2}(M;Z_2).$$

Moreover $U(1\otimes x)$ is expressed in the form

$$(*)\qquad U(1\otimes x)=\sigma(M\otimes x)+\sum x'\otimes x'',\qquad \dim x',\ \dim x''<n.$$

Applying [**16**, Proposition 2.9], we can prove (iii), (iv) immediately.

The actions of $v\in H^1(M^*;Z_2)$ and the square operation Sq^i $(i=1,2)$ on $H^*(M^*;Z_2)$ are given by Thomas [**16**, Corollary 2.10] and Bausum [**1**, Lemmas 11 and 24] as follows:

LEMMA 3.2. *There are the following relations in $H^*(M^*;Z_2)$:*

(i) $v\rho\sigma(x\otimes y)=0,\quad v\rho(u^i\otimes x^2)=\rho(u^{i+1}\otimes x^2)$;

(ii) *if* $x\in H^r(M;Z_2)$, *then*

$$Sq^1\rho(u^i\otimes x^2)=\begin{cases}(i+r)\rho(u^{i+1}\otimes x^2) & i>0,\\ r\rho(u\otimes x^2)+\rho\sigma(Sq^1x\otimes x) & i=0;\end{cases}$$

$$Sq^2\rho(u^i\otimes x^2)=\begin{cases}\binom{r+i}{2}\rho(u^{i+2}\otimes x^2)+\rho(u^i\otimes(Sq^1x)^2) & i>0,\\ \binom{r}{2}\rho(u^2\otimes x^2)+\rho(1\otimes(Sq^1x)^2)+\rho\sigma(Sq^2x\otimes x) & i=0.\end{cases}$$

For a homologically $(k-1)$-connected n-manifold M $(k\geqq 2)$, the cohomology groups $H^i(M^*;\underline{Z})$ for $2n-k-2\leqq i\leqq 2n-k$ are given in the following theorem, postponing the proof till §5:

THEOREM 3.3. *Assume that M is a homologically $(k-1)$-connected n-manifold $(k\geqq 2)$. Then*

(i) $H^{2n-k}(M^*;\underline{Z})\cong\begin{cases}H^{n-k}(M;Z_2) & \text{if } n-k \text{ is even,}\\ H^{n-k}(M;Z) & \text{if } n-k \text{ is odd;}\end{cases}$

(ii) $H^{2n-k-1}(M^*;\underline{Z})\cong\begin{cases}H^{n-k-1}(M;Z_2) & \text{if } n-k \text{ is even,}\\ H^{n-k-1}(M;Z)+H^{n-k}(M;Z_2) & \text{if } n-k \text{ is odd;}\end{cases}$

(iii) $\bar{\rho}_2H^{2n-k-2}(M^*;\underline{Z})$

$$=\{\rho(u^{k-2}\otimes x^2)\mid x\in H^{n-k}(M;Z_2)\}+\{\rho(u^{k+2}\otimes x^2)\mid x\in H^{n-k-2}(M;Z_2)\}$$
$$+[\{\rho\sigma(x\otimes y)\mid x,\ y\in H^{n-2}(M;Z_2),\ x\neq y\}] \quad \text{if } n-k \text{ is even,}$$
$$=\{\rho(u^k\otimes x^2)\mid x\in H^{n-k-1}(M;Z_2)\}+\{\rho\sigma(\rho_2x\otimes M)\mid x\in H^{n-k-2}(M;Z)\}$$
$$+[\{\rho\sigma(x\otimes y)\mid x,\ y\in H^{n-2}(M;Z_2),\ x\neq y\}] \quad \text{if } n-k \text{ is odd,}$$

where the terms in the square brackets [ ] *are present only when* $k=2$.

§4. Proof of the main theorem.

