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On pluricanonical maps for 3-folds of general type

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§ 0. Introduction.

Throughout this paper, we fix the complex number field $\boldsymbol{C}$ as the ground field. The purpose of this paper is to prove the following

MAIN THEOREM. *Let X be a nonsingular projective 3-fold whose canonical divisor K_X is nef and big* (cf. M. Reid [**12**] or § 1). *Then*
(i) *$\Phi_{|7K_X|}$ is birational with the possible exceptions of*
  a) *$\chi(\mathcal{O}_X)=0$ and $K_X^3=2$, or*
  b) *$|3K_X|$ is composed of pencils, i.e.,* $\dim \Phi_{|3K_X|}(X)=1$,
(ii) *$\Phi_{|nK_X|}$ is birational for $n \geqq 8$. Further if $\chi(\mathcal{O}_X)<0$, e.g. when K_X is ample, $\Phi_{|nK_X|}$ is birational for $n \geqq 7$.*

X. Benveniste [**1**] proved that $\Phi_{|nK_X|}$ is birational for $n \geqq 9$ under the same assumption as ours. Our proof follows mainly his ideas but improves the result to the extent that it guarantees $\Phi_{|nK_X|}$ being birational for $n \geqq 7$ if $\chi(\mathcal{O}_X)<0$.

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§ 1. Preliminaries.

Let X be a nonsingular complete variety, and $D \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$, where $\mathrm{Div}(X)$ is a free abelian group generated by Weil divisors on X. Then D is called nef if $D \cdot C \geqq 0$ for any curve C on X, and big if $\kappa(D, X)=\dim X$ (cf. Iitaka [**6**]), respectively. We denote the linear equivalence and the numerical equivalance by $\sim$ and $\approx$, respectively. For $D \in \mathrm{Div}(X)$ with $h^0(X, \mathcal{O}_X(D)) \neq 0$, $\Phi_{|D|}$ denotes the rational map associated with the complete linear system $|D|$.

PROPOSITION 1. *Let X be a nonsingular complete variety, and $D \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$. Assume the following two conditions:*
(i) *D is nef and big,*
(ii) *the fractional part of D has the support with only normal crossings.*
Then

$$H^i(X, \mathcal{O}_X(\ulcorner D \urcorner + K_X))=0 \qquad for\ i>0,$$

where $\ulcorner D \urcorner$ *is the minimum integral divisor with* $\ulcorner D \urcorner - D \geqq 0$.

For a proof, see Kawamata [8], Theorem 1.2.

PROPOSITION 2. *Let X be a nonsingular complete variety with the canonical divisor K_X. Then the following conditions are equivalent to each other.*

(i) *There exists a positive integer n such that the base locus* $\mathrm{Bs}|nK_X| = \emptyset$ *and that* $\Phi_{|nK_X|}$ *is birational.*

(ii) *K_X is nef and big.*

For a proof, see Kawamata [8], Theorem 2.6.

PROPOSITION 3. *Let X be a nonsingular projective 3-fold, and* $D \in \mathrm{Div}(X)$. *Then we have the following assertions:*

(i) $\chi(\mathcal{O}_X(D)) = (D^3/6) - (K_X \cdot D^2/4) + (D \cdot (K_X^2 + c_2)/12) + \chi(\mathcal{O}_X)$

and

$$\chi(\mathcal{O}_X) = -(c_2 \cdot K_X/24),$$

where c_2 is the second Chern class of X.

(ii) *$K_X \cdot D^2$ is even. In particular, K_X^3 is even.*

PROOF. (i) is the Riemann-Roch theorem. (ii) follows easily from (i) and the calculation

$$\chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X(-D)) = -(K_X \cdot D^2/2) + 2\chi(\mathcal{O}_X) \in \boldsymbol{Z}.$$

PROPOSITION 4. *Let X be a nonsingular projective 3-fold whose canonical divisor K_X is nef and big. Then*

(i) $P(n) := h^0(X, \mathcal{O}_X(nK_X)) = (2n-1)\{n(n-1)K_X^3/12 - \chi(\mathcal{O}_X)\}$ *for* $n \geqq 2$,

(ii) $\chi(\mathcal{O}_X) \leqq K_X^3/6$,

(iii) $h^0(X, \mathcal{O}_X(nK_X)) \geqq 5$ *for* $n \geqq 3$.

PROOF. (i) is clear from Proposition 3 (i) and Proposition 1. (ii) follows from the inequality

$$0 \leqq h^0(X, \mathcal{O}_X(2K_X)) = 3\{K_X^3/6 - \chi(\mathcal{O}_X)\}.$$

For (iii), we consider the two cases. Whenever $K_X^3 \leqq 4$, (ii) implies $\chi(\mathcal{O}_X) \leqq 0$. Therefore

$$h^0(X, \mathcal{O}_X(nK_X)) \geqq h^0(X, \mathcal{O}_X(3K_X)) \geqq (2\cdot 3-1)\{3(3-1)\cdot 2/12\} = 5.$$

Whenever $K_X^3 \geqq 6$, we have

$$h^0(X, \mathcal{O}_X(nK_X)) \geqq h^0(X, \mathcal{O}_X(3K_X)) \geqq (2\cdot 3-1)\{3(3-1)K_X^3/12 - K_X^3/6\} \geqq 10.$$

Thus we obtain (iii).

§ 2. Key steps.

The following theorem about a surface plays a crucial role in our proof of the main theorem. We replace the condition $h^0(S, \mathcal{O}_S(mR)) \geqq 7$ in Proposition 2-0 of Benveniste [**1**] by $(*)$ below, which is weaker than the former.

THEOREM 5. *Let S be a nonsingular projective surface, $R \in \operatorname{Pic} S$ a nef and big divisor on S, and m a positive integer which satisfy the following condition $(*)$.*

$(*)$ *Given arbitrary two distinct points $x_1, x_2 \in S$, letting $\pi: S'' \to S$ be the blowing-up at x_1 and x_2, $L_1 := \pi^{-1}(x_1)$ and $L_2 := \pi^{-1}(x_2)$, the linear system $|\pi^*(mR) - 2L_1 - 2L_2|$ is not empty.*

Then $\Phi_{|K_S+mR|}$ is birational in the following two cases:
(i) $R^2 \geqq 2$ *and* $m \geqq 3$,
(ii) $R^2 = 1$ *and* $m \geqq 4$.

PROOF. First, we note the following two lemmata.

LEMMA 5.1. *Let S be a nonsingular projective surface, $R \in \operatorname{Pic} S$ a divisor with $R^2 > 0$. Let $(E_i)_{i \in I}$ be the family of distinct curves such that $R \cdot E_i = 0$. Then the E_i are numerically independent in $N_1(S) := (\{1\text{-cycles}\}/\approx) \otimes \boldsymbol{R}$.*

PROOF. This follows easily from Hodge's index theorem.

LEMMA 5.2. *Let S be a nonsingular projective surface, $R \in \operatorname{Pic} S$ a nef divisor with $R^2 > 0$. Given a positive integer n, let A_n be the set of effective divisors D on S such that $R \cdot D = 0$ and $D^2 \geqq -n$. Then A_n is a finite set.*

PROOF. Let $(E_i)_{i \in I}$ be as in Lemma 5.1. Then the E_i are numerically independent. Thus $\#(I) \leqq \rho(S)$. Moreover, Hodge's index theorem asserts that the intersection matrix of $(E_i)_{i \in I}$ is negative definite. Thus the number of $D \in \bigoplus_{i \in I} \boldsymbol{Z}_+ E_i$ with $D^2 \geqq -n$ is finite, $\boldsymbol{Z}_+$ denoting the set of positive integers.

We now return to the proof of Theorem 5. Let $B_2 := \bigcup_{D \in A_2} D$ and $U := S \setminus B_2$. Then by Lemma 5.2, B_2 is a proper closed subset of S. Thus U is a nonempty Zariski open set of S. In the following argument, we shall show that $|K_S + mR| \neq \emptyset$ and that $\Phi_{|K_S+mR|}$ separates any two distinct points x_1, x_2 of U.

