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Extension of modifications of ample divisors on fourfolds: II

By Maria Lucia FANIA

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Introduction.

In [4] I looked at the following problem: Let A be an ample divisor on a connected four dimensional projective manifold X. Assume that the Kodaira dimension of X is non negative. Suppose that A is the blow up of a projective manifold A' with center R_g where R_g is a smooth curve of genus $\geqq 1$ which is contained in A'. Does there exist a four dimensional manifold X' such that A' lies on X' as a divisor and such that X is the blow up of X' with center R_g? The answer turned out to be positive.

It was hoped that the result would still hold true for the case when $g=0$, i.e., when $R_g \simeq \boldsymbol{P}^1$. I would like to express my sincere thank to the referee for providing a counterexample in the above case. I have included this counterexample later in this paper. Hence the main theorem has been modified to obtain the following:

THEOREM. *Let X be a connected four dimensional projective variety which is a local complete intersection with isolated singularities. Assume that the ω_X-dimension of the invertible sheaf ω_X is non negative. Let A be a smooth ample divisor on X. Assume that A is the blow up of a smooth projective threefold A' with center a smooth projective curve R_g of genus $g \geqq 0$ and let Y denote the exceptional divisor on A. Then*

(i) *if $g \geqq 0$ and $Y \not\simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ there exists a four dimensional variety X' which is a local complete intersection such that A' lies in X' as a divisor, such that X is the blow up of X' along R_g,*

(ii) *if $g=0$ and $Y \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$* (i) *is still true unless $N_{A/X,Y} = \mathcal{O}(a, 1)$ with $a \geqq 2$. In the case when $N_{A/X,Y} = \mathcal{O}(a, 1)$, $a \geqq 2$ there exists a four dimensional Cohen-Macaulay variety X' and a morphism $\phi: X \to X'$ such that:*

a) *the following diagram commutes*

$$\begin{tikzcd} D \arrow[r] \arrow[d] & X \arrow[d] \\ \boldsymbol{P}^1 \arrow[r] & X' \end{tikzcd}$$

where D *is as in* (1.1),

b) ϕ *maps* $X-D$ *biholomorphically onto* $X'-\boldsymbol{P}^1$.

I would like to remark that the above result is still true for X with non negative logarithmic Kodaira dimension by a simple modification of the proof given in this paper.

I would also like to thank the referee for his helpful comments.

Last, but not the least, I would like to thank Professor Sommese for his helpful suggestions.

§0. Background material and notations.

In this section we will give the notation and as well some of the results that will be needed. Good references are [**10**] and [**11**].

(0.1) Given a sheaf $\mathcal{S}$ of abelian groups on a topological space X, we denote the global sections of $\mathcal{S}$ over X by $\Gamma(\mathcal{S})$ or by $H^0(\mathcal{S})$.

(0.2) Given a projective variety X we denote the structure sheaf by $\mathcal{O}_X$. Given a coherent sheaf $\mathcal{S}$ on X, we let $h^i(\mathcal{S})$ or $h^i(X, \mathcal{S})$ denote $\dim H^i(X, \mathcal{S})$.

(0.3) Let X be a projective variety. Let D be an effective Cartier divisor on X. We denote by $[D]$ the line bundle associated to D. If L is a line bundle, we denote the linear system of Cartier divisors associated to L by $|L|$. If $D\in|L|$ and C is a curve in X, $L\cdot C=D\cdot C=c_1(L)[C]$. We denote by K_X the canonical bundle of X if X is a smooth projective variety.

(0.4) DEFINITION. A local complete intersection is a complex analytic space X with the following properties:

i) each irreducible component has the same dimension, say n,

ii) each point x has a neighborhood U with the property that it can be embedded in the ball in $\boldsymbol{C}^N$ so that the defining ideal is generated by exactly $N-n$ equations.

(0.5) Let X be a local complete intersection. We denote by ω_X the dualizing sheaf of X which is a locally invertible sheaf, see [**10**] for a proof.

