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The canonical lifting of an ordinary Jacobian variety need not be a Jacobian variety

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We denote by k a perfect field of characteristic p, with $p>0$, and by $A=W(k)$ the ring of infinite Witt vectors over k. Let C_0 be a complete, non-singular curve of genus g over k; we say that C_0 is *ordinary* if its Jacobian variety $\mathrm{Jac}(C_0)$ is an ordinary abelian variety, i.e.

$$\mathrm{Jac}(C_0)[p](\bar{k}) \cong (\boldsymbol{Z}/p)^g, \qquad \text{where } g=\mathrm{genus}(C_0)=\dim(\mathrm{Jac}(C_0)).$$

Let (X_0, λ_0) be a polarized abelian variety and suppose that X_0 is ordinary. By a theorem of Serre and Tate (cf. 1.1) it has a *canonical lifting* $(\mathcal{X}, \lambda)$ to $\mathrm{Spec}(A)$.

We study the following problem (cf. Katz [**4**], p. 138).

PROBLEM. Is the canonical lifting of the Jacobian $(X_0, \lambda_0)=\mathrm{Jac}(C_0)$ of an ordinary curve C_0 again a Jacobian?

Note that if $(\mathcal{X}, \lambda)$ is a polarized abelian variety over $\mathrm{Spec}(B)$, where B is a discrete valuation ring or a field, we say "$(\mathcal{X}, \lambda)$ is a Jacobian" if there exists a field $L\supset B$, and a complete stable curve D over L, such that its canonically polarized generalized Jacobian variety is:

$$\mathrm{Jac}(D) \cong (\mathcal{X}, \lambda)\otimes_B L .$$

Note that the answer to the problem is affirmative if $g\leqq 3$, because by A. Weil for $g=2$ (cf. [**15**], p. 37, Satz 2), and by Oort-Ueno for $g\leqq 3$ (cf. [**10**]), we know that in this case a principally polarized abelian variety is a Jacobian.

In this note we show that in general the answer to the problem is negative (cf. Cor. 2.5 below, also cf. Remark 2.6).

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§ 1. The construction by Serre and Tate of the canonical lifting of an ordinary abelian variety.

Let k be a perfect field of characteristic p, with $p>0$. Let X_0 be an abelian variety over k. We denote

$$X_0[p] = \mathrm{Ker}(p \cdot 1_{X_0} : X_0 \to X_0)$$

and we say that X_0 is *ordinary* if the p-rank of X_0 is maximal, i. e.

$$X_0 \text{ is ordinary} \quad \text{iff} \quad X_0[p](\bar{k}) \cong (\boldsymbol{Z}/p)^g, \quad \text{where } g=\dim(X_0).$$

We denote by $\bar{X}_0$ the Barsotti-Tate group of X_0.

THEOREM (1.1) (Serre and Tate, cf. Messing [**5**], Chap. V, Th. 3.3 and Cor. 3.4). *For an ordinary abelian variety X_0 there exists a unique projective abelian scheme $\mathcal{X} \to \mathrm{Spec}(A)$ lifting X_0 to $A=W(k)$ such that the splitting*

$$\bar{X}_0 = (\bar{X}_0)_{\mathrm{loc}} \oplus (\bar{X}_0)_{\mathrm{et}}$$

of the Barsotti-Tate group in local and etale parts lifts to a direct sum splitting of the Barsotti-Tate group $\bar{\mathcal{X}}$ of $\mathcal{X} \to \mathrm{Spec}(A)$; any polarization

$$\lambda_0 : X_0 \longrightarrow X_0^t$$

lifts uniquely to a polarization

$$\lambda : \mathcal{X} \longrightarrow \mathcal{X}^t$$

(here $\mathcal{X}^t$ is the dual of the abelian scheme $\mathcal{X} \to \mathrm{Spec}(A)$); the natural maps

$$\mathrm{End}_K(X) \xleftarrow{\sim} \mathrm{End}_A(\mathcal{X}) \xrightarrow{\sim} \mathrm{End}_k(X_0)$$

are bijective (here K is the field of fractions of A, and $X=\mathcal{X} \otimes_A K$ is the generic fibre of $\mathcal{X} \to \mathrm{Spec}(A)$).