CLAIM 5.3. *Any member $A \in |\pi^*(mR) - 2L_1 - 2L_2|$ is linearly 1-connected with $A^2 > 0$.*

PROOF. We note first that

$$|\pi^*(mR) - 2L_1 - 2L_2| \neq \emptyset$$

from the hypothesis, and we have

$$A^2=m^2R^2-4-4>0$$

in both of the cases (i) and (ii). Therefore it is sufficient to show that A is linearly 1-connected, i.e., for an arbitrary decomposition of A, $A\sim D_1+D_2$ where D_1 and D_2 are nonzero effective divisors, we have $D_1\cdot D_2\geqq 1$.

Let $E_i=\pi_*(D_i)$ for $i=1, 2$. Then for some integers a_i, b_i, we have

$$D_i=\pi^*(E_i)+a_iL_1+b_iL_2\,.$$

By definition,

$$a_1+a_2=b_1+b_2=-2\,.$$

Moreover,

$$D_1\cdot D_2=E_1\cdot E_2-a_1a_2-b_1b_2\,.$$

We put $\xi:=(R\cdot E_1/R^2)R-E_1$. We note here that $\xi\approx-(R\cdot E_2/R^2)R+E_2$, since $mR\sim\pi_*(A)\sim E_1+E_2$.

Case 1. $R\cdot E_1>0$ and $R\cdot E_2>0$. The assumption of this case implies

$$0\leqq\{(R\cdot E_1)-1\}\{(R\cdot E_2)-1\}=(R\cdot E_1)(R\cdot E_2)-mR^2+1\,.$$

Therefore

$$\begin{aligned}E_1\cdot E_2&=(R\cdot E_1)(R\cdot E_2)/R^2-\xi^2\\&\geqq(R\cdot E_1)(R\cdot E_2)/R^2\geqq(mR^2-1)/R^2>2\end{aligned}$$

in both of the cases (i) and (ii). Furthermore $a_1+a_2=b_1+b_2=-2$ implies $a_1a_2\leqq 1$ and $b_1b_2\leqq 1$. Thus

$$D_1\cdot D_2=E_1\cdot E_2-a_1a_2-b_1b_2\geqq 1\,.$$

Case 2. $R\cdot E_1=0$. If $a_1=-1$ or $b_1=-1$, then $x_1\in E_1$ or $x_2\in E_1$ respectively, since $\pi^*(E_1)+a_1L_1+b_1L_2$ is effective. Since x_1, $x_2\in U=S\setminus B_2$, the definition of B_2 implies $E_1^2\leqq-3$. Noting that $E_1\cdot E_2=E_1(mR-E_1)=-E_1^2$, we have

$$D_1\cdot D_2=E_1\cdot E_2-a_1a_2-b_1b_2=-E_1^2-a_1a_2-b_1b_2\geqq 1\,.$$

If $a_1\neq-1$ and $b_1\neq-1$, then $a_1a_2\leqq 0$ and $b_1b_2\leqq 0$. In the case with $E_1\not\approx 0$, we have $E_1^2\leqq-1$, which implies

$$D_1\cdot D_2=-E_1^2-a_1a_2-b_1b_2\geqq 1\,.$$

In the case with $E_1\approx 0$, i.e. $E_1=0$, since $D_1=\pi^*(E_1)+a_1L_1+b_1L_2$ is nonzero effective, we have $a_1>0$ or $b_1>0$. Thus $a_1a_2<0$ or $b_1b_2<0$ respectively. Therefore $D_1\cdot D_2=-a_1a_2-b_1b_2\geqq 1$.

The case with $R\cdot E_2=0$ can be treated similarly as in Case 2. This completes the proof of Claim 5.3.

We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{S''}(\pi^*(K_S+mR)-L_1-L_2) \longrightarrow \mathcal{O}_{S''}(\pi^*(K_S+mR)) \longrightarrow \mathcal{O}_{L_1}\oplus\mathcal{O}_{L_2} \longrightarrow 0.$$

Since $A\in|\pi^*(mR)-2L_1-2L_2|$ is a nonzero effective divisor which is linearly 1-connected by Claim 5.3, Ramanujam's vanishing theorem (cf. Ramanujam [**10**]) and Serre duality imply

$$H^1(S'', \mathcal{O}_{S''}(\pi^*(K_S+mR)-L_1-L_2)) \cong H^1(S'', \mathcal{O}_{S''}(-(\pi^*(mR)-2L_1-2L_2)))=0.$$

Therefore the induced homomorphism

$$H^0(S'', \mathcal{O}_{S''}(\pi^*(K_S+mR))) \longrightarrow H^0(L_1, \mathcal{O}_{L_1})\oplus H^0(L_2, \mathcal{O}_{L_2})$$

is surjective. Thus we complete the proof of Theorem 5.

COROLLARY 6 (cf. Bombieri [**3**]). *Let S be a nonsingular projective surface of general type with the canonical divisor K_S. Then $\Phi_{|nK_S|}$ is birational for $n\geqq 5$.*

PROOF. We may assume that S is minimal, which implies K_S is nef and $K_S^2\geqq 1$. By Riemann-Roch theorem and Proposition 1, we have

$$\begin{aligned} h^0(S, \mathcal{O}_S(mK_S))&=\chi(\mathcal{O}_S(mK_S)) \\ &=(mK_S-K_S, mK_S)/2+\chi(\mathcal{O}_S)\geqq 7 \qquad \text{for } m\geqq 4, \end{aligned}$$

noting that $\chi(\mathcal{O}_S)\geqq 1$ since S is of general type. Therefore $R:=K_S$ and $m:=n-1$ satisfy the condition $(*)$ of Theorem 5. Thus we obtain the required result.

THEOREM 7 (cf. Proposition 3-0 of Benveniste [**1**]). *Let X be a nonsingular projective 3-fold whose canonical divisor K_X is nef and big. Setting $W_n:=\Phi_{|nK_X|}(X)$ for a positive integer n, we have the following assertions:*

(i) $\dim W_n\geqq 2$ *for* $n\geqq 4$.

(ii) *If* $\dim W_3=1$, *then one of the following two cases* α), β) *holds. We consider the commutative diagram below and introduce the next notation.*

$$\begin{tikzcd} X' \arrow[r, "h_3"] \arrow[d, "f_3"'] \arrow[dr, "g_3"] & W_3' \arrow[d, "s_3"] \\ X \arrow[r, dashed, "\Phi_{|3K_X|}"'] & W_3 \end{tikzcd}$$

Here f_3 is a succession of blowing-ups with nonsingular centers such that $g_3:=\Phi_{|3K_X|}\circ f_3$ is a morphism, and $g_3=s_3\circ h_3$ is the Stein factorization. Let $b_3:=\deg(s_3)$ and S_3 be a general fiber of h_3.

Case α) $b_3\cdot\{S_3\cdot f_3^*(K_X)^2\}=2$. *In this case,* $\chi(\mathcal{O}_X)=1$ *and* $K_X^3=6$.

Case β) $b_3=1$, $S_3\cdot f_3^*(K_X)^2=1$. *In this case,* S_3 *is a nonsingular projective surface of general type. Letting* $\pi_3: S_3\to S_{3,0}$ *be the morphism onto the minimal model* $S_{3,0}$ *of* S_3, *and* $K_{3,0}$ *the canonical divisor of* $S_{3,0}$, *we have* $K_{3,0}^2=1$, *and*

$$\mathcal{O}_{S_3}(\pi_3^*(K_{3,0}))\cong\mathcal{O}_{S_3}(f_3^*(K_X)|_{S_3}).$$

(iii) $\dim W_n=3$ *for* $n\geqq 8$.

PROOF. First, we note that $\dim W_n\geqq 1$ for $n\geqq 3$, since $h^0(X, \mathcal{O}_X(nK_X))\geqq 5$ for $n\geqq 3$ by Proposition 4 (iii).

