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Rough isometries and the parabolicity of riemannian manifolds

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1. Introduction.

When we study non-compact complete riemannian manifolds, we observe that quasi-isometric deformations of their metrics do not alter global or qualitative properties of the manifolds: For example, two complete riemannian manifolds quasi-isometric to each other obviously have the same volume growth rate. On the other hand, for a non-compact complete riemannian manifold, "attaching finitely many handles" (see Fig. 1) also preserves such geometric invariants of the manifold; in other words, we may say that a local topological deformation of the manifold does not exert essential influences on global geometry. Suggested by these observations, we introduced the notion of rough isometry in [**10**]. A map $\varphi: X\rightarrow Y$, not necessarily continuous, between two metric spaces X and Y is called a *rough isometry* if the following two conditions are satisfied:

(i) for some $\varepsilon>0$, the ε-neighborhood of the image of φ in Y covers Y;

(ii) there are constants $a\geqq 1$ and $b\geqq 0$ such that

$$a^{-1}d(x_1, x_2)-b\leqq d(\varphi(x_1), \varphi(x_2))\leqq a d(x_1, x_2)+b \qquad \text{for all} \quad x_1, x_2\in X.$$

A metric space X is said to be *roughly isometric* to a metric space Y if there exists a rough isometry from X into Y. Evidently being roughly isometric is an equivalence relation among metric spaces. Also, since we do not impose continuity to rough isometries, there are a lot of pairs of complete riemannian manifolds which are roughly isometric to each other but are not homeomorphic; e.g., two manifolds in Fig. 1. Nevertheless, some geometric attributes of riemannian manifolds are inherited through rough isometries. In fact we proved the

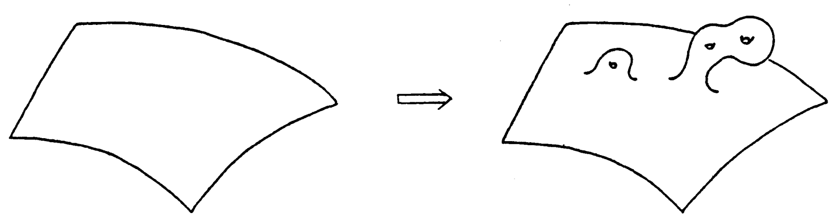

Figure 1.

following facts in [**10**].

Let X and Y be complete riemannian manifolds each of which satisfies the geometric uniformness condition that

(*) the Ricci curvature is bounded below, and the injectivity radius is positive,

and suppose that X is roughly isometric to Y. Then
(I) *X and Y have the same volume growth rate* ([**10**; Theorem 3.3]).
(II) *If* $\max\{\dim X, \dim Y\} \leqq m \leqq \infty$, *and if X satisfies the m-dimensional isoperimetric inequality* $(\mathrm{vol}\,\Omega)^{(m-1)/m} \leqq \mathrm{const}\cdot\mathrm{area}\,\partial\Omega$ *(where we adopt the convention that $(m-1)/m=1$ when $m=\infty$) for all bounded domains Ω in X with smooth boundaries $\partial\Omega$, then this m-dimensional isoperimetric inequality is also valid for Y with a suitable constant* ([**10**; Theorem 4.1]).
Moreover we proved that
(III) *If X is a complete riemannian manifold satisfying the condition* (*) *and roughly isometric to the euclidean m-space with* $m \geqq \dim X$, *then there is no positive harmonic function on X other than constants* ([**10**; Theorem 5.1]).

The aim of this article is to give the additional fact that the parabolicity is also preserved by rough isometries. By definition, a complete riemannian manifold X is said to be *parabolic* if all positive superharmonic functions on X are constant. Also there is an equivalent definition. Let $p_t(x, y)$ $(t>0, x, y \in X)$ be the minimal positive fundamental solution of the heat equation $(\partial/\partial t-\Delta)u=0$ for functions u on $(0, \infty)\times X$ (cf. [**8**], [**3**]). Then X is non-parabolic if and only if the Green function $g(x, y)=\int_0^\infty p_t(x, y)dt$ exists ([**9**]). As is well known, the euclidean n-space is non-parabolic if and only if $n \geqq 3$. Our main theorem in the present paper is

THEOREM 1. *Suppose that X and Y are complete riemannian manifolds satisfying the condition* (*) *and roughly isometric to each other. Then X is parabolic if so is Y.*

As an immediate consequence of the theorem, we get a theorem of Lyons-Sullivan [**12**]: *Suppose that X is a normal covering of a compact riemannian manifold whose deck transformation group is abelian of rank m. Then X is parabolic if and only if* $m \leqq 2$, because X is roughly isometric to the euclidean m-space unless it is compact (cf. [**10**]).

