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Remarks on the fixed point algebras of product type actions on UHF-algebras

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1. Introduction.

In this note we consider a C^*-dynamical system (A, G, α) of product type action, where A is a UHF-algebra and G is a finite group. In [**5**] and [**6**], A. Kishimoto and N. J. Munch investigated properties of the C^*-dynamical system (A, G, α). One of their results is that if G is abelian, then the space of tracial states on the fixed point algebra A^G is n-simplex where the number n is the cardinality of the subgroup of G which is weakly inner in the trace representation of A. If G is a (non-abelian) finite group, the structure of ideals in A^G was investigated in [**7**] by N. Riedel. Let τ be the unique tracial state on a UHF-algebra. Since the trace τ is α-invariant, the C^*-dynamical system (A, G, α) extends to the W^*-dynamical system $(\pi_\tau(A)'', G, \tilde{\alpha})$ where π_τ is the G. N. S. representation associated with τ. We set $K=\{g\in G;\ \tilde{\alpha}_g$ is an inner automorphism of $\pi_\tau(A)''\}$. Let $\hat{K}$ be the dual object of K. Since K is a normal subgroup of G, we obtain a G-space $(G, \hat{K})$ with the action,

$$(g\pi)(k)=\pi(g^{-1}kg)$$

for $k\in K$, $g\in G$ and $\pi\in\hat{K}$. By giving an equivalence $\sim$ by $\pi\sim\rho$ $(\pi, \rho\in\hat{K})$ iff $g\pi=\rho$ for some $g\in G$, we have an orbit space $\hat{K}/\sim$ (denoted by $\hat{K}/G$).

In this note we show that the number of extremal traces on the fixed point algebra A^G is the cardinality of the orbit space $\hat{K}/G$ and we give some conditions under which A^G is a UHF-algebra.

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2. Main results.

Let A_n be a matrix factor and π_n be a unitary representation of a finite group G into A_n. We define an action α of G on a UHF-algebra $A\equiv\bigotimes_{n=1}^{\infty}A_n$ by $\alpha_g=\bigotimes_{n=1}^{\infty}\mathrm{Ad}\pi_n(g)$.

We assume throughout that the automorphisms α_g are not inner in A except $g=e$, the unit in G.

By [7] N. Riedel § 3, we may assume that the families $J(\pi_n)$ of all irreducible subrepresentations of π_n are a common invariant set of $\hat{G}$, say Ω, for any $n \geqq 2$. By [4] R. Iltis Proposition 2.7 (vii), there is a normal subgroup H of G such that the set Ω is equal to $\{\pi \in \hat{G}; \pi|_H \text{ is trivial}\}$. By the above assumption and [2] Lemma 3.5, the invariant set Ω must be the whole space of $\hat{G}$. Since $J(\pi_3) = \hat{G}$ and $\pi_1 \otimes \pi_2$ contains the trivial representation of G, we have $J(\pi_1 \otimes \pi_2 \otimes \pi_3) = \hat{G}$. After "compressing" $A_1' = A_1 \otimes A_2 \otimes A_3$, we may assume that $J(\pi_n) = \hat{G}$ for all $n \geqq 1$. Then we can show, by [7] Theorem 3.1, that the fixed point algebra A^G is simple.

Let τ be the unique tracial state on A. We set unitary representations of G,

$$W(n, m)(g) = \bigotimes_{i=n+1}^{m} \pi_i(g) \qquad (n < m)$$

for all $g \in G$. Let $W(n, m) = \sum_{\pi \in \hat{G}} \lambda(n, m)(\pi)\pi$ be the irreducible decomposition of $W(n, m)$ where $\lambda(n, m)(\pi)$ is the multiplicity of π in $W(n, m)$. Then the finite dimensional algebra $(\otimes_{i=1}^{n} A_i) \cap \{W(0, n)(g); g \in G\}'$ is isomorphic to $\sum_{\pi \in \hat{G}}^{\oplus} B_\pi^n$ where B_π^n is a non-zero factor of type $I_{\lambda(1,n)(\pi)}$ because of $J(\pi_i) = \hat{G}$ for all $i \in N$. We define a positive operator $E(n, m)_{\rho, \pi}$,

$$E(n, m)_{\rho, \pi} = \int_G \overline{\chi_\rho(g)} \chi_\pi(g) W(n, m)(g) dg$$

where χ_π is the character of G associated with π and dg is a normalized Haar measure on G. The way how to prove the main theorem is essentially due to the one adopted in [6].