In this section, let M be a homologically $(k-1)$-connected n-manifold $(k\geqq 2)$. If its $(n-k)$-th normal Stiefel-Whitney class vanishes, then M can be embedded in Euclidean $(2n-k)$-space by Haefliger [5, §1] and there is a bijection

$$[M\subset R^{2n-k}]=H^{2n-k-1}(M^*;\underline{Z})\times\mathrm{Coker}\,\Theta$$

where

$$\Theta=\left(Sq^2+\binom{2n-k}{2}v^2\right)\bar{\rho}_2 : H^{2n-k-2}(M^*;\underline{Z})\longrightarrow H^{2n-k}(M^*;Z_2)$$

by Theorem 2.5. Since $H^{2n-k-1}(M^*;\underline{Z})$ is given in Theorem 3.3 (ii), we shall concentrate on calculating $\mathrm{Coker}\,\Theta$. Notice that there are an isomorphism

(4.1) $\qquad \chi : H^{n-k}(M;Z_2)\longrightarrow H^{2n-k}(M^*;Z_2) \qquad (\chi(x)=\rho\sigma(M\otimes x)),$

and equalities

(4.2) $\qquad \rho(u^k\otimes x^2)=\rho(U(1\otimes x))=\rho\sigma(M\otimes x) \qquad \text{for } x\in H^{n-k}(M;Z_2),$

which follow from [19, Proposition 2.6] and (*) in § 3.

Case I: *$n-k$ is even.* See Theorem 3.3 (iii) for the group $\bar{\rho}_2 H^{2n-k-2}(M^*; \underline{Z})$. If $x \in H^{n-k-2}(M; Z_2)$, then

$$\left(Sq^2+\binom{2n-k}{2}v^2\right)\rho(u^{k+2}\otimes x^2)$$
$$=\left(\binom{n}{2}+\binom{2n-k}{2}\right)\rho(u^{k+4}\otimes x^2)+\rho(u^{k+2}\otimes(Sq^1x)^2) \quad \text{by Lemma 3.2,}$$
$$=\left(\binom{n}{2}+\binom{2n-k}{2}\right)\rho(u^k\otimes((Sq^2+w_2)x)^2),$$

because there are two relations

$$\rho(u^{k+4}\otimes x^2+u^{k+2}\otimes(Sq^1x)^2+u^k\otimes((Sq^2+w_2)x)^2)=0,$$
$$\rho(u^{k+2}\otimes(Sq^1x)^2)=0,$$

which are easily proved by using [19, (2.5) and Proposition 2.6]. Therefore, by (4.2), we have

$$(4.3) \qquad \left(Sq^2+\binom{2n-k}{2}v^2\right)\rho(u^{k+2}\otimes x^2)$$
$$=\lambda\rho\sigma(M\otimes(Sq^2+w_2)x) \qquad \text{for} \quad x\in H^{n-k-2}(M; Z_2),$$

where

$$\lambda=\begin{cases} 0 & \text{for} \quad n-k\equiv 0\,(4), \\ 1 & \text{for} \quad n-k\equiv 2\,(4). \end{cases}$$

Similarly, we have a relation

$$(4.4) \qquad \left(Sq^2+\binom{2n-k}{2}v^2\right)\rho(u^{k-2}\otimes x^2)$$
$$=(1-\lambda)\rho\sigma(M\otimes x)+[\rho\sigma(w_2x\otimes x)] \qquad \text{for} \quad x\in H^{n-k}(M; Z_2).$$

Moreover the relation

$$(4.5) \qquad \left(Sq^2+\binom{2n-k}{2}v^2\right)\rho\sigma(x\otimes y)=\rho\sigma(w_2x\otimes y+w_2y\otimes x)$$
$$\text{for} \quad x,\ y\in H^{n-2}(M; Z_2) \quad \text{with } x\neq y$$

follows from Lemma 3.2. Therefore, if $k\geqq 3$ or $w_2=0$, then

$$\operatorname{Im}\Theta=\{(1-\lambda)\rho\sigma(M\otimes x)\,|\,x\in H^{n-k}(M; Z_2)\}+\{\lambda\rho\sigma(M\otimes Sq^2x)\,|\,x\in H^{n-k-2}(M; Z_2)\}$$