(0.5.1) Let X be as in (0.5). Let L be a line bundle on X. We define $\kappa(L, X)$ as in [**13**]

$$\kappa(L, X)=\begin{cases}\max\limits_{m\in N(L,X)}(\dim \phi_{|mL|}(X)) & \text{if } N(L, X)\neq\emptyset\\ -\infty & \text{if } N(L, X)=\emptyset\,.\end{cases}$$

(0.6) Let X be a local complete intersection. We denote by $\kappa(\omega_X, X)$ the so called ω_X-dimension of the invertible sheaf ω_X. It is easy to see that $\kappa(X) \leqq \kappa(\omega_X, X)$ where $\kappa(X)$ denotes the Kodaira dimension of X. For a singular variety X the Kodaira dimension of X is defined to be equal to the Kodaira dimension of a non singular model of X.

Let Y be a closed subvariety of X which is a local complete intersection. By $N_{Y/X}$ we denote the normal bundle of Y in X. If no confusion arises we will denote $N_{Y/X}$ by N_Y. If $Z\subset Y\subset X$ are closed subvariety of X, by $N_{Y/X,Z}$ we denote the normal bundle of Y in X restricted to Z.

(0.7) Let $p: X\to Y$ be a morphism and let $\mathcal{S}$ be any locally free sheaf on Y of finite rank. We denote by $p^*\mathcal{S}$ the pullback of $\mathcal{S}$. If $\mathcal{S}$ is a locally free sheaf on X of finite rank we denote by $p_{(i)}\mathcal{S}$ the i-th direct image sheaf of $\mathcal{S}$ and sometimes we denote $p_{(0)}\mathcal{S}$, the zero direct image of $\mathcal{S}$, by $p_*\mathcal{S}$.

(0.8) By F_r with $r\geqq 0$ we denote the Hirzebruch surfaces which are the unique $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ with a section E satisfying $E\cdot E=-r$. If $r\geqq 1$ we denote by $\tilde{F}_r$ the normal surface obtained from F_r by blowing down E. In case $r=1$, $\tilde{F}_1=\boldsymbol{P}^2$. If L is a line bundle in F_r then L is given by $[E]^a\otimes[f]^b$ where f is a fibre in F_r and $[E]^a\otimes[f]^b$ is ample if and only if $a>0$ and $b\geqq ar+1$. And $[E]^a\otimes[f]^b$ is spanned by global sections if and only if $a\geqq 0$ and $b\geqq ar$. Given a line bundle L on $\tilde{F}_r$, the pullback of L to F_r is of the form $([E]\otimes[f]^r)^a$ for some integer a. If we think of F_r as the projective space bundle associated to $\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}(-r)$ then a base for the group $\operatorname{Pic}(F_r)$ is $\mathcal{O}_{F_r}(1)$ and $p^*\mathcal{O}_{\boldsymbol{P}^1}(1)$ where p is the projection map of F_r onto $\boldsymbol{P}^1$. Therefore a line bundle on F_r is of the form $\mathcal{O}_{F_r}(a)\otimes p^*\mathcal{O}_{\boldsymbol{P}^1}(b)$ with a, b integers. For $r=0$, i.e., on $F_0\simeq \boldsymbol{P}^1\times\boldsymbol{P}^1$ we denote a line bundle by $q^*\mathcal{O}_{\boldsymbol{P}^1}(a)\otimes p^*\mathcal{O}_{\boldsymbol{P}^1}(b)$, where q is the other projection map of F_0 onto $\boldsymbol{P}^1$. For simplicity we will use the notation $\mathcal{O}(a, b)$ to denote $q^*\mathcal{O}_{\boldsymbol{P}^1}(a)\otimes p^*\mathcal{O}_{\boldsymbol{P}^1}(b)$. See [**10**] for further details.

(0.9) PROPOSITION. *Let X be a local complete intersection with isolated singularities. Let $\mathcal{O}_{X,x}$ be the local ring at $x\in\operatorname{Sing}(X)$. We denote* $\operatorname{Spec}(\mathcal{O}_{X,x})$ *by U and* $\operatorname{Spec}(\mathcal{O}_{X,x})-\{x\}$ *by U_x. Then the group* $\operatorname{Pic}(U_x)=0$.

For a proof combine Theorem 3.13 (ii), Exp. XI and Proposition (3.5) Exp. XI in [**9**].