Proofs of (i) and (ii). Take a positive integer $n\geqq 3$. Assuming that $\dim W_n=1$, we shall show that $n=3$. We consider the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "h_n"] \arrow[d, "f_n"'] \arrow[dr, "g_n"] & W'_n \arrow[d, "s_n"] \\ X \arrow[r, dashed, "\Phi_{|nK_X|}"'] & W_n \end{tikzcd}$$

where f_n is a succession of blowing-ups with nonsingular centers such that $g_n := \Phi_{|nK_X|}\circ f_n$ is a morphism, and $g_n=s_n\circ h_n$ is the Stein factorization. Let $b_n:=\deg(s_n)$ and S_n be a general fiber of h_n, H_n be a hyperplane section of W_n in $\boldsymbol{P}^{P(n)-1}$, and let a_n be the degree of the curve W_n in $\boldsymbol{P}^{P(n)-1}$. Then

$$f_n^*(nK_X)\sim h_n^*s_n^*(H_n)+Z_n,$$

where Z_n is the fixed part of $|f_n^*(nK_X)|$. Thus

$$f_n^*(nK_X)\approx a_nb_nS_n+Z_n.$$

Multiplying this equality by $f_n^*(K_X)^2$, we obtain

$$nK_X^3=nf_n^*(K_X)^3=a_nb_nf_n^*(K_X)^2\cdot S_n+f_n^*(K_X)^2\cdot Z_n.$$

Let $c_n:=f_n^*(K_X)^2\cdot S_n$. Since $f_n^*(K_X)$ is nef and big and since S_n is nef and $S_n\not\approx 0$, it follows that $c_n\geqq 1$ and $f_n^*(K_X)^2\cdot Z_n\geqq 0$. Thus

$$nK_X^3\geqq a_nb_nc_n.$$

Moreover, since W_n is the image of $\Phi_{|nK_X|}$, we obtain

$$a_n\geqq P(n)-1.$$

Combining these inequalities and equalities together, we have

$$(2n-1)\{n(n-1)K_X^3/12-\chi(\mathcal{O}_X)\}-nK_X^3/b_nc_n\leqq 1,$$

i.e., defining

$$R_{b_nc_n}(n):=n\{2n^2-3n+1-(12/b_nc_n)\}K_X^3/12-(2n-1)\chi(\mathcal{O}_X),$$

we obtain that $R_{b_n c_n}(n) \leqq 1$.

Now we examine the following two cases separately.

Case 1. $\chi(\mathcal{O}_X) \geqq 1$. By Proposition 4 (ii) $\chi(\mathcal{O}_X) \leqq K_X^3/6$, we have $6 \leqq K_X^3$. We define $P_{b_n c_n}(x)$ by

$$P_{b_n c_n}(x) := x\{2x^2-3x+1-(12/b_n c_n)\}/2-(2x-1).$$

α) the subcase $b_n c_n \geqq 2$. We have for $n \geqq 3$ that

$$P_2(n) \leqq P_{b_n c_n}(n) \leqq R_{b_n c_n}(n).$$

In fact, it is clear from the hypothesis $b_n c_n \geqq 2$ that $P_2(n) \leqq P_{b_n c_n}(n)$, and

$$\begin{aligned} &R_{b_n c_n}(n) - P_{b_n c_n}(n) \\ &= n\{2n^2-3n+1-(12/b_n c_n)\}(K_X^3/12-1/2)-(2n-1)\{\chi(\mathcal{O}_X)-1\} \\ &\geqq n\{2n^2-3n+1-(12/b_n c_n)\}\{\chi(\mathcal{O}_X)-1\}/2-(2n-1)\{\chi(\mathcal{O}_X)-1\} \\ &\geqq \{n(2n^2-3n+1-6)/2-(2n-1)\}\{\chi(\mathcal{O}_X)-1\} \geqq 0. \end{aligned}$$

On the other hand, a simple computation shows $P_2(n) \geqq 23$ if $n \geqq 4$. Thus $n=3$. Moreover, $P_2(3)=1$. Therefore, in this subcase, all the inequalities above must be equalities, i.e., $b_n c_n = 2$, $\chi(\mathcal{O}_X)=1$ and $K_X^3=6$.

β) the subcase $b_n c_n = 1$. By a similar computation to that in the subcase α), we have $P_1(n) \leqq R_1(n)$ for $n \geqq 4$. But $P_1(n) \geqq 11$ if $n \geqq 4$. Thus $n=3$.

Case 2. $\chi(\mathcal{O}_X) \leqq 0$. In this case,

$$n\{2n^2-3n+1-(12/b_n c_n)\}K_X^3/12-(2n-1)\chi(\mathcal{O}_X) \leqq 1$$

implies

$$R'_{b_n c_n}(n) := n\{2n^2-3n+1-(12/b_n c_n)\}K_X^3/12 \leqq 1 \qquad \text{for } n \geqq 3.$$

Define a polynomial $Q_{b_n c_n}(x)$ by

$$Q_{b_n c_n}(x) := x\{2x^2-3x+1-(12/b_n c_n)\}/6.$$

α) the subcase $b_n c_n \geqq 2$. We have

$$Q_2(n) \leqq Q_{b_n c_n}(n) \leqq R'_{b_n c_n}(n).$$

But by a simple computation $Q_2(n) \geqq 2$ for $n \geqq 3$. Thus this case does not occur.

β) the subcase $b_n c_n = 1$. We have $Q_1(n) \leqq R'_{b_n c_n}(n)$ for $n \geqq 4$. But a direct calculation shows $Q_1(n) \geqq 6$ if $n \geqq 4$. Thus $n=3$.

This completes the proofs of (i), (ii) α) and the former part of β). In what follows, we shall prove the latter part of β). We assume that $b_3 c_3 = 1$. Then $b_3 = 1$ and $c_3 = f_3^*(K_X)^2 \cdot S_3 = 1$. We put $N_3 := f_{3*}(Z_3)$ and $F_3 := f_{3*}(S_3)$.

Then

$$3K_X \approx a_3 b_3 F_3 + N_3 \tag{1}$$

and

$$f_3^*(a_3 b_3 F_3) \approx a_3 b_3 S_3 + E_3' \tag{2}$$

where E_3' is an exceptional divisor for f_3. Moreover, taking f_3 in such a way that all the centers of the blowing-ups are on $\mathrm{Bs}\,|3K_X|$, we may assume

$$\mathrm{Supp}(E_3')=\mathrm{Supp}(\text{exceptional locus of } f_3)$$

Multiplying (1) with $K_X\cdot F_3$, we have

$$3F_3\cdot K_X^2=a_3b_3K_X\cdot F_3^2+K_X\cdot F_3\cdot N_3.$$

By hypothesis, we have

$$F_3\cdot K_X^2=f_3^*(K_X)^2\cdot S_3=1,$$

and

$$K_X\cdot F_3\cdot N_3\geqq 0,$$

because K_X is nef and $F_3\cdot N_3\geqq 0$ (as a 1-cycle). Thus

$$3\geqq a_3b_3K_X\cdot F_3^2. \tag{3}$$

On the other hand, applying Proposition 4 (iii), we have

$$a_3\geqq P(3)-1\geqq 4. \tag{4}$$

Moreover, since $F_3^2\geqq 0$ (as a 1-cycle) and since K_X is nef, it follows that

$$K_X\cdot F_3^2\geqq 0. \tag{5}$$

Combining (3), (4) and (5) together, we have

$$K_X\cdot F_3^2=0. \tag{6}$$

Since S_3 is a general fiber of h_3, we obtain

$$f_3^*(K_X)\cdot S_3^2=0. \tag{7}$$

Multiplying (2) by $f_3^*(a_3b_3F_3)\cdot f_3^*(K_X)$, we have

$$a_3^2b_3^2K_X\cdot F_3^2=a_3^2b_3^2S_3^2\cdot f_3^*(K_X)+2f_3^*(K_X)\cdot f_3^*(a_3b_3F_3)\cdot E_3'-f_3^*(K_X)\cdot E_3'^2.$$

Thus the equality above with (6) and (7) implies

$$f_3^*(K_X)\cdot E_3'^2=0.$$

Proposition 2 implies that there exists a positive integer p such that $\mathrm{Bs}\,|pK_X|=\emptyset$. Then a general member $T\in|pf_3^*(K_X)|$ is a nonsingular projective surface by Bertini's theorem. Let $(E_{3,i})_{i\in I_3}$ be all the prime components of E_3'. Since E_3' is exceptional for f_3, we have

$$(f_3^*(K_X)|_T\cdot E_{3,i}|_T)_T=pf_3^*(K_X)\cdot f_3^*(K_X)\cdot E_{3,i}=0 \qquad \text{for any } i\in I_3.$$