and so

$$(4.6) \qquad \operatorname{Coker}\Theta\cong\begin{cases} 0 & \text{for} \quad n-k\equiv 0\,(4), \\ H^{n-k}(M; Z_2)/Sq^2H^{n-k-2}(M; Z_2) & \text{for} \quad n-k\equiv 2\,(4), \end{cases}$$

by (4.1), (4.2). Next, consider the case $k=2$ and $w_2\neq 0$. In general, for a simply connected n-manifold M with non-trivial second Stiefel-Whitney class w_2, the group $H^{n-2}(M; Z_2)$ can be expressed, by using Poincaré duality, in the form

$$(4.7)\qquad H^{n-2}(M; Z_2)=\sum_{1\leqq i\leqq a} Z_2\langle z_i\rangle,\qquad w_2 z_i=\begin{cases} M & \text{if } i=1, \\ 0 & \text{if } 2\leqq i\leqq a \end{cases}$$

Then a simple calculation yields that

$$\operatorname{Im}\Theta=\begin{cases} \sum_{2\leqq i\leqq a} Z_2\langle\rho\sigma(M\otimes z_i)\rangle & \text{if } n-2\equiv 0\,(4), \\ H^{2n-2}(M^*; Z_2) & \text{if } n-2\equiv 2\,(4), \end{cases}$$

and hence

$$(4.8)\qquad \operatorname{Coker}\Theta=\begin{cases} Z_2 & \text{if } k=2,\ w_2\neq 0 \text{ and } n\equiv 2\,(4), \\ 0 & \text{if } k=2,\ w_2\neq 0 \text{ and } n\equiv 0\,(4), \end{cases}$$

by (4.1). Thus we deduce the main theorem in case $n-k$ is even, from (4.6), (4.8) and Theorems 2.5, 3.3 (ii).

Case II: $n-k$ *is odd*. See also Theorem 3.3 (iii) for the group $\bar{\rho}_2 H^{2n-k-2}(M^*; \underline{Z})$. In the same way as in the case when $n-k$ is even, we have the following relations:

$$(4.9)\quad \begin{aligned} &\left(Sq^2+\binom{2n-k}{2}v^2\right)\rho(u^k\otimes x^2)=\mu\rho\sigma(M\otimes Sq^1x),\quad \mu=\begin{cases} 0 & \text{for } n-k\equiv 1\,(4), \\ 1 & \text{for } n-k\equiv 3\,(4), \end{cases} \\ &\qquad\qquad \text{if } x\in H^{n-k-1}(M; Z_2); \\ &\left(Sq^2+\binom{2n-k}{2}v^2\right)\rho\sigma(M\otimes\rho_2 x)=\rho\sigma(M\otimes Sq^2\rho_2 x) \quad \text{if } x\in H^{n-k-2}(M; Z); \\ &\left(Sq^2+\binom{2n-k}{2}v^2\right)\rho\sigma(x\otimes y)=\rho\sigma(w_2x\otimes y+w_2y\otimes x) \quad \text{if } x,\ y\in H^{n-2}(M; Z_2). \end{aligned}$$

If $w_2=0$, then (4.1) and the above relations (4.9) lead at once to the relation

$$(4.10)\quad \operatorname{Im}\Theta\cong\begin{cases} Sq^2\rho_2 H^{n-k-2}(M; Z) & \text{for } n-k\equiv 1\,(4), \\ Sq^2\rho_2 H^{n-k-2}(M; Z)+Sq^1H^{n-k-1}(M; Z_2) & \text{for } n-k\equiv 3\,(4). \end{cases}$$

If $w_2\neq 0$, it is easily verified, in the same way as in the case when $n-k$ is even, that the subgroup of $\operatorname{Im}\Theta$ determined by the last relation of (4.9) is equal to $\sum_{2\leqq i\leqq a} Z_2\langle\rho\sigma(M\otimes z_i)\rangle$. On the other hand, the following relations hold:

$$w_2Sq^2\rho_2x=Sq^2Sq^2\rho_2x=Sq^3Sq^1\rho_2x=0 \qquad \text{for } x\in H^{n-4}(M; Z),$$

$$w_2Sq^1x=Sq^1(w_2x)+(Sq^1w_2)x=w_3x \qquad \text{for } x\in H^{n-3}(M; Z_2).$$

Therefore, it is shown immediately that $\rho\sigma(M\otimes z_1)\in\operatorname{Im}\Theta$ if and only if $n-2\equiv 3\,(4)$ and $w_3\neq 0$, and hence

