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On the slender property of certain Boolean algebras

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§ 1. Introduction.

K. Eda [1] introduced the notion of the slender property in complete Boolean algebras. In this paper, we shall show that certain Boolean algebras have the slender property, thereby answer the question in [3].

Throughout this paper, we shall use the terminologies for forcing of set theory (see e.g. [5] or [6]). We denote by $\boldsymbol{Z}$ the group of all integers. We regard the set $\boldsymbol{Z}^{\omega}$ of all functions from ω to $\boldsymbol{Z}$ as the countable direct product of $\boldsymbol{Z}$. For each $n<\omega$, $\boldsymbol{e}_n$ stands for the element of $\boldsymbol{Z}^{\omega}$ which is defined by

$$\boldsymbol{e}_n(i)=\begin{cases}1 & \text{if } i=n\,,\\ 0 & \text{otherwise.}\end{cases}$$

"$\forall^{\infty}n<\omega\,(\cdots)$" means that "*For almost all natural numbers n,* $\cdots$.", and "$\exists^{\infty}n<\omega$ $(\cdots)$" means that "*For infinitely many natural numbers n,* $\cdots$.".

DEFINITION. A complete Boolean algebra $\boldsymbol{B}$ has the slender property, if it holds that

$$\|\forall\pi:(\boldsymbol{Z}^{\omega})^{\vee}\to\boldsymbol{Z}\text{ homo }\forall^{\infty}n<\omega\,(\pi(\boldsymbol{e}_n)=0)\|_{\boldsymbol{B}}=\mathbf{1}\,.$$

REMARK. This definition is different from the definition of the slender property in [1]. In [1], the slender property is defined by using other group theoretical terminologies, i.e., $\boldsymbol{B}$ has the slender property, if every homomorphism π from $\boldsymbol{Z}^{\omega}$ to the Boolean power $\boldsymbol{Z}^{(\boldsymbol{B})}$ is infinitely linear. But, both definitions are equivalent.

In relation to the slender property, Eda [1] proved the following theorem.

THEOREM 1. *Let* $\boldsymbol{B}$ *be a complete Boolean algebra.*

(i) *If* $\boldsymbol{B}$ *satisfies the* (ω,ω)*-weak distributive law, then* $\boldsymbol{B}$ *has the slender property.*

(ii) *If* $\||(2^{\omega})^{\vee}|=\omega\|_{\boldsymbol{B}}=\mathbf{1}$ *holds, then* $\boldsymbol{B}$ *does not have the slender property.*

Eda and Hibino asked in [3] whether the complete Boolean algebra adding a Cohen-generic real has the slender property. We shall answer positively this question in section 2. By Theorem 1 (ii), assuming the continuum hypothesis (CH), the complete Boolean algebra Col(ω, ω_1) consisting of all regular open sets in the (ω, ω_1)-collapsing poset does not have the slender property. In section 3, we shall consider that whether Col(ω, ω_1) has the slender property when CH is false. In section 4, we shall construct the complete Boolean algebra with the ω_1-chain condition (the ω_1-c.c.) which does not have the slender property.

In sections 2 and 3, we shall use the following terminologies. For each $f\in\omega^\omega$, $\tilde{f}$ denotes the function in ω^ω which is defined by

$$\tilde{f}(0)=1,$$

$$\tilde{f}(n+1)=\tilde{f}(n)(\textstyle\sum_{i\leq n}\tilde{f}(i)f(i)+n), \qquad \text{for } n<\omega.$$

For any elements $f, g\in\omega^\omega$, f *dominates* g (denoted by $g<^*f$), if $\forall^\infty n<\omega(g(n)<f(n))$. A subset F of ω^ω is said to *be cofinal in* ω^ω, if $\forall g\in\omega^\omega\exists f\in F(g<^*f)$.

§ 2. Main theorem.

For each cardinal κ, Fn(κ, 2) denotes the poset $\{p\,;\,\exists x\subset\kappa(|x|<\omega\ \&\ p:x\to 2)\}$ whose order is the inverse inclusion.

THEOREM 2. *The complete Boolean algebra* r.o.(Fn(κ, 2)) *consisting of all regular open subsets of* Fn(κ, 2) *has the slender property, for any cardinal* κ.

PROOF. Let κ be any cardinal. Set $P=\mathrm{Fn}(\kappa, 2)$. To get a contradiction, suppose that

(1) π is a P-name and $p_0\in P$,

(2) $\Vdash_P$"$\pi:(\boldsymbol{Z}^\omega)^\vee\to\boldsymbol{Z}$ homo",

(3) $p_0\Vdash_P$"$\exists^\infty n<\omega(\pi(\boldsymbol{e}_n)\neq 0)$".