LEMMA 2.1. *The partial embedding $B_\pi^n \to B_\rho^{n+1}$ $(\pi, \rho \in \hat{G})$ has multiplicity $\|A_{n+1}\| \tau(E(n, n+1)_{\rho, \pi})$ where $\|A_{n+1}\|$ is the rank on matrix factor A_{n+1}, i.e. $\|M_n(\boldsymbol{C})\| = n$.*

PROOF. Let $\pi \otimes \pi_{n+1} = \sum_{\omega \in \hat{G}}^{\oplus} \lambda(\omega)\omega$ be the irreducible decomposition of $\pi \otimes \pi_{n+1}$ where $\lambda(\omega)$ is the multiplicity of ω in $\pi \otimes \pi_{n+1}$. We denote by Tr a canonical trace on the full operator algebra $B(\mathcal{H})$. Then we obtain

$$\begin{aligned}
&\int_G \overline{\chi_\rho(g)} \chi_\pi(g) \operatorname{Tr}(\pi_{n+1}(g)) dg \\
&= \int_G \chi_\rho(g) \operatorname{Tr} \otimes \operatorname{Tr}(\pi \otimes \pi_{n+1}(g)) dg \\
&= \sum_{\omega \in \hat{G}} \int_G \overline{\chi_\rho(g)} \lambda(\omega) \chi_\omega(g) dg \\
&= \sum_{\omega \in \hat{G}} \lambda(\omega) \delta_{\rho, \omega} = \lambda(\rho)
\end{aligned}$$

where $\delta_{\rho, \omega}$ is Kronecker's delta. Since the unique trace τ on A is equal to $\otimes_{i=1}^{\infty}(\operatorname{Tr}/\|A_i\|)$, we get

$$\lambda(\rho)=\|A_{n+1}\|\tau(E(n, n+1)_{\rho,\bar{\pi}}).$$

REMARK 2.2. The partial embedding $B_{\pi}^{n}\rightarrow B_{\rho}^{m}$ $(n<m)$ has multiplicity $\|A_{n+1}\|\|A_{n+2}\|\cdots\|A_{m}\|\tau(E(n, m)_{\rho,\bar{\pi}})$.

By quite the same reason as given at the beginning of § 3 in [6], we may require that $W(n, \infty)(k)=\text{st-lim}_{m\to\infty}W(n, m)(k)$ exists for $k\in K$ and $n\in N$. The restriction $\pi|_{K}$ to K of an irreducible representation π of G is $\sum_{\omega\in\hat{K}}\beta_{\omega}\omega$ as an irreducible decomposition. Since K is a normal subgroup of G, the multiplicity β_{ω} is

$$\beta_{\omega}=\begin{cases}d_{\pi}>0, & \omega\in G\omega' \text{ for some } \omega'\in\hat{K}\\ 0, & \text{otherwise.}\end{cases}$$

We denote this orbit $G\omega'$ by $s(\pi)$.