Furthermore $f_3^*(K_X)|_T$ is nef, $(f_3^*(K_X)|_T)^2=pf_3^*(K_X)^3>0$ and $(E_3'|_T)_T^2=pf_3^*(K_X)\cdot E_3'^2=0$. Thus applying Hodge's index theorem on T, we have $E_{3,i}|_T=0$, i.e., $f_3^*(K_X)\cdot E_{3,i}=0$ (as a 1-cycle of X). Therefore

$$S_3\cdot E_{3,i}\cdot f_3^*(K_X)\approx 0 \qquad \text{for any } i\in I_3.$$

We set $R_3:=f_3^*(K_X)|_{S_3}$ and $G_3:=E_3''|_{S_3}$, where E_3'' is the ramification divisor for f_3, i.e., $K_{X'}\sim f_3^*(K_X)+E_3''$. Then by the way of taking f_3, $\mathrm{Supp}(E_3')=\mathrm{Supp}(E_3'')$. Since $S_3|_{S_3}\sim 0$, it follows that $K_{S_3}\sim R_3+G_3$ where K_{S_3} is the canonical divisor of S_3. Since S_3 is a general member, we may assume that G_3 is effective. R_3 being nef and big, we conclude that S_3 is a nonsingular projective surface of general type. Blowing down the exceptional curves on S_3, we obtain the minimal model $S_{3,0}$ of S_3 with the morphism $\pi_3\colon S_3\to S_{3,0}$. Then $K_{S_3}\sim\pi_3^*(K_{3,0})+L_3$, where L_3 is the ramification divisor for π_3. Thus

$$R_3+G_3\sim\pi_3^*(K_{3,0})+L_3.$$

Note that $R_3^2=f_3^*(K_X)^2\cdot S_3=1$, $R_3\cdot G_3=f_3^*(K_X)\cdot S_3\cdot E_3''=0$. Therefore, since $\pi_3^*(K_{3,0})$ is nef and big and since L_3 is effective, numerical effectivity of R_3 implies that $R_3\cdot\pi_3^*(K_{3,0})=1$ and $R_3\cdot L_3=0$. Thus

$$R_3\cdot(R_3-\pi_3^*(K_{3,0}))=0.$$

By Hodge's index theorem, we obtain

$$0\geqq(R_3-\pi_3^*(K_{3,0}))^2=R_3^2-2R_3\cdot\pi_3^*(K_{3,0})+\pi_3^*(K_{3,0})^2\geqq 0,$$

which implies $R_3\approx\pi_3^*(K_{3,0})$ and $L_3\approx G_3$. Since L_3 and G_3 are effective divisors with $R_3\cdot L_3=R_3\cdot G_3=0$, we have $L_3=G_3$ and $\pi_3^*(K_{3,0})\sim R_3=f_3^*(K_X)|_{S_3}$. This completes the proofs of (i) and (ii).

Proof of (iii). Take a positive integer $n\geqq 3$. Assuming that $\dim W_n=2$, we shall show that $n\leqq 7$. We consider the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{h_n} & W_n' \\ {\scriptstyle f_n}\downarrow & \searrow^{g_n} & \downarrow{\scriptstyle s_n} \\ X & \xrightarrow[\Phi_{|nK_X|}]{} & W_n \end{array}$$

where f_n is a succession of blowing-ups with nonsingular centers such that $g_n:=\Phi_{|nK_X|}\circ f_n$ is a morphism, and $g_n=s_n\circ h_n$ is the Stein factorization. Let C_n be a general fiber of h_n, H_n be a hyperplane section of W_n in $\boldsymbol{P}^{P(n)-1}$, $a_n:=(H_n|_{W_n})^2$, i.e., the degree of W_n in $\boldsymbol{P}^{P(n)-1}$ and $b_n:=\deg(s_n)$. Then

$$\{h_n^*s_n^*(H_n)\}^2\approx a_nb_nC_n$$

and

$$f_n^*(nK_X)\sim h_n^*s_n^*(H_n)+Z_n,$$

where Z_n is the fixed part of the linear system $|f_n^*(nK_X)|$. Squaring the equality above and then multiplying it by $f_n^*(K_X)$, we obtain

$$n^2K_X^3=\{h_n^*s_n^*(H_n)\}^2\cdot f_n^*(K_X)+\{h_n^*s_n^*(H_n)\}\cdot Z_n\cdot f_n^*(K_X)+nf_n^*(K_X)^2\cdot Z_n.$$

Since $f_n^*(K_X)$ and $h_n^*s_n^*(H_n)$ are nef, we have

$$\{h_n^*s_n^*(H_n)\}\cdot Z_n\cdot f_n^*(K_X)\geqq 0 \quad \text{and} \quad nf_n^*(K_X)^2\cdot Z_n\geqq 0.$$

Thus

$$a_nb_nf_n^*(K_X)\cdot C_n\leqq n^2K_X^3.$$

We set $c_n:=f_n^*(K_X)\cdot C_n$. Since $f_n^*(K_X)$ is nef and big and since C_n is nef and $C_n\not\equiv 0$, we have $c_n\geqq 1$. Thus

$$a_nb_nc_n\leqq n^2K_X^3.$$

Since W_n is the image of $\Phi_{|nK_X|}$, which is a surface of degree a_n in $\boldsymbol{P}^{P(n)-1}$, we have $a_n\geqq P(n)-2$. Thus it follows that

$$S_{b_nc_n}(n):=n\{2n^2-(3+12/b_nc_n)n+1\}K_X^3/12-(2n-1)\chi(\mathcal{O}_X)\leqq 2.$$

Case 1. $\chi(\mathcal{O}_X)\geqq 1$. Then we have $K_X^3\geqq 6$ by Proposition 4 (ii).

α) the subcase $b_nc_n\geqq 2$. In this case,

$$\begin{aligned}S_{b_nc_n}(n)&\geqq n\{2n^2-(3+12/b_nc_n)n+1\}/2-(2n-1)\\&\geqq n(2n^2-9n+1)/2-(2n-1) \qquad \text{if } n\geqq 5.\end{aligned}$$

In fact, it is clear from the hypotheses that

$$\begin{aligned}&n\{2n^2-(3+12/b_nc_n)n+1\}/2-(2n-1)\\&\geqq n(2n^2-9n+1)/2-(2n-1).\end{aligned}$$

Moreover,

$$\begin{aligned}&S_{b_nc_n}(n)-[n\{2n^2-(3+12/b_nc_n)n+1\}/2-(2n-1)]\\&\geqq[n\{2n^2-(3+6)n+1\}/2-(2n-1)]\{\chi(\mathcal{O}_X)-1\}\\&\geqq 0 \qquad \text{if } n\geqq 5.\end{aligned}$$

On the other hand, we have

$$n(2n^2-9n+1)/2-(2n-1)\geqq 6 \qquad \text{if } n\geqq 5.$$

Thus $n\leqq 4$.

β) the subcase $b_nc_n=1$. In this case,

$$S_{b_nc_n}(n)\geqq n(2n^2-15n+1)/2-(2n-1) \qquad \text{if } n\geqq 5.$$

But a simple calculation shows

$$n(2n^2-15n+1)/2-(2n-1)\geqq 21 \qquad \text{if } n\geqq 8.$$

Thus $n\leqq 7$.

Case 2. $\chi(\mathcal{O}_X)\leqq 0$.

α) the subcase $b_nc_n\geqq 2$. In this case,

$$S_{b_nc_n}(n)\geqq n(2n^2-9n+1)/6 \qquad \text{if } n\geqq 5.$$

On the other hand, a direct calculation shows

$$n(2n^2-9n+1)/6 \geqq 5 \quad \text{if } n \geqq 5.$$

Thus $n \leqq 4$.

β) the subcase $b_n c_n=1$. In this case,

$$S_{b_n c_n}(n) \geqq n(2n^2-15n+1)/6 \quad \text{if } n \geqq 5.$$

But by a simple calculation

$$n(2n^2-15n+1)/6 \geqq 12 \quad \text{if } n \geqq 8.$$

Thus $n \leqq 7$.

Since $\dim W_n \geqq 2$ for $n \geqq 4$ by (i), α) and β) imply that $\dim W_n=3$ for $n \geqq 8$. This completes the proof of Theorem 7.

§ 3. Proof of the main theorem.