Define the P-name σ by

$$\Vdash_P\text{“}\sigma:\omega\to\omega\ \&\ \forall n<\omega(\sigma(n)=|\pi(\boldsymbol{e}_n)|)\text{”}.$$

Since P satisfies the ω_1-c.c., there is $a\subset\kappa$ such that

$$|a|\leqq\omega \quad\text{and}\quad p_0\in P|a \quad\text{and}\quad \sigma \text{ is a } P|a\text{-name},$$

where $P|a=\{p\in P\,;\,\mathrm{dom}(p)\subset a\}$.

CONVENTION. For each $s\in 2^a$, "$s\Vdash\cdots$" mean that $\exists p\in P(p\subset s\ \&\ p\Vdash_P\cdots)$.

Set

$$S=\{s\in 2^a\,;\,\forall n<\omega\exists j<\omega(s\Vdash\text{“}\sigma(\check{n})=\check{j}\text{”})\ \&\ \exists^\infty n<\omega(s\Vdash\text{“}\sigma(\check{n})\neq 0\text{”})\}.$$

Then, by (2) and (3), we have that

(4) $S\cap[p_0]$ is comeager in $[p_0]$,

where $[p_0]=\{s\in 2^a\,;\,p_0\subset s\}$.

For each $f\in\omega^\omega$, take $p_f\in P|a$ and $q_f\in P|(\kappa\setminus a)$ such that

$$p_0\subset p_f \quad\text{and}\quad \exists k\in \boldsymbol{Z}\,(p_f\cup q_f \Vdash_P \text{“}\pi(\tilde{f}^{\vee})=\check{k}\text{”}).$$

Since $|\{p_f\,;\,f\in\omega^\omega\}|\leqq|P|a|\leqq\omega$, there is $\bar{p}\in P|a$ such that

$$\{f\in\omega^\omega\,;\,p_f=\bar{p}\}\text{ is cofinal in }\omega^\omega.$$

Since $[\bar{p}]\subset[p_0]$, by (4), there is $s\in S$ such that $\bar{p}\subset s$. Define $g:\omega\to\omega$ by

$$g(n)=\text{the unique } j<\omega \text{ such that } s\Vdash \text{“}\sigma(\check{n})=\check{j}\text{”}.$$

Then, since $s\in S$, it holds that

(5) $\exists^\infty n<\omega\,(g(n)\neq 0)$.

By the choice of $\bar{p}$, there is $f\in\omega^\omega$ such that

(6) $g<^* f$ and $p_f=\bar{p}$.

Take $\varphi:\kappa\to 2$ such that

$$s\subset\varphi \quad\text{and}\quad q_f\subset\varphi.$$

Set

$$H=\{h\in\boldsymbol{Z}^\omega\,;\,\exists p\in P\,\exists k\in\boldsymbol{Z}\,(p\subset\varphi\ \&\ p\Vdash_P\text{“}\pi(\check{h})=\check{k}\text{”})\},$$

and define $\theta: H\to\boldsymbol{Z}$ by

$$\theta(h)=\text{the unique } k\in\boldsymbol{Z} \text{ such that}$$
$$\exists p\in P\,(p\subset\varphi\ \&\ p\Vdash_P\text{“}\pi(\check{h})=\check{k}\text{”}).$$

Then, we have that

(7) H is a pure subgroup of $\boldsymbol{Z}^\omega$,

(8) θ is a homomorphism from H to $\boldsymbol{Z}$,

(9) $\forall n<\omega\,(\boldsymbol{e}_n\in H\ \&\ |\theta(\boldsymbol{e}_n)|=g(n))$.

By (6) and by the choice of φ, it holds that

(10) $\tilde{f}\in H$.

For each $n<\omega$, define $f_n\in\omega^\omega$ by

$$f_n(i)=\begin{cases}\tilde{f}(i) & \text{if } i\geqq n,\\ 0 & \text{otherwise}.\end{cases}$$

Since $\tilde{f}\in H$, we have that $\forall n<\omega\,(f_n\in H)$. Take $m_0<\omega$ such that

$$m_0\leqq\forall n<\omega\,(g(n)<f(n)).$$

Set $f^*=f_{m_0}$ and $m_1=\max(m_0, |\theta(f^*)|)$.

Claim 1. $m_1<\forall n<\omega\,(\theta(f_n)=0)$.