DEFINITION (1.2). The abelian scheme $\mathcal{X} \to \mathrm{Spec}(A)$, or its generic fibre $\mathcal{X} \otimes_A K$, described in Theorem 1.1, is called the *canonical lifting* of the ordinary abelian variety X_0.

Next we discuss the compatibility of a specialization of an ordinary abelian variety to another one and the specialization of their canonical liftings.

By a *good curve* we mean a connected, complete stable curve without rational components whose generalized Jacobian variety is an abelian variety (A rational curve meeting three different elliptic curves transversally at three different points is certainly a stable curve, which the authors do not want to regard as a good curve.); thus, a good curve is the union of smooth complete curves of *positive* genus which are connected like a tree (there are no cycles in the graph). We use the terminology "Jacobian of a good curve" if we mean the generalized Jacobian variety of that curve, i. e. the abelian variety obtained as the product of the Jacobians of the components of the good curve, with polarization equal to the sum of the canonical polarizations pulled back from the factors. We use the terminology "$C \to S$ is a good curve" if we have a proper flat family all of whose geometric fibres are good curves. A good curve C_0 in characteristic $p>0$ is called ordinary if its Jacobian is an ordinary abelian variety. Note that being ordinary is an open condition, and moreover a general elliptic curve is ordinary. Hence a general curve of genus g in positive characteristic is ordinary.

LEMMA (1.3). *Let C_0 be a good, ordinary curve over k. Let*

$$\mathcal{D}_0 \longrightarrow S = \mathrm{Spec}(R)$$

be a good curve over an irreducible k-scheme S. Let $s \in S(k)$ be a closed point and suppose that the fibre at this point is C_0:

$$\mathcal{D}_{0,s} \cong C_0 .$$

Let $k_1 = k(S)$ be the field of fractions of R, and let k_2 be a perfect field containing k_1; let

$$D_0 := \mathcal{D}_0 \otimes_R k_2 ;$$

note that D_0 is an ordinary good curve. Suppose that the canonical lifting of $\mathrm{Jac}(C_0) =: (X_0, \lambda_0)$ *is not a Jacobian. Then the canonical lifting of* $\mathrm{Jac}(D_0) =: (Z_0, \rho_0)$ *to $W(k_2)$ is not a Jacobian.*

PROOF. Let $R' \subset k_2$ be the integral closure of R in k_2; we choose a commutative diagram

$$\begin{tikzcd} k & R , \arrow[l, "\varphi"'] \arrow[d, phantom, "\cap"] \\ & R' \arrow[ul, "\psi"] \end{tikzcd}$$

where φ is given by $s \in S(k)$. By the functoriality of the construction of the ring of Witt-vectors we obtain a commutative diagram

$$\begin{tikzcd} W(k) & W(R) . \arrow[l, "\Phi"'] \arrow[d, phantom, "\cap"] \\ & W(R') \arrow[ul, "\Psi"] \end{tikzcd}$$

Let

$$(\mathcal{Y}_0, \mu_0) = \mathrm{Jac}(\mathcal{D}_0) \otimes_R R'$$

and let $\mathcal{H}_0$ be the Barsotti-Tate group of $\mathcal{Y}_0 \to \mathrm{Spec}(R')$. Then $H_0 := \mathcal{H}_0 \otimes_{R'} k_2$ is the Barsotti-Tate group of $Z_0 := \mathcal{Y}_0 \otimes_{R'} k_2$, and because k_2 is perfect, we obtain a canonical direct sum splitting

$$0 \longrightarrow H_{0,\mathrm{loc}} \longrightarrow \bar{Z}_0 = H_0 \underset{\curvearrowleft}{\longrightarrow} H_{0,\mathrm{et}} \longrightarrow 0$$