(0.10) KODAIRA VANISHING THEOREM. *Let X be an irreducible n-dimensional normal projective variety with isolated Cohen-Macaulay singularities. Let L be* *an ample line bundle on X. Then*

$$H^i(X, \omega_X \otimes L)=0 \qquad \textit{for } i>0$$

and if moreover X is Cohen-Macaulay

$$H^i(X, L^{-1})=0 \qquad \textit{for } i<n .$$

PROOF. The proof of this theorem is well known to experts but for the sake of completeness we will sketch the proof. Let $\pi: \tilde{X} \to X$ be a desingularization of X. Then there exists a spectral sequence with

$$E_2^{p,q}(t)=H^p(X, \pi_{(q)}\mathcal{O}_{\tilde{X}}(-tA)) \Longrightarrow H^{p+q}(\tilde{X}, \pi^*(-tA))$$

for every $t \in \boldsymbol{Z}$, where $\pi_{(q)}\mathcal{O}_{\tilde{X}}(-tA)$ is as in (0.7). Note that $\pi_{(q)}\mathcal{O}_{\tilde{X}}$ is supported at Sing(X) for $q>0$. Thus

$$E_2^{p,q}=0 \qquad \text{for } q>0,\ p>0,$$

$$\dim E_2^{p,0}=h^p(X, \mathcal{O}_X(-tA))=h^{n-p}(X, \omega_X \otimes [tA])=0 \qquad \text{for } p<n,\ \ t \gg 0 .$$

Hence for $t \gg 0$ and $q<n-1$, $\dim E_2^{0,q}(t)=h^q(\tilde{X}, \pi^*(-tA))=0$ by Ramanujam's vanishing theorem. This implies that

$$\pi_{(q)}\mathcal{O}_{\tilde{X}}=0 \qquad \text{for } q<n-1 .$$

Then $\dim E_2^{p,0}(t) \leqq h^p(X, \pi^*(-tA))=0$ for $p<n$ and for every $t>0$. In particular $H^p(X, [A]^{-1})=0$ for $p<n$. □

For generalizations of Kodaira Vanishing Theorem to singular varieties see also [**21**] Chapter VII.

§1.

(1.0) Throughout this section we assume X is a four dimensional connected projective variety which is a local complete intersection with isolated singularities. Let L be an ample line bundle on X with at least a smooth A in the linear system $|L|$. We also assume that the ω_X-dimension of the invertible sheaf ω_X is non negative, where ω_X denotes the dualizing sheaf of X.

(1.1) LEMMA. *Let X, A and L be as in* (1.0). *Assume that A is the blow up of a smooth projective threefold A' with center a smooth curve R_g of genus $g \geqq 0$. Let Y be the exceptional divisor of such blow up and let f be a fibre of Y. Then there exists a divisor D on X such that:*

a) *D intersects A transversely in Y, and*

b) $Y \subset D_{\text{reg}}$.

PROOF. Note that A is a smooth divisor on X therefore there exists a smooth neighborhood U of A in X. An easy computation shows that $h^1(N_f|_U)=0$ and that $h^0(N_f|_U)>0$. Thus by Kodaira-Spencer deformation theory it follows that there exist deformations of f in U. Now let $\mathcal{H}$ be the irreducible component of the Hilbert scheme of X which contains deformations of f in Y and of f in U. Since X is projective the deformations of f in U give rise to deformations of f in X. Let D be the closure of the union of all the deformations of f in X. The same argument as in [**4**], (1.1) shows that $\dim D=3$ and that D meets A transversely in Y and $Y\subset D_{\mathrm{reg}}$. $\square$

(1.2) LEMMA. *The divisor D in* (1.1) *is a normal Cartier divisor.*

PROOF. Note that $D-\mathrm{Sing}(X)$ is a Cartier divisor on $X-\mathrm{Sing}(X)$. Assume that D goes through $x\in\mathrm{Sing}(X)$. Let $\mathcal{O}_{X,x}$ be the local ring at x. We denote $\mathrm{Spec}(\mathcal{O}_{X,x})$ by U and $\mathrm{Spec}(\mathcal{O}_{X,x})-\{x\}$ by U_x. By (0.9), $\mathrm{Pic}(U_x)=0$, thus $[D\cap U_x]=\mathcal{O}_U$. Hence $[D\cap U_x]$ extends trivially to U, i.e., $[D\cap U]=\mathcal{O}_U$. Let $s\in\Gamma(U,\mathcal{O}_U)$. Thus s is a germ of a holomorphic function at x, such that the zero locus of s is equal to $D\cap U$. Therefore D is defined, locally, by a single function which means that D is a Cartier divisor. Hence D is a local complete intersection. Moreover by (1.1), D intersects A transversely in Y and Y is smooth. Thus $\mathrm{Sing}(D)\subset D-A$. But A is an ample divisor on a variety of dimension 4 therefore D has isolated singular points. We now use Serre's criterion to conclude that D is normal. $\square$

(1.3) LEMMA. *Let X, A, L, Y and D be as in* (1.1) *and* (1.2). *Assume that the genus of R_g is zero. Then* $\mathrm{Pic}(D)\simeq\mathrm{Pic}(Y)$.