(4.11) $$\operatorname{Coker}\Theta \cong \begin{cases} 0 & \text{if } n\equiv 1\,(4) \text{ and } w_3\neq 0, \\ Z_2 & \text{if } n\equiv 1\,(4),\ w_3=0, \text{ or if } n\equiv 3\,(4). \end{cases}$$

Thus (4.10), (4.11), together with Theorems 2.5, 3.3 (ii), deduce the main theorem in case $n-k$ is odd.

§ 5. Proof of Theorem 3.3.

Throughout this section, we assume that M is a homologically $(k-1)$-connected n-manifold $(k\geqq 2)$ and we compute $H^{2n-k-i}(M^*; \underline{Z})$ for $0\leqq i\leqq 2$, where $\underline{Z}=Z$ or $Z[v]$ according as k is even or odd.

Case I: *$n-k$ is even.* First we consider the odd torsion subgroup of $H^{2n-k-i}(M^*; \underline{Z})$ for $i=0, 1$. Considering the cohomology spectral sequence (cf. [**11**, Theorem 1.1]) for a fibration $M^2-\Delta M\to S^\infty\times_{Z_2}(M^2-\Delta M)\to P^\infty$, which is homotopically equivalent to $M^2-\Delta M\xrightarrow{p} M^*\xrightarrow{\xi} P^\infty$, we see that the odd torsion subgroup of $H^{2n-k-i}(M^*; \underline{Z})$ is isomorphic, by p^*, to that of

$$\{x\in H^{2n-k-i}(M^2-\Delta M; Z) \mid t^*x=(-1)^n x\}=H^{2n-k-i}(M^2-\Delta M; Z)^{(-1)^n t^*}.$$

Since M is orientable, there is a short exact sequence

$$0\longrightarrow H^i(M; Z)\xrightarrow{\phi_1} H^{n+i}(M^2; Z)\xrightarrow{\tilde{i}^*} H^{n+i}(M^2-\Delta M; Z)\longrightarrow 0,$$

where

$$\phi_1(x)=U(1\otimes x) \qquad \text{for} \quad x\in H^i(M; Z),$$

$U\in H^n(M^2; Z)$ is called the Thom class or the diagonal cohomology class of M, e. g. by [**12**], and $\tilde{i}$ is the natural inclusion. Therefore, $\tilde{i}^*$ induces an isomorphism

$$(H^{2n-k-i}(M^2; Z)/\phi_1 H^{n-k-i}(M; Z))^{(-1)^n t^*}\cong H^{2n-k-i}(M^2-\Delta M; Z)^{(-1)^n t^*}.$$

Here $\phi_1 H^{n-k-i}(M; Z)\subset H^{2n-k-i}(M^2; Z)^{(-1)^n t^*}$ by [**15**, p. 305]. On the other hand, it is easily verified that $H^{2n-k-i}(M^2; Z)^{(-1)^n t^*}$ is isomorphic to $H^{n-k-i}(M; Z)$ for $i=0, 1$. Therefore, $H^{2n-k-i}(M^2-\Delta M; Z)^{(-1)^n t^*}$ has no odd torsion subgroup and hence

(5.1) $H^{2n-k-i}(M^*; \underline{Z})$ has no odd torsion for $i=0, 1$.