Proof of Claim 1. Let n be any natural number such that $m_1<n$. By the definition of $\tilde{f}$, it holds that $\exists x\in \boldsymbol{Z}^{\omega}\,(\tilde{f}(n)x=f_n)$. Since H is pure, we have that

$$\tilde{f}(n) \text{ divides } \theta(f_n).$$

On the other hand, since $f_n=f^*-\sum_{i=m_0}^{n-1}\tilde{f}(i)\boldsymbol{e}_i$, we have that

$$\theta(f_n)=\theta(f^*)-\sum_{i=m_0}^{n-1}\tilde{f}(i)\theta(\boldsymbol{e}_i).$$

So, it holds that

$$|\theta(f_n)|\leqq|\theta(f^*)|+\sum_{i<n}\tilde{f}(i)f(i)<\tilde{f}(n).$$

Thus, we have that $\theta(f_n)=0$. q.e.d. of Claim 1.

By Claim 1, we have that

$$\theta(\boldsymbol{e}_n)=0 \qquad \text{for } m_1<\forall n<\omega$$

This contradicts (5) and (9). This completes the proof. □

Let $\mathfrak{B}$ be the Boolean algebra of all Borel subsets of the unit interval [0, 1] and $\mathfrak{I}$ the ideal of all meager sets in $\mathfrak{B}$. It is known [7] that the quotient algebra $\mathfrak{B}/\mathfrak{I}$ is isomorphic to r.o.(Fn(ω, 2)). So, we have the following corollary.

Corollary 1. *The complete Boolean algebra* $\mathfrak{B}/\mathfrak{I}$ *has the slender property.*

This answers the question in [3].

§ 3. Theorem 3.

Let Fn(ω, ω_1) be the (ω, ω_1)-collapsing poset $\{p\,;\,\exists n<\omega\,(p: n\to\omega_1)\}$ whose order is the inverse inclusion. We denote by Col(ω, ω_1) the complete Boolean algebra of all regular open sets in Fn(ω, ω_1).

By Theorem 1 (ii), assuming CH, every complete Boolean algebra that collapses ω_1 does not have the slender property. It seems to be interesting to check whether this is true when CH is false. Especially, does Col(ω, ω_1) have the slender property? The following theorem gives a partial answer to this question.

Theorem 3. *Let P be a poset and $\kappa=|P|$. Suppose that the following* (*) *holds.*

$$(*) \qquad \forall F \subset \omega^\omega (|F| \leqq \kappa \Rightarrow \exists g \in \omega^\omega \, \forall f \in F (f <^* g)).$$

Then, r.o.(P) *has the slender property.*

PROOF. Let P be a poset and $\kappa = |P|$. Assume that $(*)$ holds. To get a contradiction, suppose that

(1) π is a P-name and $\bar{p} \in P$,

(2) $\Vdash_P$ "$\pi : (\boldsymbol{Z}^\omega)^\vee \to \boldsymbol{Z}$ homomorphism",

(3) $\bar{p} \Vdash_P$ "$\exists^\infty n < \omega \, (\pi(\boldsymbol{e}_n) \neq 0)$".

For each $p \in P$, set

$$H_p = \{ h \in \boldsymbol{Z}^\omega \, ; \, \exists k \in \boldsymbol{Z} \, (p \Vdash_P \text{“}\pi(\check{h}) = \check{k}\text{”})\},$$

and define $\theta_p : H_p \to \boldsymbol{Z}$ by

$$\theta_p(h) = \text{the unique } k \in \boldsymbol{Z} \text{ such that } p \Vdash_P \text{“}\pi(\check{h}) = \check{k}\text{”}.$$

CLAIM 2. *There is* $p \in P$ *such that* $p \leq \bar{p}$ *and* $\{f \in \omega^\omega \, ; \, \tilde{f} \in H_p\}$ *is cofinal in* ω^ω.

PROOF OF CLAIM 2. Suppose not. For each $p \in P$, $p \leq \bar{p}$, take $f_p \in \omega^\omega$ such that

$$\forall g \in \omega^\omega (\tilde{g} \in H_p \Rightarrow \text{not } f_p <^* g).$$

Since $|\{f_p \, ; \, p \in P \ \& \ p \leq \bar{p}\}| \leqq |P| = \kappa$, by $(*)$, there is $\bar{f} \in \omega^\omega$ such that $\forall p \in P$ $(p \leq \bar{p} \Rightarrow f_p <^* \bar{f})$. So, it holds that

$$\forall p \in P \, \forall g \in \omega^\omega (p \leq \bar{p} \ \& \ \tilde{g} \in H_p \Rightarrow \text{not } \bar{f} <^* g).$$

But, this contradicts the fact that $\{g \in \omega^\omega \, ; \, \exists p \in P (p \leq \bar{p} \ \& \ \tilde{g} \in H_p)\} = \omega^\omega$. q.e.d. of Claim 2.