PROOF. The map $\mathrm{Pic}(D)\to\mathrm{Pic}(Y)$ is injective and the cokernel is torsion free. A proof of this was given by H. Hamm; for further details see [**8**]. Thus $\mathrm{Pic}(D)$ is either isomorphic to $\boldsymbol{Z}$ or to $\boldsymbol{Z}\oplus\boldsymbol{Z}$. If $\mathrm{Pic}(D)\simeq\boldsymbol{Z}$ then using the fact that $Y\simeq F_r$ is ample in D and the adjunction formula it is straightforward to see that this case does not occur for $r\geqq 2$. If $r=0$ or 1, let M be an ample generator of $\mathrm{Pic}(D)$. Thus $[Y]=M^\alpha$ for some $\alpha\geqq 1$ and so $[Y]|_Y\simeq M^\alpha|_Y\simeq([E]^a\otimes[f]^b)^\alpha$ with $a>0$ and $b>ar$. Let $\mathcal{L}'$ be a very ample line bundle on A'. Note $p^*\mathcal{L}'$ extends to a unique line bundle $\mathcal{L}$ on X with $\mathcal{L}_D|_Y=p^*(\mathcal{L}'|_{P^1})$. Moreover $p^*(\mathcal{L}'|_{P^1})=n[f]$ for some integer n and $\mathcal{L}_D=M^\beta$ for some β. Therefore $\mathcal{L}_D|_Y=([E]^a\otimes[f]^b)^\beta$ and by the above $n[f]\simeq([E]^a\otimes[f]^b)^\beta$. Such an isomorphism is impossible. Thus $\mathrm{Pic}(D)\simeq\boldsymbol{Z}\oplus\boldsymbol{Z}\simeq\mathrm{Pic}(Y)$. $\square$

(1.4) LEMMA. *Let X, A, L, Y and D be as in* (1.1) *and* (1.2). *Let $p: Y\to R_g$ be the restriction of the blow up map $p: A\to A'$. Then p extends to a holomorphic map $\tilde{p}$ from D to R_g but for the case $Y\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$ and $L|_Y=\mathcal{O}(a,1)$ with $a>0$, where $\mathcal{O}(a,1)$ is as in* (0.8). *Note that $p^*\mathcal{O}(1)$ is denoted by $\mathcal{O}(0,1)$ in the above exceptional case.*

PROOF. If $g \geqq 1$, then as in [4], (1.2) we see that the map p extends to a holomorphic map $\tilde{p}: D \to R_g$.

If $g=0$ then by (1.3) $\mathrm{Pic}(D) \simeq \mathrm{Pic}(Y)$. Consider in $\boldsymbol{P}^1$ the line bundle $\mathcal{O}_{P^1}(1)$. Let $\tilde{\mathcal{I}}$ be the unique extension of $p^*\mathcal{O}_{P^1}(1)$ to D. Since the image of the map associated to the linear system $|p^*\mathcal{O}_{P^1}(1)|$ is $\boldsymbol{P}^1$ we can consider such a map as the map p without loss of generality. If the sections of $p^*\mathcal{O}_{P^1}(1)$ extend to D as sections of $\tilde{\mathcal{I}}$ then the map p extends to a map $\tilde{p}: D \to \boldsymbol{P}^1$. To show that the sections extend, it is enough to prove that

$$(*) \qquad H^1(Y, (\tilde{\mathcal{I}} \otimes [Y]^{-t})|_Y)=0 \qquad \text{for all } t>0,$$

see [18] or [6]. Since the divisor Y is ample on D we have that $[Y]|_Y = \mathcal{O}_Y(a) \otimes p^*\mathcal{O}_{P^1}(b)$ with $a>0$ and $b>ar$. Thus

$$H^1(Y, \tilde{\mathcal{I}}_Y \otimes [Y]|_Y{}^{-t}) = H^1(Y, \mathcal{O}_Y(-ta) \otimes p^*{}_{P^1}(1-bt)).$$