As in [**10**], we employ the discrete approximation method to prove Theorem 1; we approximate "continuous" geometry of a riemannian manifold by combinatorial geometry of a suitable discrete subset of the manifold, which we call an ε-net. By definition, a *net* means a countable set P with a family $\{N_p\}_{p \in P}$

satisfying the following two conditions: (i) for each $p \in P$, N_p is a finite subset of P; (ii) for all $p, q \in P$, $p \in N_q$ if and only if $q \in N_p$. In other words, a net is nothing but a countable locally finite 1-dimensional simplicial complex when we combine each pair of points p and q in P with $p \in N_q$ by an edge. Suppose that P is a net. A sequence $\boldsymbol{p}=(p_0, \cdots, p_l)$ of elements of P is called a *path from* p_0 *to* p_l *of length* l if $p_k \in N_{p_{k-1}}$ holds for all $k=1, \cdots, l$, and the net P is said to be *connected* if for all $p, q \in P$, there is a path from p to q. For a net P, a linear operator L acting on functions u on P is defined by

$$Lu(p)=(\# N_p)^{-1} \sum_{q \in N_p} u(q)-u(p), \qquad p \in P,$$

where, for a set S, $\# S$ denotes the cardinality of S. It is classically known that the operator L enjoys a lot of properties which the Laplace operator on the euclidean space, or more generally, the Laplace-Beltrami operator on a riemannian manifold, possesses (see, e.g., [**4**], [**5**]). As in the case of the Laplace-Beltrami operator, a function u on a net P is said to be *superharmonic* if $Lu \leqq 0$, and a net P is called *parabolic* if every positive superharmonic function on P is constant.

Now let X be a complete riemannian manifold. Recall that a subset P of X is said to be ε-separated if $d(p, q) \geqq \varepsilon$ whenever p and q are distinct points of P. A maximal ε-separated subset P in X has a canonical structure of net; in fact we set $N_p=\{q \in P: 0<d(p, q) \leqq 3\varepsilon\}$ for $p \in P$, and we call this P an *ε-net in X*. (By a technical reason, the definition of ε-nets here is slightly different from that given in [**10**]; in fact, in [**10**], we define an ε-net by $N_p=\{q \in P: 0<d(p, q) \leqq 2\varepsilon\}$.) We can easily show that an ε-net in X is connected if X is connected. Then Theorem 1 will follow from

THEOREM 2. *Suppose that X is a complete riemannian manifold satisfying the condition* (*), *and P is an ε-net in X with an arbitrary $\varepsilon>0$. Then X is parabolic if and only if P is parabolic.*

Here, we must refer to the works of Varopoulos [**13**] and Lyons-Sullivan [**12**], who established theorems similar to Theorem 2 of ours by constructing a discrete random walk approximating the brownian motion on a riemannian manifold. But their arguments, which are probabilistic rather than analytic, are different from ours in many points; especially, in the proof of Theorem 2, we employ a criterion of parabolicity, which relates the parabolicity of a riemannian manifold to the capacity of a bounded domain in it, and it will be shown by a kind of variational arguments.

We begin the proofs of Theorem 1 and Theorem 2 by showing this criterion for parabolicity in §2 (Proposition 3). Next, in §3, we develop the elementary discrete potential theory on nets, and especially establish discrete analogues of

Theorem 1 and Proposition 3. Using these results, we prove Theorem 1 and Theorem 2 in the final section. Throughout this paper, we assume that all manifolds are connected and smooth (i.e., being differentiable of class C^∞), and that all nets are connected.

