J. Math. Soc. Japan
Vol. 38, No. 4, 1986

Real analytic actions of complex symplectic groups and other classical Lie groups on spheres

By Fuichi UCHIDA

(Received April 9, 1985)

0. Introduction.

There seems to be few works on non-compact semi-simple Lie groups acting on the sphere non-transitively. In the previous papers [7], [8] we have studied analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ (resp. $\boldsymbol{SL}(n, \boldsymbol{C})$) actions on the standard k-sphere and we have shown that such an action has been characterized by an analytic $\boldsymbol{R}_0$ (resp. $\boldsymbol{C}_0$) action on a homotopy $(k-n+1)$-sphere (resp. $(k-2n+2)$-sphere) satisfying a certain condition for $5\leqq n\leqq k\leqq 2n-2$ (resp. $n\geqq 7$ and $2n\leqq k\leqq 4n-2$). Here $\boldsymbol{R}_0$ (resp. $\boldsymbol{C}_0$) denotes the multiplicative group of all non-zero real (resp. complex) numbers.

In this paper we study analytic $\boldsymbol{Sp}(n, \boldsymbol{C})$ actions on integral homology k-spheres and we shall show in Section 5 that such an action is characterized by an analytic $\boldsymbol{C}_0$ action on an integral homology $(k-4n+2)$-sphere satisfying a certain condition for $n\geqq 7$ and $4n\leqq k\leqq 8n-2$. By an integral homology k-sphere we mean a closed orientable analytic manifold whose homology with integer coefficients is isomorphic to that of the standard k-sphere.

Our method and result are quite similar to that of the previous papers [7], [8]. One difference here is the need to show that the fixed point set of the restricted $L(n)$ action is an analytic submanifold of a given manifold with certain analytic $\boldsymbol{Sp}(n, \boldsymbol{C})$ action, where $L(n)$ is a non-compact closed subgroup of $\boldsymbol{Sp}(n, \boldsymbol{C})$ defined in Section 1. To show it, we need to study certain analytic $\boldsymbol{SL}(2, \boldsymbol{C})$ actions. Theorem 2.1 is a key result.

In the final part of Section 5, we describe transitive $\boldsymbol{Sp}(n, \boldsymbol{C})$ actions on $(4n-1)$-sphere. Finally, we study analytic $\boldsymbol{SO}(n, \boldsymbol{C})$ actions on $(2n-1)$-sphere and on the Brieskorn variety $W^{2n-1}(d)$, and analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on $(2n-1)$-sphere in Section 6.

1. Certain closed subgroups of $\boldsymbol{Sp}(n, \boldsymbol{C})$.

1.1. Let $\boldsymbol{GL}(m, \boldsymbol{C})$ and $\boldsymbol{U}(m)$ denote the group of regular matrices of degree m with complex coefficients and the group of unitary matrices of degree m, respectively. Let I_n denote the unit matrix of degree n, and we put

$$J_n=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Define $\boldsymbol{Sp}(n, \boldsymbol{C})=\{A\in \boldsymbol{GL}(2n, \boldsymbol{C}) : {}^tAJ_nA=J_n\}$ and $\boldsymbol{Sp}(n)=\boldsymbol{U}(2n)\cap \boldsymbol{Sp}(n, \boldsymbol{C})$. Then $\boldsymbol{Sp}(n, \boldsymbol{C})$ and $\boldsymbol{Sp}(n)$ are connected closed subgroups of $\boldsymbol{GL}(2n, \boldsymbol{C})$. Let $L(n)$ and $N(n)$ denote the subgroups of $\boldsymbol{Sp}(n, \boldsymbol{C})$ consisting of all matrices of the form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & X_{11} & * & X_{12} \\ 0 & 0 & 1 & 0 \\ 0 & X_{21} & * & X_{22} \end{pmatrix}, \qquad \begin{pmatrix} * & * & * & * \\ 0 & X_{11} & * & X_{12} \\ 0 & 0 & * & 0 \\ 0 & X_{21} & * & X_{22} \end{pmatrix}$$

for $X_{ij}\in M_{n-1}(\boldsymbol{C})$, respectively. Notice that $N(n)$ is the normalizer of $L(n)$, in fact, if $N(n)$ contains $gL(n)g^{-1}$ for some $g\in \boldsymbol{Sp}(n, \boldsymbol{C})$ then $g\in N(n)$; the standard $\boldsymbol{Sp}(n, \boldsymbol{C})$ action on $\boldsymbol{C}^{2n}-\{0\}$ is transitive, its isotropy groups are conjugate to $L(n)$, and each isotropy group of the restricted $\boldsymbol{Sp}(n)$ action is conjugate to $\boldsymbol{Sp}(n-1)$, where $\boldsymbol{Sp}(n-1)=L(n)\cap \boldsymbol{Sp}(n)$. Put $\boldsymbol{Sp}(n-1, \boldsymbol{C})={}^tL(n)\cap L(n)$, where ${}^tL(n)=\{{}^tA : A\in L(n)\}$.