THEOREM 8. *Let X be a nonsingular projective 3-fold whose canonical divisor K_X is nef and big. Then*

(i) *$\Phi_{|7K_X|}$ is birational with the possible exceptions of*

a) *$\chi(\mathcal{O}_X)=0$ and $K_X^3=2$, or*

b) *$|3K_X|$ is composed of pencils, i.e., $\dim \Phi_{|3K_X|}(X)=1$,*

(ii) *$\Phi_{|nK_X|}$ is birational for $n \geqq 8$.*

PROOF. We shall show that $\Phi_{|nK_X|}$ is birational in each of the following four cases:

Case 1. $\dim W_3 \geqq 2$ and $n \geqq 8$,

Case 2. $\dim W_3 \geqq 2$, $[\chi(\mathcal{O}_X) \neq 0$ or $K_X^3 \neq 2]$, and $n=7$,

Case 3. $\dim W_3=1$, β) and $n \geqq 8$,

Case 4. $\dim W_3=1$, α) and $n \geqq 8$,

where α) and β) are the cases described in Theorem 7 (ii).

Case 1. Assuming that $\Phi_{|nK_X|}$ is not birational, we shall derive a contradiction.

We have a birational morphism $f_3: X' \to X$ such that $g_3=\Phi_{|3K_X|} \circ f_3$ is a morphism. Let H_3 be a hyperplane section of $W_3 := \Phi_{|3K_X|}(X)$ in $\boldsymbol{P}^{P(3)-1}$ and S_3 a general member of $|g_3^*(H_3)|$. Since $|g_3^*(H_3)|$ is not composed of pencils by the hypothesis $\dim W_3 \geqq 2$, S_3 is a nonsingular irreducible projective surface. We set $3K_X \sim N_3+Z_3$ where Z_3 is the fixed part of $|3K_X|$, and set

$$f_3^*(N_3) \sim S_3+E_3', \quad K_{X'} \sim f_3^*(K_X)+E_3,$$

where E_3 is the ramification divisor for f_3 and E_3' is an exceptional divisor for f_3. Moreover, we put $m := n-4$ and

$$\psi_m := \Phi_{|K_{X'}+mf_3^*(K_X)+S_3|}.$$

From the relation

$$nK_{X'} \sim \{K_{X'}+mf_3^*(K_X)+S_3\}+(m+3)E_3+f_3^*(Z_3)+E_3',$$

we infer that ψ_m is not birational, since $\Phi_{|nK_{X'}|}$ is not birational. Fix an effective divisor $D_0 \in |(m+1)f_3^*(K_X)+E_3|$, and a section $t_0 \in H^0(X', \mathcal{O}_{X'}((m+1)f_3^*(K_X)+E_3))$ which determines D_0. Then there exists a nonempty Zariski open set U of X' such that $U \cap D_0 = \emptyset$, and that for an arbitrary point $x \in U$, there exists $y \in U$ distinct from x such that $\psi_m(x)=\psi_m(y)$. We may assume that $S_3 \cap U \neq \emptyset$, since S_3 is a general member.

CLAIM 8.1. *$\psi_m|_{S_3}$ is not birational.*

Proof. Take $s \in H^0(X', \mathcal{O}_{X'}(g_3^*(H_3)))$ so that s determines S_3. For an arbitrary point $x \in S_3 \cap U$, there exists $y \in U$ distinct from x such that $\psi_m(x)=\psi_m(y)$. Since $t_0 \cdot s \in H^0(X', \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)+S_3))$, there exists $a \in \boldsymbol{C}^*$ such that $t_0(x)s(x)=a \cdot t_0(y)s(y)$. By hypotheses, we have $D_0 \cap U=\emptyset$, which implies $t_0(y) \neq 0$, and $s(x)=0$. Therefore $s(y)=0$, i.e., $y \in S \cap U$. Thus $\psi_m|_{S_3}$ is not birational.

We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)) \longrightarrow \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)+S_3)$$
$$\longrightarrow \mathcal{O}_{S_3}(K_{S_3}+mR_3) \longrightarrow 0$$

where $R_3 := f_3^*(K_X)|_{S_3}$. Proposition 1 gives

$$H^1(X', \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)))=0.$$

Thus the homomorphism

$$H^0(X', \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)+S_3)) \longrightarrow H^0(S_3, \mathcal{O}_{S_3}(K_{S_3}+mR_3))$$

is surjective, which induces $\psi_m|_{S_3}=\Phi_{|K_{S_3}+mR_3|}$.

CLAIM 8.2. *$\Phi_{|K_{S_3}+mR_3|}$ is birational.*

Proof. Since $f_3^*(K_X)$ is nef and big and since S_3 is nef and $S_3 \not\equiv 0$, we have $R_3^2=f_3^*(K_X)^2 \cdot S_3 \geqq 1$. The hypothesis $n \geqq 8$ implies $m=n-4 \geqq 4$. Therefore, by Theorem 5, it is sufficient to verify the condition (*).

We consider the blowing-up of X' at arbitrary two points x_1 and x_2 of S_3, denoted by $\psi: X'' \to X'$. Let $M_1 := \psi^{-1}(x_1)$, $M_2 := \psi^{-1}(x_2)$, S_3'' the proper transform of S_3 and $\pi_3 := \psi|_{S_3''}: S_3'' \to S_3$ the restriction of ψ to S_3''. Then π_3 is the blowing-up of S_3 at x_1 and x_2 with the exceptional divisors $L_1 := \pi_3^{-1}(x_1)=M_1 \cap S_3''$ and $L_2 := \pi_3^{-1}(x_2)=M_2 \cap S_3''$. We have

$$h^0(X'', \mathcal{O}_{X''}(m\psi^*f_3^*(K_X)))=h^0(X', \mathcal{O}_{X'}(mf_3^*(K_X)))$$
$$=h^0(X, \mathcal{O}_X(mK_X)) \geqq 14.$$

In fact, in case $K_X^3 \leqq 4$, the inequality $\chi(\mathcal{O}_X) \leqq 0$ implies that

$$h^0(X, \mathcal{O}_X(mK_X)) \geqq h^0(X, \mathcal{O}_X(4K_X))$$
$$\geqq (2\cdot 4-1)\{4(4-1)K_X^3/12\} \geqq 14.$$

In case $K_X^3 \geqq 6$, the inequality $\chi(\mathcal{O}_X) \leqq K_X^3/6$ implies that

$$h^0(X, \mathcal{O}_X(mK_X)) \geqq h^0(X, \mathcal{O}_X(4K_X))$$
$$\geqq (2\cdot 4-1)\{4(4-1)-2\}K_X^3/12 \geqq 35.$$

Thus we have

$$h^0(X'', \mathcal{O}_{X''}(m\phi^* f_3^*(K_X)-2M_1-2M_2)) \geqq 14-4-4=6,$$

i.e.,

$$H^0(X'', \mathcal{O}_{X''}(m\phi^* f_3^*(K_X)-2M_1-2M_2)) \neq 0.$$

Since

$$\mathcal{O}_{X''}(m\phi^* f_3^*(K_X)-2M_1-2M_2)|_{S_3''}$$
$$= \mathcal{O}_{S_3''}(m\pi^*(R_3)-2L_1-2L_2),$$

we obtain the natural restriction homomorphism

$$H^0(X'', \mathcal{O}_{X''}(m\phi^* f^*(K_X)-2M_1-2M_2))$$
$$\longrightarrow H^0(S_3'', \mathcal{O}_{S_3''}(m\pi^*(R_3)-2L_1-2L_2)).$$

We claim that this is not a zero homomorphism. Assume the contrary. Then we have

$$S_3'' \subset \mathrm{Bs}\,|m\phi^* f_3^*(K_X)-2M_1-2M_2|,$$

which implies $h^0(X'', \mathcal{O}_{X''}(S_3''))=1$. On the other hand, we have

$$h^0(X'', \mathcal{O}_{X''}(S_3''))=h^0(X'', \mathcal{O}_{X''}(\phi^* g_3^*(H_3)-M_1-M_2))$$
$$\geqq 5-1-1=3,$$

which leads to a contradiction. This completes the proof of Claim 8.2.

Claim 8.1 and Claim 8.2 are contradictory to each other. Thus we complete the proof in Case 1.