2. Capacity and parabolicity.

Let X be a complete riemannian manifold, and Ω a non-empty bounded domain in X with smooth boundary. The *capacity* of Ω is defined by

$$\operatorname{cap}(\Omega)=\inf\left\{\int_X |\nabla u|^2 dx \;:\; u\in C_0^\infty(X),\ u|_\Omega=1\right\}.$$

Then we get

PROPOSITION 3. *X is non-parabolic if and only if* $\operatorname{cap}(\Omega)>0$.

PROOF (cf. [**6**; §1]). First we prove the "only if" part. Suppose that X is non-parabolic. Fix a point p in Ω and put $v(x)=\log g(p, x)$, where g denotes the Green function. Since $g(p,\cdot)$ is harmonic except at p, we have $\Delta v=-|\nabla v|^2$ on $X-\Omega$. Thus for an arbitrary $u\in C_0^\infty(X)$ with $u=1$ on Ω, we get, by Green's formula, that

$$\begin{aligned}\int_{X-\Omega} u^2|\nabla v|^2 dx &= -\int_{X-\Omega} u^2 \Delta v\, dx \\ &= -\int_{\partial\Omega}\frac{\partial v}{\partial \nu}dx + 2\int_{X-\Omega} u\langle \nabla u, \nabla v\rangle dx \\ &\leqq -\int_{\partial\Omega}\frac{\partial v}{\partial \nu}dx + 2\int_{X-\Omega}|u|\,|\nabla u|\,|\nabla v|\, dx \\ &\leqq -\int_{\partial\Omega}\frac{\partial v}{\partial \nu}dx + \int_{X-\Omega}|\nabla u|^2 dx + \int_{X-\Omega} u^2|\nabla v|^2 dx\,;\end{aligned}$$

i.e.,

$$\int_X |\nabla u|^2 dx \geqq \int_{\partial\Omega}\frac{\partial v}{\partial \nu}dx,$$

where $\partial/\partial\nu$ denotes the "inward" normal derivative on the boundary of Ω. This shows that $\operatorname{cap}(\Omega)\geqq\int_{\partial\Omega}\frac{\partial v}{\partial\nu}dx>0$.

Next we show the "if" part. Assume $\operatorname{cap}(\Omega)>0$. Take an increasing sequence of bounded domains Ω_k in X with smooth boundaries so that they cover X and each of them contains $\bar{\Omega}$. Then for each k there is a function $u_k\in C^\infty(\bar{\Omega}_k-\Omega)$ which is harmonic on $\Omega_k-\bar{\Omega}$ and satisfies the Dirichlet condition $u_k=1$ on $\partial\Omega$ and $u_k=0$ on $\partial\Omega_k$. Note that $\operatorname{cap}(\Omega)=\lim_{k\to\infty}\int_{\Omega_k-\Omega}|\nabla u_k|^2 dx$. By the Harnack inequality and the Schauder estimate, we can find a subsequence $\{u_j\}$ of $\{u_k\}$ which converges, with respect to the $C^{2,\alpha}$-norm on any compact subset in $X-\Omega$, to a positive function $u\in C^{\infty}(X-\Omega)$ harmonic on $X-\bar{\Omega}$ and with $u=1$ on $\partial\Omega$ (cf. [7]). Obviously the extension of u by $u=1$ on Ω is a positive superharmonic function on X, and therefore, to prove the non-parabolicity of X, it is sufficient to show that u is not constant. By Green's formula, we get

$$\int_{\partial\Omega}\frac{\partial u}{\partial\nu}dx=\lim_{j\to\infty}\int_{\partial\Omega}\frac{\partial u_j}{\partial\nu}dx=\lim_{j\to\infty}\int_{\Omega_j-\Omega}|\nabla u_j|^2dx=\operatorname{cap}(\Omega)>0,$$

and this implies that u is non-constant. $\square$

From the proposition above, we can immediately conclude that the parabolicity of complete riemannian manifolds is a quasi-isometry invariant; that is, *one of two complete riemannian manifolds quasi-isometric to each other is parabolic if so is the other*. This fact was proved by Lyons and Sullivan [**12**] using the Kelvin-Nevanlinna-Royden criterion for parabolicity, which says that a complete riemannian manifold is non-parabolic if and only if there is a vector field on the manifold satisfying some suitable conditions. Compared with this criterion, Proposition 3 of ours has an advantage that it characterizes the non-parabolicity by the non-vanishing capacity, which is defined as the infimum of some quantities, and does not require the existence of functions or vector fields for which some conditions are to be satisfied.