LEMMA 2.3.

$$\lim_{m\to\infty}\tau(E(n, m)_{\rho,\bar{\pi}})=\int_{K}\overline{\chi_{\rho}(g)}\chi_{\pi}(g)\tau(W(n, \infty)(g))dg$$

$$\lim_{n\to\infty}(\lim_{m\to\infty}\tau(E(n, m)_{\rho,\bar{\pi}}))=\begin{cases}\dfrac{|K|}{|G|}d_{\rho}d_{\pi}|s(\pi)|, & s(\pi)=s(\rho)\\ 0, & \textit{otherwise}\end{cases}$$

where $|\cdot|$ *is the cardinality of a set.*

PROOF. By [6] Lemma 2.2, we have

$$\begin{aligned}&\lim_{m\to\infty}\tau(E(n, m)_{\rho,\bar{\pi}})\\&=\lim_{m\to\infty}\int_{G}\overline{\chi_{\rho}(g)}\chi_{\pi}(g)\tau\Big(\bigotimes_{i=n+1}^{m}\pi_{i}(g)\Big)dg\\&=\lim_{m\to\infty}\int_{G}\overline{\chi_{\rho}(g)}\chi_{\pi}(g)\Big(\prod_{i=n+1}^{m}\tau(\pi_{i}(g))\Big)dg\\&=\int_{K}\overline{\chi_{\rho}(g)}\chi_{\pi}(g)\tau(W(n, \infty)(g))dg.\end{aligned}$$

Since

$$\lim_{n\to\infty}\prod_{i=n}^{\infty}\tau(\pi_{i}(g))=1 \qquad \text{for } g\in K,$$

we have

$$\lim_{n\to\infty}(\lim_{m\to\infty}\tau(E(n, m)_{\rho,\bar{\pi}}))=\int_{K}\overline{\chi_{\rho}(g)}\chi_{\pi}(g)dg.$$

By the orthogonality of characters of a compact group, we obtain

$$\int_{K}\overline{\chi_{\rho}(g)}\chi_{\pi}(g)dg=\begin{cases}\dfrac{|K|}{|G|}d_{\rho}d_{\pi}|s(\pi)|, & s(\pi)=s(\rho)\\ 0, & \text{otherwise.}\end{cases}$$

Let τ' be another normalized trace on A^{G}. Then for minimal projections

F_π^n $(\pi\in\hat{G})$ in the matrix factors B_π^n, their positive values $\tau'(F_\pi^n)$ of the trace τ' are denoted by ξ_π^n. By Lemma 2.1, the vectors $\xi^n=(\xi_\pi^n)_{\pi\in\hat{G}}$ and $\xi^{n+1}=(\xi_\pi^{n+1})_{\pi\in\hat{G}}$ satisfy a relation,

$$\xi_\pi^n=\sum_{\rho\in\hat{G}}\|A_{n+1}\|\tau(E(n,\ n+1)_{\rho,\bar{\pi}})\xi_\rho^{n+1}. \tag{2.0}$$

Then by setting $\eta_\pi^n=(\prod_{i=1}^n\|A_i\|)\xi_\pi^n$, we have

$$\eta_\pi^n=\sum_{\rho\in\hat{G}}\tau(E(n,\ n+1)_{\rho,\bar{\pi}})\eta_\rho^{n+1},$$

that is,

$$\eta^n=\eta^{n+1}C(n,\ n+1)$$

where $\eta^n=(\eta_\pi^n)_{\pi\in\hat{G}}$ and the matrix $C(n,\ n+1)=(\tau(E(n,\ n+1)_{\rho,\bar{\pi}}))_{\rho,\pi\in\hat{G}}$.

REMARK 2.4. For $n<m<1$,

$$\eta^n=\eta^m C(n,\ m) \tag{2.1}$$

$$C(m,\ 1)C(n,\ m)=C(n,\ 1)$$

where the matrix $C(n,\ m)=(\tau(E(n,\ m)_{\rho,\bar{\pi}}))_{\rho,\pi\in\hat{G}}$.