(e.g. use [7], p. I. 1-4, Lemma 1.1). This induces a direct sum splitting

$$0 \longrightarrow \mathcal{H}_{0,\mathrm{loc}} \longrightarrow \bar{\mathcal{Y}}_0 = \mathcal{H}_0 \underset{\curvearrowleft}{\longrightarrow} \mathcal{H}_{0,\mathrm{et}} \longrightarrow 0$$

over R'. From an appropriate version of the Grothendieck-Mumford deformation theory (generalize [8], p. 242, Th. 2.3.3), and from the proof of the Serre-Tate theorem (both methods in the situation of a residue class ring R' instead of a residue class field) we conclude that there exists a principally polarized abelian scheme

$$(\mathcal{Y}, \mu) \qquad \text{over } \mathrm{Spec}(W(R'))$$

such that

$$(\mathcal{Y}, \mu) \otimes_{W(R')} R' = (\mathcal{Y}_0, \mu_0),$$

with the property that the Barsotti-Tate group $\mathcal{H}$ of $\mathcal{Y} \to \mathrm{Spec}(W(R'))$ splits as a direct sum in the way required in the Serre-Tate construction. From this it follows that the Barsotti-Tate group of

$$Y \otimes_{W(R')} W(k), \qquad \text{with } \Psi : W(R') \to W(k),$$

splits in the same way; hence by the uniqueness of the Serre-Tate lifting (cf. Th. 1.1) it follows that

$$(\mathcal{Y}, \mu) \otimes_{W(R')} W(k) \cong (\mathcal{X}, \lambda),$$

where $(\mathcal{X}, \lambda)$ is the canonical lifting of $(X_0, \lambda_0) = \mathrm{Jac}(C_0)$ to $W(k)$. Suppose there would exist a field $L \supset W(k_2)$ and a good curve D defined over L such that

$$\mathrm{Jac}(D) = (\mathcal{Y}, \mu) \otimes_{W(R')} L .$$

Consider the Hilbert scheme $H = H_g$ of tricanonically embedded stable curves as in [3], with $g = \dim X_0$, and let $H^0 \subset H$ be the subscheme corresponding with good stable curves. We obtain a morphism

$$H^0 \longrightarrow \mathcal{A}_{g,1}$$

(where $\mathcal{A}_{g,1}$ is the coarse moduli scheme of principally polarized abelian schemes). If a Jacobian variety has good reduction, the corresponding curve has good, stable reduction (by [3], Th. 2.4 and Th. 2.5), hence the image of H^0 in $A_{g,1}$

is closed. From $(\mathcal{Y}, \mu)/W(R')$ we deduce a morphism

$$y : S \longrightarrow \mathcal{A}_{g,1}, \qquad \text{where } S := \operatorname{Spec} W(R').$$

Because of the existence of D the generic point of S is mapped under y in the image of H^0 in $\mathcal{A}_{g,1}$, and because this image is closed in $\mathcal{A}_{g,1}$ we conclude that y factors through this image. We define T by the cartesian square

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "f"'] & H^0 \arrow[d] \\ S \arrow[r, "y"] & \mathcal{A}_{g,1}. \end{tikzcd}$$

It follows that the morphism f is surjective. Via g we pull back the universal curve over H^0, and we obtain a good, stable curve

$$\mathcal{D} \longrightarrow T, \qquad \text{such that } \operatorname{Jac}(\mathcal{D}) \cong (\mathcal{Y}, \mu)\times_S T.$$

Because f is surjective we can choose a field K, and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & T \arrow[d, "f"] \\ \operatorname{Spec} W(k) \arrow[r] & S = \operatorname{Spec} W(R'). \end{tikzcd}$$

We conclude that

$$(\mathcal{X}, \lambda)\otimes_{W(k)} K = (\mathcal{Y}, \mu)\otimes_{W(R')} K \cong \operatorname{Jac}(D)\times_T \operatorname{Spec}(K),$$

which contradicts the assumption that the canonical lifting of $\operatorname{Jac}(C_0)=(X_0, \lambda_0)$ is not a Jacobian. This shows that for any field $L\supset W(k_2)$ the polarized abelian variety $(\mathcal{Y}, \mu)\otimes L$ is not a Jacobian, and the lemma is proved.