In order to study $H^{2n-k-i}(M^*; \underline{Z})$, consider the Bockstein exact sequence associated with $0\to\underline{Z}\xrightarrow{\times 2}\underline{Z}\xrightarrow{\rho_2} Z_2\to 0$,

(5.2) $$\cdots\to H^{i-1}(M^*; Z_2)\xrightarrow{\bar{\beta}_2} H^i(M^*; \underline{Z})\xrightarrow{\times 2} H^i(M^*; \underline{Z})\xrightarrow{\bar{\rho}_2} H^i(M^*; Z_2)\to\cdots.$$

By using the relations in (2.4) and Lemma 3.2, we have the following relations:

$\bar{\rho}_2\bar{\beta}_2\rho\sigma(x\otimes y)=0$ if $k=2$ and $x, y\in H^{n-2}(M; Z_2)$,

$\bar{\rho}_2\bar{\beta}_2\rho(u^i\otimes x^2)=\rho(u^{i+1}\otimes x^2)$

for $(i, \dim x)=(k-1, n-k), (k, n-k-1), (k-3, n-k), (k+1, n-k-2)$.

These relations, (4.2), (5.1) and the exact sequence (5.2), together with Lemma 3.1, lead to Theorem 3.3 in case $n-k$ is even.

Case II: *$n-k$ is odd.* The group Z_2 acts on SM, the tangent sphere bundle over M, via the antipodal map on each fibre S^{n-1}. Let

$$PM=SM/Z_2, \qquad (\Lambda^2M, \Delta M)=(M^2/Z_2, \Delta M/Z_2),$$
$$i : M^*=\Lambda^2M-\Delta M\subset(\Lambda^2M, \Delta M),$$

and let

$$j : PM \longrightarrow M^*$$

be the embedding such that j^*v is the first Stiefel-Whitney class of the double covering $SM\to PM$. We write j^*v as $v\in H^1(PM; Z_2)$ if no confusion can arise. Then there exists a long exact sequence, cf. [**19**, Lemma 1.3],

(5.3)
$$\cdots\to H^{i-1}(PM; \underline{Z}) \xrightarrow{\delta} H^i(\Lambda^2M, \Delta M; \underline{Z}) \xrightarrow{i^*} H^i(M^*; \underline{Z}) \xrightarrow{j^*} H^i(PM; \underline{Z})\to\cdots.$$

The cohomology of PM has been given by Rigdon [**13**, § 9] as follows:

LEMMA 5.4 (Rigdon). *Assume that M is a homologically $(k-1)$-connected n-manifold $(k\geqq 2)$ and that $n-k$ is odd. Then*

(i) $$H^{2n-k}(PM; \underline{Z})=\begin{cases} 0 & \textit{if } k \textit{ is even,} \\ Z_2\langle\tilde{\beta}_2(v^{n-k-1}M)\rangle & \textit{if } k \textit{ is odd;}\end{cases}$$

(ii) $$H^{2n-k-1}(PM; \underline{Z})=\begin{cases} \{\beta_2(v^{n-2}x+v^{n-k-2}Sq^kx)\,|\,x\in H^{n-k}(M; Z_2)\} & \\ \qquad +Z_2\langle\beta_2(v^{n-k-2}M)\rangle & \textit{if } k \textit{ is even,} \\ \{\tilde{\beta}_2(v^{n-2}x)\,|\,x\in H^{n-k}(M; Z_2)\} & \textit{if } k \textit{ is odd;}\end{cases}$$

(iii) $$H^{2n-k-2}(PM; \underline{Z})=\begin{cases} \{\beta_2(v^{n-2}x)\,|\,x\in H^{n-k-1}(M; Z_2)\} & \textit{if } k \textit{ is even,} \\ \{\tilde{\beta}_2(v^{n-2}x+v^{n-k-3}Sq^{k+1}x)\,|\,x\in H^{n-k-1}(M; Z_2)\} & \\ \qquad +Z_2\langle\tilde{\beta}_2(v^{n-k-3}M)\rangle & \textit{if } k \textit{ is odd.}\end{cases}$$

In the above lemma, and also from now on, $\tilde{\beta}_2$ denotes the Bockstein operator twisted by v.

The cohomology of $(\Lambda^2M, \Delta M)$ has been investigated by Larmore [**9**].