We compute

$$\begin{aligned}|G|^{-1}\sum_{\pi\in\hat{G}}\dim\pi\ \eta_\pi^n&=|G|^{-1}\sum_{\pi\in\hat{G}}\dim\pi\ (\sum_{\rho\in\hat{G}}\tau(E(n,\ n+1)_{\rho,\bar{\pi}})\eta_\rho^{n+1})\\&=\sum_{\rho\in\hat{G}}(|G|^{-1}\sum_{\rho\in\hat{G}}\dim\pi\ \tau(E(n,\ n+1)_{\rho,\bar{\pi}}))\eta_\rho^{n+1}\\&=\sum_{\rho\in\hat{G}}\Big(\int_G\overline{\chi_\rho(g)}(|G|^{-1}\sum_{\pi\in\hat{G}}\dim\pi\ \chi_\pi(g))\tau(W(n,\ n+1)(g))dg\Big)\eta_\rho^{n+1}.\end{aligned}$$

Since the left regular representation of G is $\sum_{\pi\in\hat{G}}(\dim\pi)\pi$,

$$\begin{aligned}|G|^{-1}\sum_{\pi\in\hat{G}}\dim\pi\,\eta_\pi^n&=\sum_{\rho\in\hat{G}}\int_G\overline{\chi_\rho(g)}\delta_{g,e}\tau(W(n,\ n+1)(g))dg\,\eta_\rho^{n+1}\\&=\sum_{\rho\in\hat{G}}|G|^{-1}\dim\rho\ \eta_\rho^{n+1}.\end{aligned}$$

Therefore we have

$$|G|^{-1}\dim\rho\ \eta_\rho^n\leqq\sum_{\rho\in\hat{G}}|G|^{-1}\dim\rho\ \eta_\rho^n=\sum_{\rho\in\hat{G}}|G|^{-1}\dim\rho\ \eta_\rho^1$$

and

$$\sup_{\rho\in\hat{G}}|\eta_\rho^n|\leqq\sum_{\rho\in\hat{G}}\dim\rho\ \eta_\rho^1$$

for all $n\in N$. Hence we may take a subsequence $\{\eta^{n_p}\}$ of $\{\eta^n\}$ which converges to a vector $\eta=(\eta_\pi)_{\pi\in\hat{G}}$. It follows from (2.1) that

$$\lim_{n_q\to\infty}\eta^{n_p}-\eta^{n_q}=\lim_{n_q\to\infty}\eta^{n_q}(C(n_p,\ n_q)-I)$$

where I is an identity matrix. By Lemma 2.3, we get

$$0=\lim_{n_p\to\infty}(\lim_{n_q\to\infty}\eta^{n_p}-\eta^{n_q})=\eta(C-I)$$

where the matrix C is equal to $((|K|/|G|)d_\rho d_\pi|s(\pi)|\delta_{s(\pi),s(\rho)})_{\rho,\pi\in\hat{G}}$. Then the vector η satisfies a relation,

$$\eta_\pi=(|K|/(|G|)\sum_{s(\rho)=s(\pi)}d_\rho d_\pi|s(\pi)|\eta_\rho.$$

We set

$$x_{s(\pi)}=\sum_{\rho\in\hat{G},\,s(\rho)=s(\pi)}d_\rho\eta_\rho.$$

Hence we obtain a vector $(x_{s(\pi)})_{s(\pi)\in\hat{K}/G}$ such that

$$\eta_\pi=(|K|/|G|)d_\pi|s(\pi)|x_{s(\pi)}. \tag{2.2}$$

On the other hand, since $\eta^{n_p}=\eta^{n_q}C(n_p, n_q)$ $(n_p<n_q)$, $\eta^{n_p}=\lim_{n_q\to\infty}\eta^{n_p}C(n_p, n_q)$ $=\eta C(n_p, \infty)$. Therefore, for all n, we have

$$\begin{aligned}\eta^n&=\eta^{n_p}C(n, n_p)\\&=\eta C(n_p, \infty)C(n, n_p)\\&=\eta C(n, \infty).\end{aligned} \tag{2.3}$$

THEOREM 2.5. *Let (A, G, α) and K be as above. Then the number of extremal traces on the fixed point algebra A^G equals the cardinality of the orbit space $\hat{K}/G$.*