For later use we recall the following facts.

LEMMA (1.4). *Let $\mathcal{C}\to S$ be a flat, proper, smooth curve over a scheme S. We obtain an S-morphism*

$$\mathbf{Aut}_S(\mathcal{C}) \longrightarrow \mathbf{Aut}_S(\operatorname{Jac}(\mathcal{C}/S))$$

from the relative automorphism group scheme of $\mathcal{C}\to S$ to the same for the canonically polarized Jacobian $\operatorname{Jac}(\mathcal{C}/S)\to S$. *This morphism is a closed immersion* (cf. [11], Prop. 2.3).

THEOREM (1.5). *Let k be a perfect field of characteristic $p>0$. Let $A=W(k)$ (thus A is a complete discrete valuation ring of unequal characteristics in which p does not ramify). Let $GL_n(A)$ denote the group of $(n\times n)$-matrices in A with determinant a unit in A. Let $\sigma\in GL_n(A)$ be such that $\sigma^p=1$ (the identity matrix). Then there exists a matrix $P\in GL_n(A)$ such that*

$$\sigma \sim P^{-1}\sigma P = \mathrm{diag}(Q, \cdots, Q, R, \cdots, R, E),$$

where E is a unit matrix, $E=\mathrm{diag}(1, \cdots, 1)$, where Q is a $(p-1)\times(p-1)$-matrix

$$Q = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} \in GL_{p-1}(A),$$

and where R is a $(p\times p)$-matrix

$$R = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in GL_p(A)$$

(cf. Curtis-Reiner [**2**], Th. 74.3).

REMARK (1.6). With the notations of (1.5) we write

$$Q\otimes_A k = Q_0 \in GL_{p-1}(k), \quad \text{and}$$
$$R\otimes_A k = R_0 \in GL_p(k).$$

It is easy to see that Q_0, respectively R_0, can be put in a Jordan form of size $p-1$, respectively p:

$$Q_0 \sim \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix} \in GL_{p-1}(k) \qquad \text{and} \qquad R_0 \sim \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix} \in GL_p(k).$$

§ 2. Counter examples.

We start by constructing a Galois covering

$$C_0 \longrightarrow \boldsymbol{P}^1$$

over a field of characteristic $p>0$ with Galois group cyclic of order p. We choose an integer $r\geqq 2$, we take $\alpha_1, \cdots, \alpha_r \in k$, with $\alpha_i \neq \alpha_j$ if $i\neq j$, and we take H and f such that

$$H, G \in k[X], \quad \mathrm{degree}(G) = r, \quad G/H = f \in k(X),$$
$$H = (X-\alpha_1)\times \cdots \times(X-\alpha_r), \quad \text{and}$$
$$G(\alpha_i) \neq 0 \quad \text{for } 1\leqq i\leqq r \quad (\text{i. e. } (G, H)=1 \text{ in } k[x]).$$

We define C_0 as the normalization of the projective plane curve defined by

$$Y^p - H^{p-1}Y = H^{p-1}G\,;$$

thus the function field of C_0 equals

$$k(C_0) = k(X)\left(\frac{y}{H}\right), \qquad \text{with } \left(\frac{y}{H}\right)^p - \frac{y}{H} = f = \frac{G}{H}.$$

We define

$$\sigma_0 X = X \qquad \text{and} \qquad \sigma_0 Y = Y + H\,.$$

This gives

$$\sigma_0 \in \mathrm{Aut}(k(C_0)/k(X)) = \mathrm{Aut}(C_0 \to \boldsymbol{P}^1)\,,$$

clearly we have that the order of σ_0 equals p, and

$$C_0 \longrightarrow C_0/\langle\sigma_0\rangle \cong \boldsymbol{P}^1$$

(and in such a case the curve C_0 is called an Artin-Schreier curve, e.g. cf. [**14**], p. 176). We denote by σ_0^* the representation of σ_0 in $H^0(C_0, \Omega_{C_0}^{\otimes 2})$. The curve C_0 and $\sigma_0 \in \mathrm{Aut}_k(C_0)$ satisfy the following conditions.