Case 2. We fix the notation as in Case 1. We can carry out the same argument as in Case 1 up to the proof of Claim 8.2, which we modify as follows. In this case, we have $m=n-4=3$. Since $\chi(\mathcal{O}_X)\neq 0$ or $K_X^3\neq 2$, we have

$$h^0(X'', \mathcal{O}_{X''}(3\phi^* f_3^*(K_X)))=h^0(X', \mathcal{O}_{X'}(3f_3^*(K_X)))$$
$$=h^0(X, \mathcal{O}_X(3K_X)) \geqq 10.$$

In fact, in case $K_X^3=2$, we have $\chi(\mathcal{O}_X)<0$, which implies

$$h^0(X, \mathcal{O}_X(3K_X)) \geqq (2\cdot 3-1)\{3(3-1)K_X^3/12+1\}=10.$$

In case $K_X^3=4$, we have $\chi(\mathcal{O}_X)\leqq 0$, which implies

$$h^0(X, \mathcal{O}_X(3K_X)) \geqq (2\cdot 3-1)\{3(3-1)K_X^3/12\}=10.$$

In case $K_X^3 \geqq 6$, the inequality $\chi(\mathcal{O}_X) \leqq K_X^3/6$ implies

$$h^0(X, \mathcal{O}_X(3K_X)) \geqq (2\cdot 3-1)\{3(3-1)-2\}K_X^3/12 \geqq 10.$$

Thus

$$h^0(X'', \mathcal{O}_{X''}(3\phi^* f_3^*(K_X)-2M_1-2M_2)) \geqq 10-4-4=2.$$

Moreover, we have

$$\begin{aligned} h^0(X'', \mathcal{O}_{X''}(S_3'')) &= h^0(X'', \mathcal{O}_{X''}(\phi^* g_3^*(H)-M_1-M_2)) \\ &\geqq 10-1-1=8. \end{aligned}$$

Therefore, it is sufficient to show that $R_3^2 \geqq 2$ in order to apply Theorem 5.

CLAIM 8.3. $R_3^2 \geqq 2$.

Proof. We have a priori $R_3^2 = f_3^*(K_X)^2 \cdot S_3 \geqq 1$. Assuming that $R_3^2=1$, we shall derive a contradiction. Multiplying $3K_X \sim N_3+Z_3$ by $K_X\cdot N_3$, we have

$$3K_X^2\cdot N_3 = K_X\cdot N_3^2 + K_X\cdot N_3\cdot Z_3.$$

Thus, noting that $K_X^2\cdot N_3 = f_3^*(K_X)^2\cdot S_3 = R_3^2 = 1$, we have $3=K_X\cdot N_3^2 + K_X\cdot N_3\cdot Z_3$. Since $|S_3|$ is not composed of pencils, $f_3^*(K_X)$ is nef and big, and since S_3 is nef, we have

$$\begin{aligned} K_X\cdot N_3^2 &= f_3^*(K_X)\cdot f_3^*(N_3)^2 = f_3^*(K_X)\cdot f_3^*(N_3)\cdot S_3 \\ &= f_3^*(K_X)\cdot S_3^2 + f_3^*(K_X)\cdot S_3\cdot E_3' \geqq 1. \end{aligned}$$

Moreover, $K_X\cdot N_3^2$ is even by Proposition 3 (ii), and $K_X\cdot N_3\cdot Z_3 \geqq 0$ because $N_3\cdot Z_3 \geqq 0$ as a 1-cycle. Therefore we conclude that $K_X\cdot N_3^2=2$ and $K_X\cdot N_3\cdot Z_3=1$. Since $2=K_X\cdot N_3^2 = f_3^*(K_X)\cdot S_3^2 + f_3^*(K_X)\cdot S_3\cdot E_3'$, $f_3^*(K_X)\cdot S_3^2 \geqq 1$ and since $f_3^*(K_X)\cdot S_3\cdot E_3' \geqq 0$, we have the following two cases:

(A) $f_3^*(K_X)\cdot S_3^2=1$ and $f_3^*(K_X)\cdot S_3\cdot E_3'=1$, or

(B) $f_3^*(K_X)\cdot S_3^2=2$ and $f_3^*(K_X)\cdot S_3\cdot E_3'=0$.

We consider an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X', \mathcal{O}_{X'}(f_3^*(Z_3)+E_3')) &\longrightarrow H^0(X', \mathcal{O}_{X'}(3f_3^*(K_X))) \\ &\xrightarrow{r} H^0(S_3, \mathcal{O}_{S_3}(3R_3)). \end{aligned}$$

Since $f_3^*(Z_3)+E_3'$ is the fixed part of $|3f_3^*(K_X)|$, we have

$$\dim_{\mathbf{C}}(\operatorname{Im} r) = P(3)-1 \geqq 9.$$

Subcase: $\dim g_3(S_3)=1$. In this case,

$$a_3 := g_3(S_3)\cdot H_3 \geqq P(3)-2 \geqq 8,$$

but we have $D \approx a_3 F$, where F is a general fiber of $g_3|_{S_3}$ and $D := g_3^*(H_3)|_{S_3}$. Thus

$$R_3 \cdot D \geqq a_3 \geqq 8.$$

On the other hand,

$$R_3 \cdot D = f_3^*(K_X) \cdot S_3^2 = 1 \quad \text{or} \quad = 2$$

in the case (A) or (B), respectively. This is a contradiction.

Subcase: $\dim g_3(S_3) = 2$. In this case,

$$D^2 \geqq (H_3|_{g_3(S_3)})^2 \geqq P(3) - 3 \geqq 7.$$

When (A) holds, $R_3 \cdot D = f_3^*(K_X) \cdot S_3^2 = 1$, which leads to $R_3 \cdot (D - R_3) = 0$. Thus we have by Hodge's index theorem

$$(D - R_3)^2 = D^2 - 2R_3 \cdot D + R_3^2 \leqq 0,$$

i.e., $D^2 \leqq 1$, which contradicts $D^2 \geqq 7$. When (B) holds, $R_3 \cdot D = f_3^*(K_X) \cdot S_3^2 = 2$, which leads to $R_3 \cdot (D - 2R_3) = 0$. Thus we have by Hodge's index theorem

$$(D - 2R_3)^2 = D^2 - 4R_3 \cdot D + 4R_3^2 \leqq 0,$$

i.e., $D^2 \leqq 4$, which contradicts $D^2 \geqq 7$.

This completes the proof of Claim 8.3, and thus the proof in Case 2.

Case 3. We take a birational morphism $f_3 : X' \to X$ such that $g_3 = \Phi_{|3K_X|} \circ f_3$ is a morphism. Moreover, we use the same notation as in Case 1 except that S_3 denotes a general fiber of $g_3 : X' \to W_3$. Then S_3 is a nonsingular projective surface of general type as claimed in Theorem 7 (ii) β).

Assuming that $\Phi_{|nK_X|}$ is not birational, we shall derive a contradiction. Under the assumption above, $\psi_m := \Phi_{|K_{X'} + mf_3^*(K_X) + g_3^*(H_3)|}$ is not birational as in Case 1.

CLAIM 8.4. *$\psi_m|_{S_3}$ is not birational.*

Proof. Let $(s_i)_{i \in I}$ be a base of the $\boldsymbol{C}$-vector space $H^0(X', \mathcal{O}_{X'}(g_3^*(H_3)))$. Then

$$t_0 \cdot s_i \in H^0(X', \mathcal{O}_{X'}(K_{X'} + mf_3^*(K_X) + E_3))$$

and the $t_0 \cdot s_i$ are linearly independent over $\boldsymbol{C}$. We take D_0, t_0 and U as in Case 1. For an arbitrary point $x \in S_3 \cap U$, there exists $y \in U$ distinct from x such that $\psi_m(x) = \psi_m(y)$. Thus there exists $a \in \boldsymbol{C}^*$ such that $t_0(x)s_i(x) = a \cdot t_0(y)s_i(y)$. By hypothesis, we have $D_0 \cap U = \emptyset$, which implies $t_0(y) \neq 0$. Since $g_3 = \Phi_{|g_3^*(H_3)|}$, we have $g(x) = g(y)$, i.e., $y \in g^{-1}g(x) = S_3$. Thus $\psi_m|_{S_3}$ is not birational.