PROOF. We have already proved (2.2). For an orbit $s(\pi)\in\hat{K}/G$, we set, for a positive number x,

$$x_{s(\rho)}=\begin{cases}x, & s(\rho)=s(\pi)\\ 0, & \text{otherwise},\end{cases}$$

and we define a vector $\eta_{s(\pi)}=(d_\pi|s(\pi)|\delta_{s(\pi),s(\rho)})_{\rho\in\hat{G}}$ and

$$\eta^n=(|K|x/|G|)\eta_{s(\pi)}C(n, \infty)$$

where $C(n, \infty)=\lim_{m\to\infty}C(n, m)$. Therefore we also set

$$\xi^n=\Big(1\Big/\prod_{i=1}^n\|A_i\|\Big)\eta^n.$$

Since $C(n+1, \infty)C(n, n+1)=C(n, \infty)$ by (2.1), we get

$$\begin{aligned}\xi^n&=\Big(1\Big/\prod_{i=1}^n\|A_i\|\Big)(|K|x/|G|)\eta_{s(\pi)}C(n, \infty)\\&=\Big(1\Big/\prod_{i=1}^n\|A_i\|\Big)(|K|x/|G|)\eta_{s(\pi)}C(n+1, \infty)C(n, n+1)\end{aligned} \tag{2.4}$$

$$=\left(1\Big/\prod_{i=1}^{n}\|A_i\|\right)\eta^{n+1}C(n,\ n+1)$$

$$=\|A_{n+1}\|\xi^{n+1}C(n,\ n+1),$$

which is the relation (2.0). If $\pi_1=\sum_{\rho\in\hat{G}}\lambda(0,\ 1)(\rho)\rho$ as an irreducible decomposition, then

$$\sum_{\rho\in\hat{G}}^{\oplus}B_\rho^1=\sum_{\rho\in\hat{G}}^{\oplus}M_{\lambda(0,1)(\rho)}(\boldsymbol{C})\otimes 1_{\dim\rho}\,.$$

Since $\|A_1\|\xi^1=(x|K|/|G|)\eta_{s(\pi)}C(1,\ \infty)$ and x is an arbitrary positive number, we can decide x uniquely such that $\sum_{\rho\in\hat{G}}\xi_\rho^1\lambda(0,\ 1)(\rho)=1$. Hence for each $\sum_{\pi\in\hat{G}}B_\pi^n$, we set a trace $\tau_{s(\pi)}^n$ by

$$\tau_{s(\pi)}^n=\sum_{\rho\in\hat{G}}(\xi_\rho^n)\,\mathrm{Tr}$$

where Tr are canonical traces on $M_{\lambda(1,n)(\rho)}(\boldsymbol{C})$ for all $\rho\in\hat{G}$. Then $\{\tau_{s(\pi)}^n\}$ gives a tracial state (denoted by $\tau_{s(\pi)}$) on A^G by (2.4). By (2.2) and (2.3), the tracial states $\{\tau_{s(\pi)}\}_{s(\pi)\in\hat{K}/G}$ are extremal on A^G.

PROPOSITION 2.6. *The center of the fixed point algebra* $\{x\in\pi_\tau(A)'':\tilde{\alpha}_g(x)=x,\ g\in G\}$ *is* $|\hat{K}/G|$*-dimensional.*

PROOF. At first, we must compute $\eta=(\eta_\pi)_{\pi\in\hat{G}}$ in (2.2) for the restricted trace $\tau|_{A^G}$ of the unique trace τ to A^G. By easy computation, we have

$$\xi_\pi^n=\dim\pi\Big/\prod_{i=1}^{n}\|A_i\|$$

$$\eta_\pi^n=\dim\pi\,,$$

therefore $\eta_\pi=\dim\pi$ for all $\pi\in\hat{G}$. Then we may set $x_{s(\pi)}$ in (2.2) by

$$x_{s(\pi)}=\frac{|G|\dim\pi}{|K|d_\pi s(\pi)}$$

which is dependent only on the orbit $s(\pi)$. Hence the trace $\tau|_{A^G}$ is of the form $\sum_{s(\pi)\in\hat{K}/G}a_{s(\pi)}\tau_{s(\pi)}$, $a_{s(\pi)}>0$, $\sum_{s(\pi)\in\hat{K}/G}a_{s(\pi)}=1$. Since, by Theorem 2.5, the center of $\pi_\tau(A)''^G$ is less than $|\hat{K}/G|$-dimensional, it must be $|\hat{K}/G|$-dimensional. Note that the minimal projections of its center correspond to $\{\tau_{s(\pi)}\}_{s(\pi)\in\hat{K}/G}$.