3. Discrete potential theory.

As was seen in [**10**], a net is rich in combinatorial geometry, and some geometric properties of complete riemannian manifolds satisfying the condition (*) are approximated by corresponding combinatorial properties for ε-nets in the manifolds. Theorem 2 suggests just that this is also the case with the parabolicity, and to see this, we study potential theoretic aspects of nets in this section (cf. Dodziuk [**4**]).

Let P be a net, and put

$$\nu(p)=\#N_p,\qquad\text{and}\qquad \pi(p,q)=\begin{cases}\nu(p)^{-1} & \text{if } q\in N_p\\ 0 & \text{otherwise}.\end{cases}$$

Recall that the linear operator L acting on functions u on P is defined by

$$Lu(p)=\sum_{q\in N_p}\pi(p,q)u(q)-u(p),\qquad p\in P.$$

Also, as was mentioned in the introduction, a function u on P is said to be superharmonic if $Lu\leqq 0$, and the net P is called parabolic if it does not carry positive superharmonic functions other than constants. (It is familiar to probabilists that the net P is parabolic if and only if the Markov chain on P with the transition probability π is recurrent.) For each $k=0, 1, \cdots$, define a function $\pi_k: P\times P\to \boldsymbol{R}$ inductively by

$$\pi_0(p,q)=\begin{cases}1 & \text{if } p=q\\ 0 & \text{if } p\neq q,\end{cases}\qquad \pi_{k+1}(p,q)=\sum_{r\in P}\pi_k(p,r)\pi(r,q).$$

This corresponds to the heat kernel of a riemannian manifold. Moreover, the *Green function* g of P is defined by

$$g(p,q)=\sum_{k=0}^{\infty}\pi_k(p,q),$$

if it exists. Since we have been assuming that the net P is connected, it is easy to see that $g(p,q)<\infty$ for all $p,q\in P$ if $g(p_0,q_0)<\infty$ for some $p_0,q_0\in P$. Moreover if $g<\infty$ then for each fixed $q\in P$, we have

$$Lg_q(p)=\begin{cases}-1 & \text{if } p=q\\ 0 & \text{if } p\neq q,\end{cases}\tag{1}$$

where $g_q(p)=g(p,q)$. First we prove a discrete counterpart of Itô's theorem [9].

LEMMA 4. *P is non-parabolic if and only if $g<\infty$.*

PROOF. The "if" part is trivial from (1). We prove the "only if" part. Let u be a non-constant positive superharmonic function on P, and put $f=-Lu\geqq 0$. We may assume $f\not\equiv 0$. (In fact, in the case when $Lu=0$, take a real number a between $\inf u$ and $\sup u$, and define a function u' on P by $u'(p)=u(p)$ if $u(p)\leqq a$, and $u'(p)=a$ if $u(p)>a$. This u' is a non-constant positive superharmonic function on P with $Lu'\not\equiv 0$.) Then we get

$$\begin{aligned}\sum_{j=0}^{k}\sum_{q\in P}\pi_j(p,q)f(q)&=-\sum_{j=0}^{k}\sum_{q\in P}\pi_j(p,q)\{\sum_{r\in P}\pi(q,r)u(r)-u(q)\}\\ &=-\sum_{j=0}^{k}\sum_{q\in P}\{\pi_{j+1}(p,q)-\pi_j(p,q)\}u(q)\\ &=u(p)-\sum_{q\in P}\pi_{k+1}(p,q)u(q)\\ &\leqq u(p),\end{aligned}$$

and this shows that $\sum_{q\in P}g(p,q)f(q)=\sum_{j=0}^{\infty}\sum_{q\in P}\pi_j(p,q)f(q)$ is absolutely summable. Thus we conclude $g<\infty$. □