LEMMA 5.5 (Larmore). *Assume that M is a homologically $(k-1)$-connected n-manifold $(k\geqq 2)$ and that $n-k$ is odd. Then*

(i) $H^{2n-k}(\Lambda^2 M, \Delta M; \underline{Z}) \cong \begin{cases} H^{n-k}(M; Z) + Z_2\langle \beta_2(v^{n-k-1}\Lambda M)\rangle & \textit{if } k \textit{ is even}, \\ H^{n-k}(M; Z) & \textit{if } k \textit{ is odd}; \end{cases}$

(ii) $H^{2n-k-1}(\Lambda^2 M, \Delta M; \underline{Z}) \cong \begin{cases} H^{n-k-1}(M; Z) & \textit{if } k \textit{ is even}. \\ H^{n-k-1}(M; Z) + Z_2\langle \tilde{\beta}_2(v^{n-k-2}\Lambda M)\rangle & \textit{if } k \textit{ is odd}; \end{cases}$

(iii) $i^*\bar{\rho}_2 H^{2n-k-2}(\Lambda^2 M, \Delta M; \underline{Z}) = \{\rho\sigma(\rho_2 x \otimes M) \mid x \in H^{n-k-2}(M; Z)\}$
$+[\{\rho\sigma(x \otimes y) \mid x, y \in H^{n-2}(M; Z_2), x \neq y\}]$,

where the term in the square brackets is present only when $k=2$.

PROOF. The cohomology groups $H^{2n-k-i}(\Lambda^2 M, \Delta M; \underline{Z})$ for $i=0, 1, 2$ are given directly by [**9**, Theorem 20]. Their $i^*\bar{\rho}_2$-images are easily obtained by using the relations

$$\delta(v^i x) = v^{i+1}\Lambda x, \qquad i^*(\Lambda x \Lambda y) = \rho\sigma(x \otimes y) + \rho\sigma(xy \otimes 1) \tag{5.6}$$

in [**18**, Lemma 1.5], [**19**, Lemma 3.3] and the two congruences mod Im δ

$$\tilde{\rho}_2\tilde{\beta}_r(\Lambda x) \equiv \Lambda(\rho_2\beta_r x) \qquad \text{if } x \in H^*(M; Z_r),$$
$$\tilde{\rho}_2\tilde{\beta}_r\Delta(x, \rho_r y) \equiv \tilde{\rho}_2\Delta(\beta_r x, y) \qquad \text{if } x \in H^*(M; Z_r),\ y \in H^*(M; Z),$$

which are easily proved.

REMARK. The author has proved this lemma in the same way as he proved the propositions in [**18**, §5], i.e., by using the results on pp. 908–915 in [**9**]. He thinks that the expression "*r is a power of* 2 *or*" in I(iv), II(v) of [**9**, Theorem 20] should be omitted.

Using the first relation of (5.6) and the relation

$$j^*\rho(u^r \otimes x^2) = \sum_{0 \leq i \leq q} v^{r+q-i} Sq^i x \qquad \text{if } x \in H^q(M; Z_2),$$

in [**16**, §2], we have the following relations:

$$\tilde{\rho}_2\delta\tilde{\beta}_2(v^{n-k-1}M) = \delta(v^{n-k}M) = v^{n-k+1}\Lambda M \neq 0 \qquad \text{if } k \text{ is odd},$$
$$\delta\beta_2(v^{n-k-2}M) = \beta_2(v^{n-k-1}\Lambda M) \qquad \text{if } k \text{ is even},$$
$$\delta\tilde{\beta}_2(v^{n-k-3}M) = \tilde{\beta}_2(v^{n-k-2}\Lambda M) \qquad \text{if } k \text{ is odd},$$

$$j^*\bar{\beta}_2\rho(u^{k-2} \otimes x^2) = \begin{cases} \beta_2(v^{n-2}x + v^{n-2-k}Sq^k x) & \text{if } k \text{ is even and } \dim x = n-k, \\ \tilde{\beta}_2(v^{n-2}x) & \text{if } k \text{ is odd and } \dim x = n-k, \end{cases}$$

$$j^*\bar{\beta}_2\rho(u^{k-1} \otimes x^2) = \begin{cases} \beta_2(v^{n-2}x) & \text{if } k \text{ is even and } \dim x = n-k-1, \\ \tilde{\beta}^2(v^{n-2}x + v^{n-k-3}Sq^{k+1}x) & \text{if } k \text{ is odd and } \dim x = n-k-1. \end{cases}$$