THEOREM (2.1). (i) *The genus of C_0 equals $g=(p-1)(r-1)$.*
(ii) *C_0 is non-hyperelliptic if and only if $r \geqq 3$.*
(iii) *The representation matrix σ_0^* has $(2r-3)$ Jordan blocks of size p and r Jordan blocks of size $(p-3)$.*
(iv) *Because the poles of $f \in k(X)$ are simple, the curve C_0 is ordinary.*

(For the last property, cf. Subrao [**14**], Prop. 3.2, further cf. [**11**], Prop. 2.3, Cor. 2.4, and Th. 2.6.)

LEMMA (2.2). *Let $f \in k(X)$ and C_0 be as above. There does not exist a proper, flat curve $\boldsymbol{C}$ over $S=\mathrm{Spec}(A)$, with $A=W(k)$, and an automorphism*

$$\sigma \in \mathrm{Aut}_S(\boldsymbol{C})$$

such that

$$(\boldsymbol{C}, \sigma)\otimes_A k = (C_0, \sigma_0)\,.$$

PROOF. Assume $(\boldsymbol{C}, \sigma)$ as indicated would exist. The A-module $H^0(\boldsymbol{C}, \Omega_{\boldsymbol{C}/A}^{\otimes 2})$ is free of rank $(3g-3)$; the automorphism σ acts on this A-module, thus we obtain a representation

$$\sigma^* \in GL_{3g-3}(A)$$

of σ. Note that the canonical map

$$\mathrm{Aut}_S(\boldsymbol{C}) \longrightarrow \mathrm{Aut}_k(C_0)$$

is injective, hence the order of σ^* divides p. Thus, by Theorem (1.5), the matrix σ^* is conjugate to a matrix

$$\sigma^* \sim \mathrm{Diag}(Q, \cdots, Q, P, \cdots, P, E)$$

as in (1.5). Hence (cf. Remark 1.6) we have that $\sigma^*\otimes_A k=\sigma_0^*$ has only Jordan blocks of sizes p, $(p-1)$ and 1. This contradicts Theorem (2.1, iii), which proves the lemma.

LEMMA (2.3). *Let k be an algebraically closed field, C_0 a non-hyperelliptic curve over k, let $(X_0, \lambda_0)=\mathrm{Jac}(C_0)$ be its Jacobian variety, and let $(\mathcal{X}, \lambda)\to\mathrm{Spec}(A)$ be a lifting of this over $A=W(k)$. If the generic fibre $(X, \lambda)=(\mathcal{X}, \lambda)\otimes_A K$ is a Jacobian, then C_0 can be lifted to a curve $\mathcal{C}\to\mathrm{Spec}(A)$ such that $\mathrm{Jac}(\mathcal{C})\cong(\mathcal{X}, \lambda)$. Moreover, if C_0 is ordinary and if $(\mathcal{X}, \lambda)$ is the canonical lifting of (X_0, λ_0), then*

$$\mathrm{Aut}_A(\mathcal{C}) \xrightarrow{\sim} \mathrm{Aut}_k(C_0)\,.$$

PROOF. Let $\mathcal{M}$, respectively $\mathcal{A}$, be the formal moduli spaces of C_0, respectively of (X_0, λ_0). By this we mean the following. Let $\mathcal{R}$ be the category of local, artinian A-algebras, with $A=W(k)$, and let

$$\mathrm{Def}(C_0) : \mathcal{R} \longrightarrow \mathcal{S}ets\,, \qquad \text{resp.} \quad \mathrm{Def}(X_0, \lambda_0) : \mathcal{R} \longrightarrow \mathcal{S}ets$$

be the corresponding deformation functors. These are pro-representable:

$$\mathcal{M} \cong \mathrm{Spf}\, A[[t_1, \cdots, t_{3g-3}]]\,, \qquad \text{with } g = \mathrm{genus}(C_0)\,,$$
$$\mathcal{A} \cong \mathrm{Spf}\, A[[t_{ij}] : 1\leqq i\leqq j\leqq g]]\,.$$