It is an easy check to verify that the hypothesis of the Ramanujam's vanishing theorem for the divisor $\mathcal{O}_Y(ta) \otimes p^*\mathcal{O}_{P^1}(tb-1)$ are satisfied except for the case where $Y = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $[Y]|_Y = \mathcal{O}(a, 1)$ and $t=1$. Therefore $(*)$ follows from [16]. Thus the map p extends to D except for the case when $Y = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $[Y]|_Y = \mathcal{O}(a, 1)$ with $a>0$. In the latter case, as we will see in (1.5), using the adjunction process we will be able to get a holomorphic map defined on D which, although is not an extension along the ruling of Y defined by p, is an extension along the "other" ruling. We denote by $q: Y \to \boldsymbol{P}^1$ the map that defines the "other" ruling, see (0.8). □

(1.5) LEMMA. *Let X, A, L, Y and D be as in* (1.1) *and* (1.2). *Assume that $Y \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $N_{A/X,Y} = \mathcal{O}(a, 1)$ with $a \geqq 2$. Then the map q above extends to a holomorphic map $\phi: D \to \phi(D)$ with $\phi(D) \simeq \boldsymbol{P}^1$.*

PROOF. Let us denote by $[Y]_D$ the line bundle on D associated to the divisor Y. For convenience we call it M. If we tensor the residue sequence for Y with M^2 we have

$$0 \longrightarrow \omega_D \otimes M^2 \longrightarrow \omega_D \otimes M^3 \longrightarrow (\omega_D \otimes M^3)|_Y \longrightarrow 0.$$

Note that

$$(\omega_D \otimes M^3)|_Y = (\omega_D \otimes M)|_Y \otimes M^2|_Y = \mathcal{O}(-2, -2) \otimes \mathcal{O}(2a, 2) = \mathcal{O}(2a-2, 0)$$

is spanned by global sections. Moreover $H^1(D, \omega_D \otimes M^2)=0$ by Kodaira vanishing theorem, see (0.10). Therefore

$$(*) \qquad \Gamma(D, \omega_D \otimes M^3) \longrightarrow \Gamma(Y, (\omega_D \otimes M^3)|_Y) \longrightarrow 0.$$

Note that $(\omega_D \otimes M^3)|_Y = \mathcal{O}(2a-2, 0) = q^*\mathcal{O}(2a-2)$, where q is as in (1.4). Thus the rational map $\tilde{\phi}$ defined by $|(\omega_D \otimes M^3)|_Y|$ is $q: Y \to \boldsymbol{P}^1$ followed by the $(2a-2)$-

fold Veronese embedding of $\boldsymbol{P}^1$. By (*) it is clear that the above map is the restriction of ϕ, the rational map defined by $|\omega_D \otimes M^3|$. Then from [**6**], (2.7) it follows that the map ϕ is a morphism and that $\phi(D)=\tilde{\phi}(Y)=\boldsymbol{P}^1$. □

It should be noted that in the case $g=0$ and $Y \simeq F_r$ with $r \geqq 2$, Badescu's result [**2**] for smooth threefolds can be carried over for local complete intersections, but we prefer to give a much easier proof.

(1.6) LEMMA. *The triples $(D, R_g, \tilde{p})$, where $g \geqq 0$, and $(D, \boldsymbol{P}^1, \phi)$ are $\boldsymbol{P}^2$-bundles.*

PROOF. As in [**4**], (1.3) the fibres F of $\tilde{p}$ are smooth and isomorphic either to F_r with $r \geqq 0$ or to $\boldsymbol{P}^2$. Assume that $F \simeq F_r$. We will distinguish two cases:

1) Sing(X) is not contained in D. Since deformation theory is a local theory and X is a projective variety we can argue as in [**4**], (1.4) to show that $F \simeq F_r$ does not occur.

2) Sing(X) is contained in D. If $F \simeq F_r$ does not contain any $x \in$ Sing(X) then as in 1) we see that $F \simeq F_r$ cannot occur. Assume that there exists $x \in$ Sing(X) with $x \in F_r$. We consider the deformations of the fibre f of F_r which misses the singular point x. Again as in 1), $F \simeq F_r$ does not occur.