Take $p^* \in P$ such that

(4) $p^* \leq \bar{p}$ and $\{f \in \omega^\omega \, ; \, \tilde{f} \in H_{p^*}\}$ is cofinal in ω^ω.

By induction on $n < \omega$, using (2) and (3), take $p_n \in P (n < \omega)$ and $m_n < \omega \, (n < \omega)$ such that, for any $n < \omega$,

(5) $p_0 \leq p^*$ & $p_{n+1} \leq p_n$,

(6) $m_n < m_{n+1}$,

(7) $\forall i \leqq m_n \, \exists k \in \boldsymbol{Z} \, (p_n \Vdash_P \text{“}\pi(\boldsymbol{e}_i) = \check{k}\text{”})$,

(8) $p_n \Vdash_P$ "$\pi(\boldsymbol{e}_{m_n}) \neq 0$".

Set $H = \bigcup_{n<\omega} H_{p_n}$ and $\theta = \bigcup_{n<\omega} \theta_{p_n}$. Define $g \in \omega^\omega$ by

$$g(n) = |\theta(\boldsymbol{e}_n)| \qquad \text{for any } n<\omega .$$

Then, the following (9)$\sim$(12) hold.

(9) H is a pure subgroup of $\boldsymbol{Z}^\omega$.

(10) $\theta : H \to \boldsymbol{Z}$ homomorphism.

(11) $\forall n<\omega\,(\boldsymbol{e}_n \in H)$ & $\exists^\infty n<\omega\,(g(n)\neq 0)$.

(12) $\exists f \in \omega^\omega\,(g<^* f$ & $\tilde{f}\in H)$.

By a similar argument as in the proof of Theorem 2, we can derive a contradiction from (9)$\sim$(12). □

It is well-known [5] that the statement "$\forall F\subset\omega^\omega\,(|F|<2^\omega \Rightarrow \exists g\in\omega^\omega\,\forall f\in F\,(f<^*g))$" is consistent with ZFC$+2^\omega=\omega_2$. So, we have the following corollary.

COROLLARY 2. *The statement* "Col(ω, ω_1) *has the slender property*" *is consistent with* ZFC$+2^\omega=\omega_2$.

I do not know whether the statement "Col(ω, ω_1) does not have the slender property" is consistent with ZFC$+2^\omega=\omega_2$.

§4. A certain complete Boolean algebra with the ω_1-c.c. which does not have the slender property.

In [2], Eda showed an example of a complete Boolean algebra with the ω_1-c.c. which does not have the slender property. His example is complicated, since it was constructed by using a complete Boolean algebra B which satisfies $\|\mathrm{MA}+\neg\mathrm{CH}\|_B=\mathbf{1}$. In this section, we shall construct a complete Boolean algebra with those properties more directly.

Define the poset P by $(H, \theta)\in P$ if and only if

$$\exists n<\omega\,\exists h_0, \cdots, h_{n-1}\in H\,(H=\textstyle\bigoplus_{i<n}\langle h_i\rangle \;\&\; \boldsymbol{Z}^\omega=H\oplus\boldsymbol{Z}^{\omega\setminus n})$$
$$\& \quad \theta : H\to\boldsymbol{Z} \text{ homomorphism},$$

and, for any (H, θ), $(G, \sigma)\in P$,

$$(H, \theta)\leq(G, \sigma) \qquad \text{if and only if} \qquad H\supset G \;\&\; \theta\supset\sigma ,$$

where $\langle h\rangle$ denotes the subgroup generated by h, $\bigoplus_{i<n}H_i$ the direct product of H_i $(i<n)$ and $\boldsymbol{Z}^{\omega\setminus n}$ the subgroup $\{h\in\boldsymbol{Z}^\omega;\ \forall i<n\,(h(i)=0)\}$ of $\boldsymbol{Z}^\omega$.

THEOREM 4. (i) *P satisfies the ω_1-c.c.*

(ii) r.o.(P) *does not have the slender property.*

In the proof of this theorem, we need the following lemma.

LEMMA 1. $\forall h\in \boldsymbol{Z}^{\omega}\,\exists(H,\,\theta)\in P(h\in H)$.