On considering the exact sequence (5.3), it follows, from Lemmas 5.4, 5.5 and the above relations, that $j^*: H^{2n-k-i}(M^*; \underline{Z}) \to \mathrm{Im}\, j^*$ is a split epimorphism for $i=1, 2$. Further, the relation

$$\bar{\rho}_2 \bar{\beta}_2 \rho(u^{k-1} \otimes x^2) = \rho(u^k \otimes x^2) \qquad \text{for} \quad x \in H^{n-k-1}(M; Z_2)$$

follows from Lemma 3.2. Hence, the theorem is established in case $n-k$ is odd.

References

[1] D. R. Bausum, Embeddings and immersions of manifolds in Euclidean space, Trans. Amer. Math. Soc., **213** (1975), 263-303.

[2] R. Greenblatt, The twisted Bockstein coboundary, Proc. Cambridge Philos. Soc., **61** (1965), 295-297.

[3] A. Haefliger, Points multiples d'une application et produit cyclique réduit, Amer. J. Math., **83** (1961), 57-70.

[4] A. Haefliger, Plongements différentiables dans le domaine stable, Comment. Math. Helv., **37** (1962), 155-176.

[5] A. Haefliger, Plongements de variétés dans le domaine stable, Séminaire Bourbaki, **15** (1962/63), n° 245.

[6] A. Haefliger and M. W. Hirsch, On the existence and classification of differentiable embeddings, Topology, **2** (1962), 129-135.

[7] A. Haefliger and M. W. Hirsch, Immersions in the stable range, Ann. of Math., **75** (1962), 231-241.

[8] I. M. James and E. Thomas, Note on the classification of cross-sections, Topology, **4** (1966), 351-359.

[9] L. L. Larmore, The cohomology of $(\Lambda^2 X, \Delta X)$, Canad. J. Math., **25** (1973), 908-921.

[10] J. F. McClendon, Higher order twisted cohomology operations, Invent. Math., **7** (1969), 183-214.

[11] J. F. McClendon, Obstruction theory in fiber spaces, Math. Z., **120** (1971), 1-17.

[12] J. W. Milnor and J. D. Stasheff, Characteristic classes, Ann. of Math. Studies, **76**, Princeton Univ. Press, 1974.

[13] R. D. Rigdon, Immersions and embeddings of manifolds in Euclidean space, Thesis, Univ. of California at Berkeley, 1970.

[14] H. Samelson, A note on the Bockstein operator, Proc. Amer. Math. Soc., **15** (1964), 450-453.

[15] E. H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

[16] E. Thomas, Embedding manifolds in Euclidean space, Osaka J. Math., **13** (1976), 163-186.

[17] G. W. Whitehead, Elements of Homotopy Theory, Graduate Texts in Math., **61**, Springer-Verlag, New York, 1978.

[18] T. Yasui, On the map defined by regarding embeddings as immersions, Hiroshima Math. J., **13** (1983), 457-476.

[19] T. Yasui, Enumerating embeddings of n-manifolds in Euclidean $(2n-1)$-space, J. Math. Soc. Japan, **36** (1984), 555-576.

[20] G. Yo, Cohomology operations and duality in a manifold, Sci. Sinica, **12** (1963), 1469-1487.

[21] G. Yo, Cohomology mod p of deleted cyclic product of a manifold, Sci. Sinica, **12**

(1963), 1779-1794.

Tsutomu YASUI
Department of Mathematics
Faculty of Education
Yamagata University
Kojirakawacho, Yamagata 990
Japan