Let

$$\tau : \mathcal{M} \longrightarrow \mathcal{A}$$

be the Torelli map (e.g. cf. [**6**], 7.4). Because C_0 is non-hyperelliptic τ is a closed immersion (this follows from Noether's theorem, cf. [**9**], p. 172, Th. 7.2 and Cor. 2.8).

Suppose there exist a field $L\supset A$ and a good curve D over L such that

$$\mathrm{Jac}(D) = (\mathcal{X}, \lambda)\otimes_A L$$

(where $(\mathcal{X}, \lambda)$ is a lifting of (X_0, λ_0) to $W(k)=A$). Choose a discrete valuation ring $B\subset L$ containing A; let b be the residue class field of B; suppose b is algebraically closed (if not, extend $B\subset L$ to such a situation). By Deligne and Mumford [**3**], Th. 2.4, we conclude that D admits a stable model

$$\mathcal{D} \longrightarrow \mathrm{Spec}(B)$$

and because $\mathrm{Jac}(D)$ extends to an abelian scheme $\mathcal{X}\otimes_A B$ over $\mathrm{Spec}(B)$ this is a good curve, and $\mathrm{Jac}(\mathcal{D})=(\mathcal{X}, \lambda)\otimes_A B$ (cf. [**3**], Th. 2.5). From

$$\mathrm{Jac}(D_0) = \mathrm{Jac}(\mathcal{D})\otimes_B b \cong (X_0, \lambda_0)\otimes_k b = \mathrm{Jac}(C_0)\otimes_k b$$

(and because b is algebraically closed) we conclude by Torelli's theorem that

$$\mathcal{D}\otimes_B b =: D_0 \cong C_0\otimes_k b$$

(cf. [**1**]). Thus $\mathcal{D}$ is a deformation of $C_0 \otimes_k b$, and hence we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spf}\, B \arrow[r] \arrow[d, "\psi"'] & \mathrm{Spf}\, A \arrow[d, "\varphi"] \arrow[dl, dashed, "\psi'"'] \\ \mathcal{M} \arrow[r, "\tau"] & \mathcal{A} \end{tikzcd}$$

where φ is given by $(\mathcal{X}, \lambda)$, and ψ is given by $\mathcal{D}$. Since τ is a closed immersion, this implies that we have ψ' as indicated, leaving the diagram commutative, and hence we conclude that $\mathcal{D}$ has a model, say $\mathcal{C}$, over $\mathrm{Spec}\, A$; because k is algebraically closed, from

$$\mathcal{C} \otimes_A b \cong \mathcal{D} \otimes_B b \cong C_0 \otimes_k b$$

it follows that its closed fibre is isomorphic with C_0:

$$\mathcal{C} \longrightarrow \mathrm{Spec}\, A\,, \qquad \mathcal{C} \otimes_A B \cong \mathcal{D}\,, \qquad \text{and} \quad \mathcal{C} \otimes_A k \cong C_0\,.$$

Thus it follows that $\mathcal{C} \to \mathrm{Spec} A$ is smooth. Now suppose moreover that C_0 is ordinary and that $(\mathcal{X}, \lambda)$ is the canonical lifting of (X_0, λ_0). Let $\sigma_0 \in \mathrm{Aut}(C_0) \subset \mathrm{Aut}(\mathrm{Jac}(C_0))$. By Theorem (1.1) any $\sigma_0 \in \mathrm{Aut}(\mathrm{Jac}(C_0))$ admits a unique lifting to an automorphism, say σ, of $(\mathcal{X}, \lambda)$. By Torelli's theorem (cf. [**1**]) either

$$+\sigma \otimes \bar{K} \in \mathrm{Aut}(\mathcal{C} \otimes \bar{K}) \qquad \text{or} \qquad -\sigma \otimes \bar{K} \in \mathrm{Aut}(\mathcal{C} \otimes \bar{K})$$