Let $\pi_3 : S_3 \to S_{3,0}$ be the morphism onto the minimal model $S_{3,0}$ with the canonical divisor $K_{3,0}$ as in Theorem 7 (ii) β). Since

$$\mathcal{O}_{S_3}(\pi_3^*(K_{3,0})) \cong \mathcal{O}_{S_3}(f_3^*(K_X)|_{S_3}) \quad \text{and} \quad \mathcal{O}_{X'}(g_3^*(H_3))|_{S_3} \cong \mathcal{O}_{X'}(S_3)|_{S_3} \cong \mathcal{O}_{S_3},$$

we have an exact sequence

$$
\begin{aligned}
0 &\longrightarrow \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)+g_3^*(H_3)-S_3) \\
&\longrightarrow \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)+g_3^*(H_3)) \\
&\longrightarrow \mathcal{O}_{S_3}(K_{S_3}+m\pi_3^*(K_{3,0})) \longrightarrow 0.
\end{aligned}
$$

Moreover, since $mf_3^*(K_X)+g_3^*(H_3)-S_3$ is nef and big, Proposition 1 gives

$$H^1(X', \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)+g_3^*(H_3)-S_3))=0.$$

Thus the homomorphism

$$H^0(X', \mathcal{O}_{X'}(K_{X'}+mf_3^*(K_X)+g_3^*(H_3))) \longrightarrow H^0(S_3, \mathcal{O}_{S_3}(K_{S_3}+m\pi_3^*(K_{3,0})))$$

is surjective, which implies

$$\psi_m|_{S_3}=\Phi_{|K_{S_3}+m\pi_3^*(K_{3,0})|}=\Phi_{|(m+1)K_{S_3}|}.$$

But since $m+1=n-3\geqq 5$, $\Phi_{|(m+1)K_{S_3}|}$ is birational by Corollary 6. Thus we come to a contradiction. This completes the proof in Case 3.

Case 4. We consider the following diagram:

$$
\begin{tikzcd}
 & X' \arrow[r, "h_3"] \arrow[dl, "g_m"'] \arrow[d, "f_3"'] \arrow[dr, "g_3"] & W_3' \arrow[d, "s_3"] \\
W_m & X \arrow[l, dashed, "\Phi_{|mK_X|}"] \arrow[r, dashed, "\Phi_{|3K_X|}"'] & W_3
\end{tikzcd}
$$

where f_3 is a succession of blowing-ups with nonsingular centers such that $g_3:=\Phi_{|3K_X|}\circ f_3$ and $g_m:=\Phi_{|mK_X|}\circ f_3$ are morphisms, and $g_3=s_3\circ h_3$ is the Stein factorization. Let S_3 be a general fiber of h_3, H_3 a hyperplane section of W_3 in $\boldsymbol{P}^{P(3)-1}$, H_m a hyperplane section of W_m in $\boldsymbol{P}^{P(m)-1}$, and let S_m be a general member of $|g_m^*(H_m)|$. We set

$$a_3:=\deg_{W_3}(H_3), \qquad b_3:=\deg(s_3) \quad \text{and} \quad c_3:=f_3^*(K_X)^2\cdot S_3.$$

We put $3K_X\sim N_3+Z_3$ where Z_3 is the fixed part of $|3K_X|$, $f_3^*(N_3)\sim h_3^*s_3^*(H_3)+E_3'$, and $K_{X'}\sim f_3^*(K_X)+E_3$, where E_3 is the ramification divisor for f_3 and E_3' is an exceptional divisor for f_3. Then $h_3^*s_3^*(H_3)\approx a_3b_3S_3$. Thus

$$f_3^*(3K_X) \approx a_3b_3S_3+E_3'+f_3^*(Z_3).$$

Multiplying this by $f_3^*(K_X)^2$, we have $3K_X^3\geqq a_3b_3c_3$. Since $b_3c_3=2$ as in Theorem 7 (ii) α), $a_3\geqq P(3)-1=(2\cdot 3-1)\{3(3-1)K_X^3/12-\chi(\mathcal{O}_X)\}-1=9$, and since $K_X^3=6$, we have $a_3=9$. Therefore

$$f_3^*(3K_X) \approx 18S_3(\text{or } 9S_3)+E_3'+f_3^*(Z_3).$$

Assuming that $\Phi_{|nK_X|}$ is not birational, we shall derive a contradiction. Since $m:=n-4\geqq 4$, Theorem 7 (i) implies that $|mK_X|$ is not composed of pencils. Thus S_m is a nonsingular projective surface. We set

$$\psi_m:=\Phi_{|K_{X'}+3f_3^*(K_X)+S_m|}.$$

Since

$$K_{X'}+(m+3)f_3^*(K_X)\sim K_{X'}+3f_3^*(K_X)+S_m+Z_m$$

where Z_m is the fixed part of $|mf^*(K_X)|$, ψ_m is not birational.

CLAIM 8.5. *$\psi_m|_{S_m}$ is not birational.*

Proof. This can be done as in the former cases.

We have an exact sequence

$$0\longrightarrow \mathcal{O}_{X'}(K_{X'}+3f_3^*(K_X))\longrightarrow \mathcal{O}_{X'}(K_{X'}+3f_3^*(K_X)+S_m)$$
$$\longrightarrow \mathcal{O}_{S_m}(K_m+3R_m)\longrightarrow 0,$$

where K_m is the canonical divisor of S_m and $R_m:=f_3^*(K_X)|_{S_m}$. Proposition 1 gives

$$H^1(X', \mathcal{O}_{X'}(K_{X'}+3f_3^*(K_X)))=0.$$

Thus the homomorphism

$$H^0(X', \mathcal{O}_{X'}(K_{X'}+3f_3^*(K_X)+S_m))\longrightarrow H^0(S_m, \mathcal{O}_{S_m}(K_m+3R_m))$$

is surjective, which implies $\psi_m|_{S_m}=\Phi_{|K_m+3R_m|}$.

CLAIM 8.6. *For a general member S_m, we get $h^0(S_m, \mathcal{O}_{S_m}(3R_m))\geqq 10$ and $R_m^2\geqq 3$. Thus applying Theorem 5, we obtain that $\Phi_{|K_m+3R_m|}$ is birational.*

Proof. Since $|S_m|$ is not composed of pencils, we have $h_3(S_m)=W_3'$. Moreover, S_m and S_3 are nef. Combining these together, we obtain that $f_3^*(K_X)\cdot S_m\cdot S_3\geqq 1$. Restricting the numerical equivalence $f_3^*(3K_X)\approx 18S_3$(or $9S_3$)$+E_3'+f_3^*(Z_3)$ to S_m, we have

$$3R_m\approx 18S_3|_{S_m}(\text{or } 9S_3|_{S_m})+E_3'|_{S_m}+f_3^*(Z_3)|_{S_m}.$$

We may assume that $E_3'|_{S_m}$ and $f_3^*(Z_3)|_{S_m}$ are effective, since S_m is a general member. Thus multiplying the above by R_m, we have $3R_m^2\geqq 18$ or 9. Thus $R_m^2\geqq 3$. We have an exact sequence

$$0\longrightarrow \mathcal{O}_{X'}(3f_3^*(K_X)-S_m)\longrightarrow \mathcal{O}_{X'}(3f_3^*(K_X))\longrightarrow \mathcal{O}_{S_m}(3R_m)\longrightarrow 0,$$

which leads to the long exact cohomology sequence

$$0\longrightarrow H^0(X', \mathcal{O}_{X'}(3f_3^*(K_X)-S_m))\longrightarrow H^0(X', \mathcal{O}_{X'}(3f_3^*(K_X)))$$
$$\longrightarrow H^0(S_m, \mathcal{O}_{S_m}(3R_m)).$$

Since $|3f_3^*(K_X)|$ is composed of pencils in Case 4, and since $|S_m|$ is not composed of pencils, we have

$$H^0(X', \mathcal{O}_{X'}(3f_3^*(K_X)-S_m))=0.$$

Thus the homomorphism

$$H^0(X', \mathcal{O}_{X'}(3f_3^*(K_X))) \longrightarrow H^0(S_m, \mathcal{O}_{S_m}(3R_m))$$

is injective. Furthermore

$$\begin{aligned} h^0(X', \mathcal{O}_{X'}(3f_3^*(K_X)))&=h^0(X, \mathcal{O}_X(3K_X)) \\ &=(2\cdot 3-1)\{3(3-1)K_X^3/12-\chi(\mathcal{O}_X)\}=10. \end{aligned}$$

Thus $h^0(S_m, \mathcal{O}_{S_m}(3R_m))\geqq 10$.

Claim 8.5 and Claim 8.6 are contradictory to each other. Thus we finish the proof in Case 4, which completes the proof of the main theorem.

COROLLARY 9. *We fix the situation as in Theorem* 8. *Assume further that* $\chi(\mathcal{O}_X)<0$. *Then* $\Phi_{|nK_X|}$ *is birational for* $n\geqq 7$.