For functions u and v on P, we define functions $\langle Du, Dv\rangle$ and $|Du|$ on P by

$$\langle Du, Dv\rangle(p)=\sum_{q\in N_p}\{u(q)-u(p)\}\{v(q)-v(p)\},$$

$$|Du|(p)=\sqrt{\langle Du, Du\rangle(p)}$$

for $p\in P$. Then we get

LEMMA 5 (Green's formula). *Let u and v be functions on P, and assume that at least one of them has a finite support. Then the following identity holds:*

$$\sum_{p\in P}(2\nu uLv+\langle Du, Dv\rangle)(p)=0.$$

The lemma follows from direct computation, and the proof is omitted.

We are now in a position to give a discrete version of Proposition 3. For a finite subset S of P, the *capacity* of S is defined by

$$\mathrm{cap}(S)=\inf\{\sum_{p\in P}|Du|^2(p) : u \text{ is a function on } P \text{ with finite support, and } u=1 \text{ on } S\}.$$

PROPOSITION 6. *Let S be a non-empty finite subset of a net P. Then P is non-parabolic if and only if* $\mathrm{cap}(S)>0$.

PROOF. Take an increasing sequence of finite subsets S_k of P so that $S\subset S_k$, $P=\bigcup S_k$, and, for each k, let u_k be a function on P which minimizes the quantity $\sum_{p\in P}|Dv|^2(p)$ among all functions v on P with $v=1$ on S and $v=0$ on $P-S_k$. Obviously $0\leqq u_k\leqq 1$, $u_k=1$ on S, $u_k=0$ on $P-S_k$, and $\mathrm{cap}(S)=\lim_{k\to\infty}\sum_{p\in P}|Du_k|^2(p)$. Moreover we can see that $Lu_k=0$ on S_k-S as follows. Let w be an arbitrary function on P such that its support lies in S_k-S, and put $u_{k,t}=u_k+tw$, $t\in(-1, 1)$. Then $\sum_{p\in P}|Du_{k,t}|^2(p)$ is minimized at $t=0$, and hence, by Lemma 5, we get

$$0=\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\sum_{p\in P}|Du_{k,t}|^2(p)=\sum_{p\in P}\langle Du_k, Dw\rangle(p)=-2\sum_{p\in S_k-S}(\nu wLu_k)(p).$$

Since this must hold for any w, we have $Lu_k=0$ on S_k-S. Now we can find a subsequence $\{u_j\}$ of $\{u_k\}$ which converges pointwise to a function u on P. It is easy to see that u is a positive superharmonic function such that $u=1$ on S and $Lu=0$ on $P-S$. In addition, by Lemma 5, we have

$$(2)\qquad -2\sum_{p\in S}\nu(p)Lu(p)=-2\lim_{j\to\infty}\sum_{p\in S}\nu(p)Lu_j(p)$$
$$=\lim_{j\to\infty}\sum_{p\in P}|Du_j|^2(p)=\mathrm{cap}(S).$$

Now the "if" part of the proposition follows from (2) directly, since u is non-constant if $\mathrm{cap}(S)>0$. We prove the "only if" part. Assume P is non-parabolic. Then, by Lemma 4, the Green function g exists. Note that it is sufficient to show that $\mathrm{cap}(S)>0$ only for S consisting only of one element of P, say q. By the choice of u and the maximum principle, we get $g(p, q)/g(q, q)\geqq u(p)$ for all $p\in P$. If u were identically equal to 1, then $g(p, q)\geqq g(q, q)$ in contradiction to (1). Hence u is non-constant, and consequently $\mathrm{cap}(\{q\})>0$ by (2). □

Finally we prove a discrete analogue of Theorem 1 by virtue of the proposition above. For a net P, denote by $\delta(p, q)$ the minimum of the lengths of paths from p to q. Obviously δ is a metric on P, which we call the *combinatorial metric* of P. Also a net P is said to be *uniform* if $\sup_{p\in P}\# N_p<\infty$.