EXAMPLE 2.7. Let S_3 be a symmetric group of three elements. It is well known that S_3 has two one-dimensional irreducible representations ι and sgn, and one two-dimensional irreducible representation π (See [3] 27.61). Let A_n be a $(n^2+n^2+2)\times(n^2+n^2+2)$-matrix factor and π_n be a representation of S_3 into A_n with $\pi_n=n^2\iota\oplus n^2\,\mathrm{sgn}\oplus\pi$. Then we have, by [3] (27.61),

$$(1/2n^2+2)\operatorname{Tr}(\pi_n(g))=\begin{cases} 1 & g=e \\ \dfrac{n^2-n^2+1\cdot 0}{2n^2+2}=0 & g=(1,2),\ (1,3) \text{ or } (2,3) \\ \dfrac{2n^2-1}{2n^2+2} & g=(1,2,3) \text{ or } (1,3,2). \end{cases}$$

Since the normal subgroup K of S_3 is $\{g\in S_3 : \sum_{n=1}^{\infty} 1-\tau(\pi_n(g))<+\infty\}$, K is the alternating subgroup $\mathfrak{A}_3$ of S_3. By an easy computation, the dual group $\hat{\mathfrak{A}}_3$ of $\mathfrak{A}_3$ consists of three points and the orbit space $\hat{\mathfrak{A}}_3/S_3$ consists of two orbits. Therefore this fixed point AF-algebra A^{S_3} is simple and it has two extremal tracial states.

REMARK 2.8. Let (A, G, α) be as in Theorem 2.5. If G is abelian, the orbit space $\hat{K}/G$ is equal to $\hat{K}$. Since $|\hat{K}|=|K|$, Theorem 4.2 in [6] follows from Theorem 2.5.

REMARK 2.9. Let (A, G, α) be as in Theorem 2.5. The fixed point algebra $\pi_\tau(A)''^G$ is a factor if and only if the automorphisms $\tilde{\alpha}_g$ are not inner in $\pi_\tau(A)''$ except $g=e$.

Next we want to get conditions under which the fixed point algebra A^G is a UHF-algebra. Let $B(l^2(G))$ be the full operator algebra on $l^2(G)$ and $|G|^{-1}\operatorname{Tr}$ be the normalized trace on $B(l^2(G))$. We define a left regular representation λ of G on $l^2(G)$ by $(\lambda(g)\xi)(h)=\xi(g^{-1}h)$ for $g, h\in G$ and $\xi\in l^2(G)$. The action α of G on $B(l^2(G))$ is defined by $\alpha_g(x)=\operatorname{Ad}\lambda(g)(x)$ for $x\in B(l^2(G))$. The infinite tensor product $\otimes_{n=1}^{\infty} B(l^2(G))$ of $B(l^2(G))$ is denoted by A_G and the tensor product type action $\otimes_{n=1}^{\infty}\alpha_g$ is by α_g^G.

LEMMA 2.10. *The fixed point algebra $(A_G)^G$ is isomorphic to A_G.*

PROOF. Only in this lemma, we use the same notations (A, G, α) for (A_G, G, α^G). By using Lemma 2.1, we compute the multiplicity of partial embedding $B_\pi^n\to B_\rho^{n+1}$ $(\pi, \rho\in\hat{G})$ as follows,

$$\begin{aligned}&\|B(l^2(G))\|\,|G|^{-1}\operatorname{Tr}(E(n, n+1)_{\rho,\bar{\pi}})\\ &=\operatorname{Tr}\Big(\int_G \overline{\chi_\rho(g)}\chi_\pi(g)\lambda(g)dg\Big)\\ &=\int_G \overline{\chi_\rho(g)}\chi_\pi(g)\operatorname{Tr}(\lambda(g))dg=\dim\pi\dim\rho\end{aligned}$$

because of $\operatorname{Tr}(\lambda(g))=|G|\delta_{g,e}$. Then the Bratteli diagram for $(A_G)^G$ is Figure 1.