REMARK. When K_X is ample, we have the inequality $\chi(\mathcal{O}_X)\leqq -K_X^3/64<0$ (cf. Yau [**13**]).

PROOF OF COROLLARY 9. When $\dim W_3\geqq 2$, we know that $\Phi_{|nK_X|}$ is birational for $n\geqq 7$ as in Case 1 and Case 2 of the proof of Theorem 8, noting that our assumption $\chi(\mathcal{O}_X)<0$ implies the condition $\chi(\mathcal{O}_X)\neq 0$ of Case 2.

When $\dim W_3=1$, we have the two cases α) and β) as in Theorem 7. The case α) does not occur because the derived condition of this case that $\chi(\mathcal{O}_X)=1$ and $K_X^3=6$ contradicts the assumption $\chi(\mathcal{O}_X)<0$.

Therefore the remaining case to be considered is the one with $\dim W_3=1$ and β) as described in Theorem 7. Since $\dim W_3=1$ and $b_3c_3=1$, putting $n=3$ in the following formula stated in the first part of the proof of Theorem 7,

$$(2n-1)\{n(n-1)K_X^3/12-\chi(\mathcal{O}_X)\}-nK_X^3/b_nc_n\leqq 1,$$

we obtain that

$$-2-10\chi(\mathcal{O}_X)\leqq K_X^3.$$

Case: $\dim W_2\geqq 2$. We use the same notation and argument as in Case 1 of the proof of Theorem 8, replacing the number 3 there by the number 2 here and letting $m:=n-3$ in this case. We shall derive a contradiction assuming that $\Phi_{|nK_X|}$ is not birational. Under this assumption ϕ_m is not birational and we can show that $\phi_m|_{S_2}$ is not birational as in Claim 8.1. Since

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{X'}(K_{X'}+mf_2^*(K_X)) &\longrightarrow \mathcal{O}_{X'}(K_{X'}+mf_2^*(K_X)+S_2) \\ &\longrightarrow \mathcal{O}_{S_2}(K_{S_2}+mR_2) \longrightarrow 0 \end{aligned}$$

is exact and

$$H^1(X', \mathcal{O}_{X'}(K_{X'}+mf_2^*(K_X)))=0$$

by Proposition 1, we have $\phi_m|_{S_2}=\Phi_{|K_{S_2}+mR_2|}$. Therefore it is sufficient to show that $\Phi_{|K_{S_2}+mR_2|}$ is birational as in Claim 8.2. Since $m:=n-3\geqq 4$,

$$h^0(X'', \mathcal{O}_{X''}(m\phi^* f_2^*(K_X)))\geqq 14.$$

Since $8\leqq -2-10\chi(\mathcal{O}_X)\leqq K_X^3$,

$$\begin{aligned} h^0(X'', \mathcal{O}_{X''}(S_2''))&=h^0(X'', \mathcal{O}_{X''}(\phi^* g_2^*(H_2)-M_1-M_2)) \\ &\geqq h^0(X, \mathcal{O}_X(2K_X))-1-1 \\ &\geqq (2\cdot 2-1)\{2(2-1)8/12+1\}-1-1=5. \end{aligned}$$

The remaining part of the argument is just the same as in Claim 8.2, and we are done.

Case: $\dim W_2=1$. We use the notation as in the first part of the proof of Theorem 7. Putting $n=2$ in the formula

$$(2n-1)\{n(n-1)K_X^3/12-\chi(\mathcal{O}_X)\}-nK_X^3/b_n c_n\leqq 1,$$

we obtain

$$(1-4/b_2c_2)K_X^3/2-3\chi(\mathcal{O}_X)\leqq 1,$$

which implies $b_2c_2\leqq 3$ since $\chi(\mathcal{O}_X)<0$.

CLAIM 9.1. *S_2 is a nonsingular projective surface of general type, and thus letting $\pi_2: S_2\to S_{2,0}$ be the morphism onto the minimal model $S_{2,0}$ of S_2,*

$$\mathcal{O}_{S_2}(\pi_2^*(K_{2,0}))=\mathcal{O}_{S_2}(f_2^*(K_X)|_{S_2})$$

where $K_{2,0}$ is the canonical divisor of $S_{2,0}$.

Proof. We apply the argument of the proof of the latter part of β) in Theorem 7 replacing the number 3 there by the number 2 here. We will name the corresponding formulas with the same numbers. We obtain

$$2c_2\geqq a_2b_2K_X\cdot F_2^2 \tag{3}$$

and

$$a_2\geqq P(2)-1\geqq 3.$$

Since

$$K_X\cdot F_2^2\geqq 0 \tag{5}$$

and $K_X\cdot F_2^2$ is even by Proposition 3 (ii), we have

$$K_X\cdot F_2^2=0. \tag{6}$$

The remaining argument goes without any changes and we finally have the result that

$$S_2\cdot E_{2,i}\cdot f_2^*(K_X)=0 \qquad \text{for any } i\in I_2.$$

Therefore with the formula

$$K_{S_2} \sim R_2+G_2 \sim \pi_2^*(K_{2,0})+L_2,$$

the uniqueness of the Zariski decomposition implies $R_2 \sim \pi_2^*(K_{2,0})$, i.e.,

$$\mathcal{O}_{S_2}(\pi_2^*(K_{2,0}))=\mathcal{O}_{S_2}(f_2^*(K_X)|_{S_2}).$$

This completes the proof of Claim 9.1.

Now we back to the proof of Corollary 9. Note that if H_2 is a general hyperplane section, $g_2^*(H_2)$ is a disjoint union of $S_{2,j}$'s $(1 \leqq j \leqq a_2 b_2)$, each of which is of the same kind as S_2 in Claim 9.1. We use the notations $R_{2,j}$, $\pi_{2,j}$ and $K_{2,0,j}$ for $S_{2,j}$ to signify R_2, π_2 and $K_{2,0}$ for S_2. Since

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{X'}(K_{X'}+mf_2^*(K_X)) \\ &\longrightarrow \mathcal{O}_{X'}(K_{X'}+mf_2^*(K_X)+g_2^*(H_2)) \\ &\longrightarrow \textstyle\bigoplus_{j=1}^{a_2 b_2} \mathcal{O}_{S_{2,j}}(K_{2,j}+mR_{2,j}) \longrightarrow 0 \end{aligned}$$

is exact, and since Proposition 1 gives

$$H^1(X', \mathcal{O}_{X'}(K_{X'}+mf_2^*(K_X)))=0,$$

we have that

$$\begin{aligned} &H^0(X', \mathcal{O}_{X'}(K_{X'}+mf_2^*(K_X)+g_2^*(H_2))) \\ &\longrightarrow \textstyle\bigoplus_{j=1}^{a_2 b_2} H^0(S_{2,j}, \mathcal{O}_{S_{2,j}}(K_{S_{2,j}}+mR_{2,j})) \end{aligned}$$

is surjective. This means that $\Phi_{|K_{X'}+mf_2^*(K_X)+g_2^*(H_2)|}$ separates the fibers of g_2 and the components on a fiber at least on some nonempty Zariski open subset of X'. Furthermore,

$$\Phi_{|K_{S_2}+mR_{2,j}|}=\Phi_{|(m+1)K_{S_{2,j}}|}$$

since $R_{2,j}=\pi_{2,j}^*(K_{2,0,j})$ by Claim 9.1. Since $m:=n-3 \geqq 4$, $\Phi_{|(m+1)K_{S_{2,j}}|}$ is birational by Corollary 6. Thus $\Phi_{|K_{X'}+mf_2^*(K_X)+g_2^*(H_2)|}$ restricted to $S_{2,j}$ is birational, which altogether with the consideration above implies $\Phi_{|nK_X|}$ is birational for $n \geqq 7$. This completes the proof of Corollary 9.

REMARKS. (i) There is a conjecture that $\chi(\mathcal{O}_X)<0$ under the assumption about X in Theorem 8 (cf. Miyaoka [**9**]). Once this is established, with Corollary 9 we can get the result that $\Phi_{|nK_X|}$ is birational for $n \geqq 7$ under the situation of Main Theorem.

(ii) When X has only terminal singularities, and when X is Gorenstein and $\boldsymbol{Q}$-factorial, we can carry out the same argument as above taking some special resolution $f: X' \to X$ as in Corollary (2.12) of M. Reid [**11**].

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