Thus $F \simeq \boldsymbol{P}^2$. Moreover an easy numerical computation shows that the restriction of the line bundle L to $\boldsymbol{P}^2$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^2}(1)$. We now note that the map $\tilde{p}: D \to R_g$ is flat and that its fibres are smooth. Moreover the line bundle $L|_D$ in D is such that its restriction to the fibres of $\tilde{p}$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^2}(1)$. Hence by Hironaka's theorem $\tilde{p}: D \to R_g$ is a $\boldsymbol{P}^2$-bundle, see [**12**].

As for the triple $(D, \boldsymbol{P}^1, \phi)$ we let F be the generic fibre of ϕ. Since F is smooth and $\omega_D|_F \simeq K_F$ we get that $(K_F \otimes M_F{}^3) \simeq \mathcal{O}_F$. Therefore by Kobayashi-Ochiai theorem it follows that $F \simeq \boldsymbol{P}^2$ and $M_F \simeq \mathcal{O}_F(1)$.

If F is singular then as in [**4**] (1.3) F is either F_r with $r \geqq 0$ or $\tilde{F}_r$ with $r \geqq 1$. Assume that $F = \tilde{F}_r$. We note that Y intersects F transversely in h, where h is a fibre of $\tilde{\phi}: Y \to \boldsymbol{P}^1$. Moreover $h\,(\simeq \boldsymbol{P}^1)$ is ample in F thus $h = (E+rf)^\alpha$ for some integer $\alpha>0$. Since $F \cap Y = h$ we get that $N_{h/F} = N_{Y/D,h}$. Moreover $N_{h/F} = \mathcal{O}_h(r\alpha^2)$ and $N_{Y/D} = M|_Y = \mathcal{O}(a, 1)$. Thus $\mathcal{O}_h(r\alpha^2) = \mathcal{O}_h(1)$, i.e., $r\alpha^2 = 1$ from which it follows that $r=1$ and $\alpha=1$, i.e., $F = \boldsymbol{P}^2$.

Note that the map ϕ is flat and its generic fibre $F \simeq \boldsymbol{P}^2$ and there exists a line bundle on D whose restriction to F is the hyperplane bundle. Moreover all the fibres of ϕ are smooth. Hence by Hironaka's theorem $\phi: D \to \boldsymbol{P}^1$ is a $\boldsymbol{P}^2$-bundle, see [**12**]. □

The following example that was pointed out to me by the referee shows that the map $p: Y \to R_g$ does not always extend in the case $g=0$ and $Y \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

(1.7) EXAMPLE. Let M be any projective smooth fourfold and let $x \in M$ be a point in M. Let M_1 be the blow up of M with center x and let E_1 be the exceptional divisor over x. Take a line l in $E_1 \simeq \boldsymbol{P}^3$. Let X be the blow up of M_1 with center l and let E_2 be the resulting exceptional divisor. Denote by D the proper transform of E_1 on X. Let H be a sufficiently ample line bundle on M and consider the linear system $\Lambda = |H_X - 2E_1|_X - E_2|$. By $E_1|_X$ we denote the total transform, so $E_1|_X = D + E_2$. It can be easily seen that the base locus of Λ is empty and $[\Lambda]$ is ample. So any general member A of Λ is an ample smooth divisor on X. Set $Y = D \cap A$. Then via the map $D \to E_1 \simeq \boldsymbol{P}^3$, Y is mapped isomorphically onto a smooth quadric containing l. So $Y \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, l being a fibre. The normal bundle $N_{Y/A}$ is $[D]_Y = [E_1 - E_2]$, so of bidegree $\mathcal{O}(-1, -2)$. Hence Y can be blown down to $\boldsymbol{P}^1$ in such a way that l is mapped to a point. However, D is not blown down smoothly to a curve.

(1.8) THEOREM. *Let X be a connected four dimensional projective variety which is a local complete intersection with isolated singularities. Assume that the ω_X-dimension of the invertible sheaf ω_X is non negative. Let A be a smooth ample divisor on X. Assume that A is the blow up of a smooth projective three-fold A' with center a smooth projective curve R_g of genus $g \geqq 0$ and let Y denote the exceptional divisor on A. Then*

(i) *if $g \geqq 0$ and $Y \not\simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ there exists a four dimensional variety X' which is a local complete intersection such that A' lies in X' as a divisor, such that X is the blow up of Y' along R_g,*