PROOF. This lemma follows immediately from the proof of Theorem 19.2 in [**4**, p. 94]. □

PROOF OF THEOREM 4. First, we shall show (i). So, let W be any subset of P such that $|W|=\omega_1$. For each $p=(H,\,\theta)\in W$, take $n_p<\omega$ and $h_0^p, \cdots, h_{n_p-1}^p \in H$ such that

$$H=\bigoplus_{i<n_p}\langle h_i^p\rangle\,,$$

and set

$$s_p=\langle h_i^p\restriction n_p \mid i<n_p\rangle\,,$$

$$t_p=\langle \theta(h_i^p) \mid i<n_p\rangle\,.$$

CLAIM 3. There are $W'\subset W$, $n<\omega$, s and t such that

(1) $|W'|=\omega_1$,

(2) $\forall p\in W'\,(n_p=n\ \&\ s_p=s\ \&\ t_p=t)$.

PROOF OF CLAIM 3. This claim follows immediately from the fact that $|\{n_p\,;\,p\in W\}\cup\{s_p\,;\,p\in W\}\cup\{t_p\,;\,p\in W\}|\leqq\omega$. q.e.d. of Claim 3.

Take $W'\subset W$, n, s and t which satisfy (1) and (2) in Claim 3. We shall show that

(*) W' are pairwise compatible.

In order to show (*), let $p=(H,\,\theta)$ and $q=(H',\,\theta')$ be any elements in W'. For each $i<n$, set

$$g_i=h_i^p-h_i^q\,.$$

Since $g_0, \cdots, g_{n-1}$ are in $\boldsymbol{Z}^{\omega\setminus n}$, by Lemma 1, there are a subgroup G of $\boldsymbol{Z}^{\omega\setminus n}$ and $j<\omega$ such that

(3) $G\oplus\boldsymbol{Z}^{\omega\setminus(n+j)}=\boldsymbol{Z}^{\omega\setminus n}$,

(4) $g_0, \cdots, g_{n-1}\in G$.

Let τ be the homomorphism from G to $\{0\}\subset\boldsymbol{Z}$. Set

$$\sigma=\theta\oplus\tau\ :\ H\oplus G\longrightarrow\boldsymbol{Z}\,.$$

Then, it is easy to see that $(H\oplus G,\,\sigma)\in P\ \&\ (H\oplus G,\,\sigma)\leq(H,\,\theta),\ (H',\,\theta')$.

Next, we shall show (ii). Define the P-name $\tilde{\pi}$ by

$$\mathrm{dom}(\tilde{\pi})=\{(h,\,k)^{\check{}}\,;\ h\in\boldsymbol{Z}^{\omega}\ \&\ k\in\boldsymbol{Z}\},$$

$$\tilde{\pi}((h,\,k)^{\check{}})=\{(H,\,\theta)\,;\ h\in H\ \&\ \theta(h)=k\}.$$

By Lemma 1, it holds that

$$\forall h\in \boldsymbol{Z}^{\omega}\ \forall p\in P\,\exists q\in P\,\exists k\in \boldsymbol{Z}\,(q\Vdash_P \text{“}(h,\ k)^{\check{}}\in\tilde{\pi}\text{”}).$$

From this, we have that

$$\Vdash_P \text{“}\tilde{\pi}:(\boldsymbol{Z}^{\omega})^{\check{}}\rightarrow \boldsymbol{Z}\ \text{homomorphism”}.$$

To complete the proof we show that $\Vdash_P$“$\exists^{\infty}n<\omega\,(\tilde{\pi}(\boldsymbol{e}_n)\neq 0)$”. Let $n<\omega$ and $p=(H,\ \theta)\in P$. It suffices to show that

$$\exists m<\omega\,\exists q\in P\,(q\leq p\ \&\ n\leqq m\ \&\ q\Vdash_P\text{“}\tilde{\pi}(\boldsymbol{e}_m)\neq 0\text{”}).$$

Take $m<\omega$ and $\bar{p}=(\bar{H},\ \bar{\theta})\in P$ such that

$$\bar{p}\leq p\quad \&\quad n\leqq m\quad \&\quad \boldsymbol{Z}^{\omega}=\bar{H}\oplus\boldsymbol{Z}^{\omega\setminus m}.$$

Set $G=\bar{H}\oplus\langle\boldsymbol{e}_m\rangle$, and define $\tau: G\rightarrow\boldsymbol{Z}$ by

$$\tau(h+k\boldsymbol{e}_m)=\bar{\theta}(h)+k\qquad \text{for } \forall h\in\bar{H} \text{ and } \forall k\in\boldsymbol{Z}.$$

Set $q=(G,\ \tau)\in P$. Then, it holds that

$$q\leq\bar{p}\leq p\qquad\text{and}\qquad q\Vdash_P\text{“}\tilde{\pi}(\boldsymbol{e}_m)=1\text{”}.$$

This completes the proof. □

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