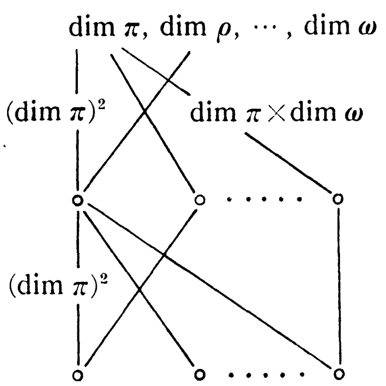

Figure 1.

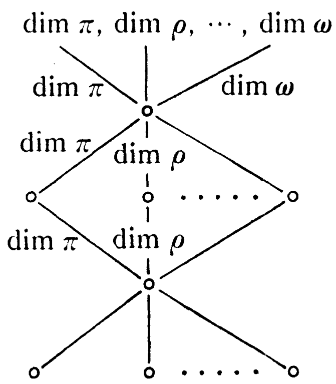

Figure 2.

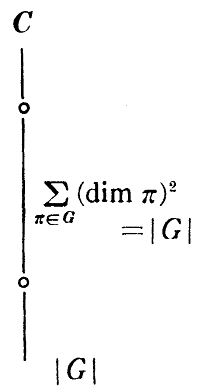

Figure 3.

We transform Figure 1 to Figure 2 and Figure 3 without changing the corresponding algebra. Since Figure 3 is a Bratteli diagram for A_G, $(A_G)^G$ is isomorphic to A_G.

THEOREM 2.11. *Let* (A, G, α) *be as in Theorem* 2.5. *Then the following statements are equivalent,*

(i) A^G *is isomorphic to* A

(ii) A^G *is a UHF-algebra*

(iii) (A, G, α) *is isomorphic to* $(A_0 \otimes A_G, G, \iota \otimes \alpha^G)$ *where* ι *is the trivial automorphism of a UHF-algebra* A_0

(iv) *there exists a subsequence* $\{n_k\}_{k=1}^{\infty}$ *of non-negative integers such that* $n_1=0$ *and* $C(n_k, n_{k+1})=(|G|^{-1}\dim\rho \dim\pi)_{\rho,\pi\in\hat{G}}$.

PROOF. By Lemma 2.10, the implications (iii)⇒(i)⇒(ii) are clear. Suppose (ii) holds. By [1] 2.5 and 2.6, there are an increasing sequence $\{B(k)\}$ of type I factor and $\{n_k\}$ of non-negative integer ($n_1=0$) such that $\sum_{\pi\in\hat{G}}^{\oplus} B_\pi^{n_k} \subset B(k) \subset \sum_{\pi\in\hat{G}}^{\oplus} B_\pi^{n_{k+1}}$. Let a_π^k (resp. b_π^k) be the multiplicity of $B_\pi^{n_k} \to B(k)$ (resp. $B(k) \to B_\pi^{n_{k+1}}$). Since the multiplicity $a_\pi^k b_\rho^k$ of $B_\pi^{n_k} \to B_\rho^{n_{k+1}}$ is $(\prod_{i=n_k+1}^{n_{k+1}} \|A_i\|)\tau(E(n_k, n_{k+1})_{\rho,\pi})$,

$$\sum_{\pi\in\hat{G}} \dim\pi\, a_\pi^k b_\rho^k \tag{2.5}$$

$$=\Big(\prod_{i=n_k+1}^{n_{k+1}} \|A_i\|\Big)\int_G \overline{\chi_\rho(g)}\Big(\sum_{\pi\in\hat{G}} \dim\pi\, \chi_\pi(g)\Big)\tau(W(n_k, n_{k+1}(g))dg$$

$$=\Big(\prod_{i=n_k+1}^{n_{k+1}} \|A_i\|\Big)\int_G \overline{\chi_\rho(g)}|G|\delta_{g,e}\tau(W(n_k, n_{k+1})(g))dg$$

$$=\Big(\prod_{i=n_k+1}^{n_{k+1}} \|A_i\|\Big)\dim\rho\,.$$

Therefore $b_\rho^k = b^k \dim\rho$ for all $\rho\in\hat{G}$ (some constant b^k). Similarly we obtain