(ii) *if $g=0$ and $Y \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$,* (i) *is still true unless $N_{A/X,Y} = \mathcal{O}(a, 1)$ with $a \geqq 2$. In the case when $N_{A/X,Y} = \mathcal{O}(a, 1)$, $a \geqq 2$ there exists a four dimensional Cohen-Macaulay variety X' and a morphism $\phi\colon X \to X'$ such that:*

a) *the following diagram commutes*

$$\begin{array}{ccc} D & \longrightarrow & X \\ \downarrow & & \downarrow \\ \boldsymbol{P}^1 & \longrightarrow & X' \end{array}$$

where D is as in (1.1),

b) *ϕ maps $X - D$ biholomorphically onto $X' - \boldsymbol{P}^1$.*

PROOF. (i) The divisor D in (1.1) is smooth since it is a $\boldsymbol{P}^2$-bundle over a smooth curve, R_g. Note that a smooth Cartier divisor on a local complete intersection X does not go through the singular set of X. Moreover by (1.6) D is a $\boldsymbol{P}^2$-bundle over R_g, and $[D]|_D \simeq \mathcal{O}_{\boldsymbol{P}^2}(-1)$. Thus we can smoothly blow down D, see [**15**]. Therefore there exists a four dimensional variety X' and a holomorphic map p from X to X' such that X is the blow up of X' along the curve R_g. Thus it is clear that X' is a local complete intersection.

(ii) From (1.6) we see that for $g=0$ and $Y \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ the map p extends unless $N_{A/X,Y} \simeq \mathcal{O}(a, 1)$ with $a \geqq 2$. In this case we can find a morphism $\phi: D \to \boldsymbol{P}^1$ which is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$. Thus to prove (ii) it is enough to show that $N_{D/X,F}$ is negative for all fibres F of ϕ, see [**3**].

Let h denote the fibre of $\tilde{\phi}$. If we show that $N_{D/X,h}$ is negative it will follow that $N_{D/X,F}$ is negative. In fact since $F \simeq \boldsymbol{P}^2$ then $N_{D/X,\boldsymbol{P}^2} \simeq \mathcal{O}_{\boldsymbol{P}^2}(\beta)$ for some integer β. Thus $N_{D/X,h} \simeq \mathcal{O}_h(\alpha\beta)$ since $h \in |\mathcal{O}_{\boldsymbol{P}^2}(\alpha)|$ with α being a positive integer. By assumption $N_{D/X,h} = \mathcal{O}_h(-n)$ with $n \in \boldsymbol{Z}$, $n > 0$. Thus $\alpha\beta = -n$ which implies that β is negative, i.e., $N_{D/X,F}$ is negative.

We claim that $N_{D/X,h}$ is not spanned by global sections. Assume otherwise, i.e., $N_{D/X} \cdot h \geqq 0$. Since $F \cap Y = h$ and such intersection is transverse in D it follows that $N_{h/F} = N_{Y/D,h} = \mathcal{O}(a, 1)|_h = \mathcal{O}_h(1)$. From $h \subset F \subset D$ we get

$$0 \longrightarrow N_{h/F} \longrightarrow N_{h/D} \longrightarrow N_{F/D,h} \longrightarrow 0.$$

From the long exact cohomology sequence associated to it, it follows that $h^1(h, N_{h/D}) = 0$. From $h \subset D \subset X$ we get the following sequence

$$0 \longrightarrow N_{h/D} \longrightarrow N_{h/X} \longrightarrow N_{D/X,h} \longrightarrow 0.$$

Using the long exact cohomology sequence associated to the above sequence, the fact that $N_{D/X,h}$ is spanned by global sections and $h^1(N_{h/D}) = 0$ we get that $N_{h/X}$ is spanned by global sections and that $h^1(N_{h/X}) = 0$. Since $N_{h/X}$ is spanned by global sections, using Kodaira-Spencer theory, the deformations of h in X fill out a dense subset. An easy computation shows that this is impossible since $\kappa(X) \neq -\infty$.

Clearly $\phi_{(*)}\mathcal{O}_X = \mathcal{O}_{X'}$ and $\phi_{(i)}\mathcal{O}_X = 0$ for $i > 0$. From this it follows that X' is Cohen-Macaulay, for a proof see [**5**], (0.9). □

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Maria Lucia FANIA
Dipartimento di Matematica
Università degli Studi dell'Aquila
Via Roma, 33
67100 L'Aquila
Italy