(here $\bar{K}$ is an algebraic closure of the field of fractions K of A). In the latter case

$$\{\text{specialization of } (-\sigma \otimes \bar{K})\} \cdot \sigma_0^{-1} \in \mathrm{Aut}(C_0)\,,$$

a contradiction to the fact that C_0 is non-hyperelliptic; thus $\sigma \otimes \bar{K} \in \mathrm{Aut}(\mathcal{C} \otimes \bar{K})$. By (1.4) from $\sigma \in \mathrm{Aut}(\mathrm{Jac}(\mathcal{C}))$ we conclude that $\sigma \in \mathrm{Aut}_A(\mathcal{C})$, and the lemma is proved.

THEOREM (2.4). *Let $r \geqq 3$, $p \geqq 5$, and define C_0 as in the beginning of this section. Note that C_0 is ordinary and non-hyperelliptic* (2.1, ii *and* iv). *Let $(X_0, \lambda_0) = \mathrm{Jac}(C_0)$, and let $(\mathcal{X}, \lambda)$ be its canonical lifting to $A = W(k)$, with generic fibre (X, λ). This principally polarized abelian variety is not a Jocobian.*

PROOF. We may assume k to be algebraically closed. By the construction of C_0 we have an automorphism σ_0 of C_0 of order p. Assume (X, λ) is a Jacobian. Then by Lemma (2.3) we conclude that (C_0, σ_0) lifts to $W(k) = A$, and this contradicts Lemma (2.2). Q.E.D.

COROLLARY (2.5). *Let D_0 be a generic curve of genus g over a field L.*

Assume char$(L)=p\geqq 5$, *and assume* $g\geqq 2(p-1)$. *The canonical lifting of* Jac(D_0) *is not a Jacobian.*

PROOF. Choose $r=3$, and let C_0 be defined as in the beginning of this section (choose some G and $\alpha_1, \alpha_2, \alpha_3$ as indicated); then

$$\text{genus}(C_0)=2(p-1)$$

by (2.1, i), and by Theorem (2.4) the canonical lifting of Jac(C_0) is not a Jacobian. If $g=2(p-1)$ we take $C_0'=C_0$; if $g>2(p-1)$ choose a good ordinary curve of genus $g-2(p-1)$ (e.g. the join of copies of ordinary elliptic curves), and let C_0' be the join of this curve with C_0 with one transversal crossing; the Jacobian of C_0' is the product of the Jacobians for that curve and C_0. Note that taking canonical liftings commutes with taking products of polarized abelian varieties. Hence it follows from Theorem (2.4) that the canonical lifting of Jac(C_0') is not a Jacobian (of any good curve). We choose $\mathcal{D}_0\to S$, $s\in S(k)$, $\mathcal{D}_{0,s}\cong C_0'$ with the properties as in Lemma (1.3) such that $D_0:=\mathcal{D}_0\otimes_R k_2$ is isomorphic with the D_0 given in the corollary; this choice is possible because in Deligne-Mumford [**3**] irreducibility of the Hilbert scheme $H_g\otimes \boldsymbol{F}_p$ of stable curves in characteristic $p>0$ has been proved. We apply Lemma (1.3), and the corollary follows.

REMARK (2.6). We were informed that B. Dwork and A. Ogus obtained a negative answer to the problem stated in the introduction for every $g>3$ and $p>2$.

REMARK (2.7). We have seen that a curve C_0 with an automorphism σ_0 as constructed in the beginning of this section does not admit a lifting $(\mathcal{C}, \sigma)$ to an *unramified* p-adic ring (cf. Lemma 2.2). We can give another proof of this fact, avoiding the matrix computation used in this note, but using the fact that ramification is needed if we want to lift an Artin-Schreier covering with small multiplicities in its conductor to a characteristic zero domain (cf. [**13**], Appendix).

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