$a_\pi^k=a^k\dim\pi$ for all $\pi\in\hat{G}$ (some constant a^k). By (2.5), we get $a^kb^k=(\prod_{i=n_k+1}^{n_{k+1}}\|A_i\|)/|G|$. The matrix $C(n_k, n_{k+1})=(\prod_{i=n_k+1}^{n_{k+1}}\|A_i\|)^{-1}(a^kb^k\dim\pi\dim\rho)_{\rho,\pi\in\hat{G}}$ is equal to $(|G|^{-1}\dim\rho\dim\pi)_{\rho,\pi\in\hat{G}}$. Suppose (iv) holds. Then we have

$$|G|^{-1}\dim\rho=\int_G\overline{\chi_\rho(g)}\tau(W(n_k, n_{k+1})(g))dg,$$

which implies that the representation $W(n_k, n_{k+1})$ of G is equivalent to $(\prod_{i=n_k+1}^{n_{k+1}}\|A_i\|)|G|^{-1}$-multiple of left regular representation λ. Therefore $\otimes_{i=n_k+1}^{n_{k+1}}A_i=A(k)\otimes B(l^2(G))$ and $\mathrm{Ad}\,W(n_k, n_{k+1})$ is transferred to $\iota\otimes\mathrm{Ad}\,\lambda$ for all k where $A(k)$ is a matrix factor. Hence $A=A_0\otimes A_G$ where $A_0=\otimes_{k=1}^\infty A(k)$ and α is transferred to $\iota\otimes\alpha_G$.

EXAMPLE 2.12. Let A_n be a $(a_n+b_n+2c_n)\times(a_n+b_n+2c_n)$ matrix factor and π_n be a representation of the symmetric group S_3 into A_n with $\pi_n=a_n\iota\oplus b_n\mathrm{sgn}\oplus c_n\pi$. If we take $a_n=n$, $b_n=(n-1)$ and $c_n=1$ for all $n\in\boldsymbol{N}$, then we have

$$(1/2n+2)\,\mathrm{Tr}(\pi_n(g))=\begin{cases}1 & g=e\\ 1/2n+2 & g=(1,2),\ (1,3)\ \text{or}\ (2,3)\\ 2n-2/2n+2 & g=(1,2,3)\ \text{or}\ (1,3,2).\end{cases}$$

Therefore the normal subgroup K for the action α induced by π_n on $A=\otimes_{n=1}^\infty A_n$ is trivial. On the other hand, since the left regular representation λ of S_3 is $\iota\oplus\mathrm{sgn}\oplus 2\pi$ and $\pi\otimes\pi=\iota\oplus\mathrm{sgn}\oplus\pi$, the tensor product representations $\otimes_{n=k}^l\pi_n$ of $\{\pi_n\}_{n=k}^l$ are not any multiple of λ. Hence the fixed point algebra A^{S_3} is not a UHF-algebra with a unique tracial state by the proof of Theorem 2.11.

EXAMPLE 2.13. If we take $a_n=b_n=n$ and $c_{2k}=1$, $c_{2k+1}=2(2k+1)$ for k, $n\in\boldsymbol{N}$, then, by an easy computation, we have $\pi_{2k}\otimes\pi_{2k+1}$ is a $2(2k+1)^2$-multiple of λ (π_{2k} is not any multiple of λ). Therefore this fixed point algebra is a UHF-algebra.

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After we typed out this manuscript, N. J. Munch informed us that Theorem 2.11 appears in [**8**].

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