COROLLARY 7. *Suppose that P and Q are uniform nets roughly isometric to each other* (*with respect to their combinatorial metrics*). *Then P is parabolic if so is Q.*

PROOF. Let $\varphi: P\rightarrow Q$ be a rough isometry, and suppose that P is non-parabolic. Then, for a non-empty finite subset S of P, we have $\mathrm{cap}(S)>0$ by Proposition 6. We show that $\mathrm{cap}(\varphi(S))>0$, which implies the non-parabolicity of Q. Let v be an arbitrary function on Q of finite support with $v=1$ on $\varphi(S)$, and put $u=v\circ\varphi$. Obviously $u=1$ on S, and hence, it suffices to show that $\sum_{p\in P}|Du|^2(p)\leqq c_1\sum_{q\in Q}|Dv|^2(q)$ with some constant c_1 independent of v. By the definition, there exists a constant c_2 such that for all $p,\ p'\in P$ with $\delta(p,\ p')=1$ there is a length-minimizing path $\boldsymbol{q}=(q_0,\ \cdots,\ q_l)$ in Q from $q_0=\varphi(p)$ to $q_l=\varphi(p')$ of length $l\leqq c_2$. From this we get

$$(u(p')-u(p))^2\leqq c_2\{(v(q_0)-v(q_1))^2+\cdots+(v(q_{l-1})-v(q_l))^2\}$$

and hence, with the uniformness of P, we get

$$|Du|^2(p)\leqq c_3\sum_{\delta(q,\varphi(p))<c_2}|Dv|^2(q).$$

Again, from the uniformness assumption on P and Q, we obtain a constant c_1 such that $\sum_{p\in P}|Du|^2(p)\leqq c_1\sum_{q\in Q}|Dv|^2(q)$. $\square$

4. Proofs of Theorem 1 and Theorem 2.

To prove Theorem 2 we first show

LEMMA 8 (The local Poincaré inequality). *Let X be a complete riemannian n-manifold whose Ricci curvature is bounded below by a constant $-(n-1)K^2$ ($K\geqq 0$). Then for the geodesic ball $B=B_r(p)$ in X with center at $p\in X$ and of radius $r>0$, there is a constant $\beta=\beta(n,\ K,\ r)>0$ such that*

$$\int_B|\nabla u|dx\geqq\beta\int_B|u-u^*|dx\qquad \textit{for all}\quad u\in C^\infty(\bar{B}),$$

where u^* denotes the integral mean of u over B; $u^*=(\mathrm{vol}\,B)^{-1}\int_B u\,dx$.

PROOF. It is sufficient to prove the lemma only for $u\in C^\infty(\bar{B})$ such that $u^*=0$, the critical points of u are of finite number, and that $\mathrm{vol}\{x\in B: u(x)>0\}\leqq\mathrm{vol}\{x\in B: u(x)<0\}$. Put $D_t=\{x\in B: u(x)>t\}$. Since $\mathrm{vol}\,D_t\leqq(\mathrm{vol}\,B)/2$ for $t\geqq 0$, Buser's local isoperimetric inequality [**1**; §5] implies that $\mathrm{area}(\partial D_t\cap B)\geqq c_1\mathrm{vol}\,D_t$ for $t\geqq 0$, where $c_1=c_1(n,\ K,\ r)>0$ is a constant, and hence we get

$$\begin{aligned}\int_B|\nabla u|dx&\geqq\int_{D_0}|\nabla u|dx=\int_0^\infty\mathrm{area}(\partial D_t\cap B)dt\\&\geqq c_1\int_0^\infty\mathrm{vol}\,D_t dt=c_1\int_{D_0}u\,dx=\frac{c_1}{2}\int_B|u|dx.\end{aligned}$$

$\square$

Other necessities to prove Theorem 2 are concerned with nets in manifolds. Suppose that P is an ε-net in a complete riemannian manifold X. First of all, we must note that $B_{\varepsilon/2}(p)$'s ($p\in P$) are disjoint, and that $\{B_\varepsilon(p)\}_{p\in P}$ covers X, since P is a maximal ε-separated subset of X. Moreover, if the Ricci curvature of X is bounded below, then the following hold:

(3) *for any $r>0$, there is a constant $\mu(r)$ such that $\#(P\cap B_r(x))\leqq\mu(r)$ for all $x\in X$; especially P is uniform;*

(4) *the net P with its combinatorial metric is roughly isometric to X.*

The proofs of these facts are found in [**10**; §2]. Also, before giving the proof of Theorem 2, we mention what the condition (*) implies. Suppose that X is a complete riemannian manifold satisfying (*). Then the following volume estimates for geodesic balls are known:

$$\mathrm{vol}\, B_r(p)\geqq V_0(r) \qquad \text{for } p\in X \text{ and } r\in(0, (\mathrm{inj}\, X)/2]; \tag{5}$$

$$\mathrm{vol}\, B_r(p)\leqq V_1(r) \qquad \text{for } p\in X \text{ and } r>0, \tag{6}$$

where $V_0(r)$ and $V_1(r)$ are constants independent of $p\in X$, and $\mathrm{inj}\, X$ denotes the injectivity radius of X. The first inequality (5) is a theorem of Croke [**2**: Proposition 14], and the second is a consequence of a well-known comparison theorem.

PROOF OF THEOREM 2. Let X be a complete riemannian manifold satisfying (*). Then, by (3) and (4), any two nets in X are uniform and roughly isometric to each other (recall that to be roughly isometric is an equivalence relation). Hence, from Corollary 7, one of them is parabolic if and only if so is the other, and this makes it possible to prove Theorem 2 only for an ε-net P in X with $0<\varepsilon\leqq(\mathrm{inj}\, X)/2$. First we show that P is non-parabolic if so is X. Assume that X is non-parabolic, and take a non-empty bounded domain Ω in X with smooth boundary. Then, by Proposition 3, Ω has a positive capacity. We will show that the finite subset $S=\{p\in P: B_{2\varepsilon}(p)\cap\Omega\neq\emptyset\}$ of P also has a positive capacity, which implies the non-parabolicity of P by Proposition 6. For each $p\in P$, take a function $\eta_p\in C_0^\infty(X)$ such that $0\leqq\eta_p\leqq 1$, $\eta_p=1$ on $B_\varepsilon(p)$, $\eta_p=0$ outside of $B_{2\varepsilon}(p)$, and that $|\nabla\eta_p|\leqq c_1$ on X, where c_1 is a constant independent of p, and put $\xi_p(x)=\eta_p(x)/\sum_{q\in P}\eta_q(x)$ for $x\in X$. Note that there is a constant c_2 such that $|\nabla\xi_p|\leqq c_2$. In fact, we have

$$\begin{aligned}|\nabla\xi_p| &\leqq |\nabla\eta_p|(\sum_q \eta_q)^{-1}+\eta_p\sum_q|\nabla\eta_q|(\sum_q \eta_q)^{-2}\\ &\leqq |\nabla\eta_p|+\sum_q|\nabla\eta_q|\leqq(\nu_0+2)c_1,\end{aligned}$$

where $\nu_0=\sup_{p\in P}\#N_p<\infty$. Suppose that u^* is an arbitrary function on P of finite support with $u^*=1$ on S, and set

$$u(x)=\sum_{p\in P}u^*(p)\xi_p(x), \qquad x\in X.$$

Then, at $x\in B_\varepsilon(p)$, we have

$$\nabla u(x)=\sum_{q\in N_p\cup\{p\}}u^*(q)\nabla\xi_q(x)=\sum_{q\in N_p-\{p\}}(u^*(q)-u^*(p))\nabla\xi_q(x)$$

since $\sum_{q\in N_p\cup\{p\}}\nabla\xi_q(x)=0$, and this shows that

$$|\nabla u(x)|^2\leqq c_2^2\nu_0|Du^*|^2(p).$$

Hence with (6) we get

$$\int_X|\nabla u|^2dx\leqq\sum_{p\in P}\int_{B_\varepsilon(p)}|\nabla u|^2dx\leqq c_3\sum_{p\in P}|Du^*|^2(p)$$

with $c_3=c_2^2\nu_0V_1(\varepsilon)$. On the other hand, $u=1$ on Ω, and therefore, we have $c_3\sum_{p\in P}|Du^*|^2(p)\geqq\mathrm{cap}(\Omega)>0$. This proves $\mathrm{cap}(S)>0$.

Next we show the non-parabolicity of X under the assumption that P is non-parabolic. Fix a non-empty finite subset S of P. Then, by Proposition 6, $\mathrm{cap}(S)>0$. Also let Ω be a bounded domain in X with smooth boundary such that $B_{4\varepsilon}(p)\subset\Omega$ for $p\in S$. For an arbitrary function $u\in C_0^\infty(X)$ with $u=1$ on Ω, define a function u^* on P by

$$u^*(p)=\frac{1}{\mathrm{vol}\,B_{4\varepsilon}(p)}\int_{B_{4\varepsilon}(p)}u\,dx.$$

Obviously $u^*=1$ on S. Also for $p\in P$ we have

$$V_1(4\varepsilon)\int_{B_{4\varepsilon}(p)}|\nabla u|^2dx\geqq\left\{\int_{B_{4\varepsilon}(p)}|\nabla u|\,dx\right\}^2\geqq\beta^2\left\{\int_{B_{4\varepsilon}(p)}|u(x)-u^*(p)|\,dx\right\}^2,$$

by (6), the Schwarz inequality and Lemma 8. Hence for $p, q\in P$ with $\delta(p, q)=1$, we get

$$\begin{aligned}
&2\beta^{-2}V_1(4\varepsilon)\int_{B_{7\varepsilon}(p)}|\nabla u|^2dx\\
&\geqq\beta^{-2}V_1(4\varepsilon)\left\{\int_{B_{4\varepsilon}(p)}|\nabla u|^2dx+\int_{B_{4\varepsilon}(q)}|\nabla u|^2dx\right\}\\
&\geqq\left\{\int_{B_{4\varepsilon}(p)}|u(x)-u^*(p)|\,dx\right\}^2+\left\{\int_{B_{4\varepsilon}(q)}|u(x)-u^*(q)|\,dx\right\}^2\\
&\geqq\frac{1}{2}\left\{\int_{B_{4\varepsilon}(p)}|u(x)-u^*(p)|\,dx+\int_{B_{4\varepsilon}(q)}|u(x)-u^*(q)|\,dx\right\}^2\\
&\geqq\frac{1}{2}\left\{\int_{B_{4\varepsilon}(p)\cap B_{4\varepsilon}(q)}|u^*(q)-u^*(p)|\,dx\right\}^2\\
&\geqq\frac{1}{2}V_0(\varepsilon)^2(u^*(q)-u^*(p))^2,
\end{aligned}$$

because of $B_\varepsilon(p)\subset B_{4\varepsilon}(p)\cap B_{4\varepsilon}(q)$ and (5). Especially this implies

$$c_4\int_{B_{7\varepsilon}(p)}|\nabla u|^2dx\geqq|Du^*|^2(p)$$

with a suitable constant c_4 independent of u and p. Moreover, from (3), we get

$$\mu(7\varepsilon)\int_X|\nabla u|^2dx\geqq\sum_{p\in P}\int_{B_{7\varepsilon}(p)}|\nabla u|^2dx,$$

and therefore we obtain

$$c_5\int_X|\nabla u|^2dx\geqq\sum_{p\in P}|Du^*|^2(p).$$

This shows that $c_6\mathrm{cap}(\Omega)\geqq\mathrm{cap}(S)>0$, and consequently, implies the non-parabolicity of X as Proposition 3 suggests. This completes the proof of Theorem 2. □

PROOF OF THEOREM 1. Let X and Y be complete riemannian manifolds satisfying $(*)$ and roughly isometric to each other, and take nets P and Q in X and Y, respectively. Note that, by (4), P and Q are roughly isometric to each other with respect to their combinatorial metrics, and that both of P and Q are uniform as (3) suggests. Hence, by Corollary 7, P is parabolic if so is Q. Also, as is seen in Theorem 2, P (resp. Q) is parabolic if and only if X (resp. Y) is parabolic. Thus the parabolicity of Y implies those of Q and P, and consequently that of X. This completes the proof of Theorem 1. □

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