THEOREM 1.1 (Uchida [**10**], Theorem 1.3). *Let G be a closed proper subgroup of $\boldsymbol{Sp}(n, \boldsymbol{C})$ which contains $\boldsymbol{Sp}(n-1)$ for $n\geqq 4$. Suppose that each isotropy group of the restricted $\boldsymbol{Sp}(n)$ action on the homogeneous space $\boldsymbol{Sp}(n, \boldsymbol{C})/G$ contains a subgroup conjugate to $\boldsymbol{Sp}(n-1)$. Then $L(n)\subset hGh^{-1}\subset N(n)$ for an element h of the centralizer of $\boldsymbol{Sp}(n-1, \boldsymbol{C})$ in $\boldsymbol{Sp}(n, \boldsymbol{C})$.*

REMARK. Let a, b, c, d be complex numbers with $ad-bc=1$. Put

$$M\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & 0 & b & 0 \\ 0 & I_{n-1} & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix}.$$

Then $M\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of the centralizer of $\boldsymbol{Sp}(n-1, \boldsymbol{C})$ in $\boldsymbol{Sp}(n, \boldsymbol{C})$. In fact, the centralizer consists of all matrices of the form $\pm M\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1.2. Let X be a set with a transformation group G. Denote by $F(H, X)$ the set of fixed points of the restricted H action for a subgroup H of G.

LEMMA 1.2. *Let X be a Hausdorff space with a non-trivial continuous $\boldsymbol{Sp}(n, \boldsymbol{C})$ action. Suppose that $n\geqq 4$ and each isotropy group of the restricted $\boldsymbol{Sp}(n)$ action contains a subgroup conjugate to $\boldsymbol{Sp}(n-1)$. Then*

$$F(\boldsymbol{Sp}(n), X) = F(\boldsymbol{Sp}(n, \boldsymbol{C}), X), \qquad F(\boldsymbol{Sp}(n-1), X) = F(\boldsymbol{Sp}(n-1, \boldsymbol{C}), X)$$

and

$$F(ML(1)\cdot \boldsymbol{Sp}(n-1, \boldsymbol{C}), X) = F(L(n), X),$$

where $ML(1)$ consists of all matrices of the form $M\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

PROOF. It is only necessary to show that $F(\boldsymbol{Sp}(n), X)$ (resp. $F(\boldsymbol{Sp}(n-1), X)$, $F(ML(1)\cdot\boldsymbol{Sp}(n-1, \boldsymbol{C}), X))$ is contained in $F(\boldsymbol{Sp}(n, \boldsymbol{C}), X)$ (resp. $F(\boldsymbol{Sp}(n-1, \boldsymbol{C}), X)$, $F(L(n), X))$. Let G denote the isotropy group at $x \in X$. Suppose first that G contains $\boldsymbol{Sp}(n)$ but G is a proper subgroup of $\boldsymbol{Sp}(n, \boldsymbol{C})$. Then G satisfies the condition of Theorem 1.1, and hence $hGh^{-1} \subset N(n)$ for some h. But $N(n)$ does not contain any subgroup conjugate to $\boldsymbol{Sp}(n)$, this is a contradiction. Therefore, if G contains $\boldsymbol{Sp}(n)$, then G coincides with $\boldsymbol{Sp}(n, \boldsymbol{C})$. This shows that $F(\boldsymbol{Sp}(n), X)$ is equal to $F(\boldsymbol{Sp}(n, \boldsymbol{C}), X)$. In the following, suppose that G is a proper subgroup of $\boldsymbol{Sp}(n, \boldsymbol{C})$. Suppose that G contains $\boldsymbol{Sp}(n-1)$. Then, $\boldsymbol{Sp}(n-1, \boldsymbol{C}) \subset L(n) \subset hGh^{-1}$ for an element h of the centralizer of $\boldsymbol{Sp}(n-1, \boldsymbol{C})$, and hence G contains $\boldsymbol{Sp}(n-1, \boldsymbol{C})$. This shows that $F(\boldsymbol{Sp}(n-1), X)$ is equal to $F(\boldsymbol{Sp}(n-1, \boldsymbol{C}), X)$. Suppose next that G contains $ML(1)\cdot\boldsymbol{Sp}(n-1, \boldsymbol{C})$. Then there is an element $h=M\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $L(n) \subset hGh^{-1} \subset N(n)$. In particular, $hML(1)h^{-1}$ is contained in $N(n)$ Then we see that $c=0$ by a routine work, and hence $h \in N(n)$. Therefore G contains $L(n)$, and hence $F(L(n), X)$ is equal to $F(ML(1)\cdot\boldsymbol{Sp}(n-1, \boldsymbol{C}), X)$.

q. e. d.

COROLLARY 1.3. *Under the hypotheses of Lemma 1.2, the equality*

$$X = \boldsymbol{Sp}(n, \boldsymbol{C})\cdot F(L(n), X) = \{gx : g \in \boldsymbol{Sp}(n, \boldsymbol{C}),\ x \in F(L(n), X)\}$$

holds. Moreover if $F(\boldsymbol{Sp}(n, \boldsymbol{C}), X)$ is empty, then there is an equivariant homeomorphism

$$X \cong (\boldsymbol{Sp}(n, \boldsymbol{C}) \times F(L(n), X))/N(n),$$

where the normalizer $N(n)$ of $L(n)$ acts naturally on $F(L(n), X)$.

1.3. Let X be an analytic manifold with a non-trivial analytic $\boldsymbol{Sp}(n, \boldsymbol{C})$ action. Suppose the hypotheses of Lemma 1.2 hold. Then each connected component of $F(\boldsymbol{Sp}(n, \boldsymbol{C}), X)$ (resp. $F(\boldsymbol{Sp}(n-1, \boldsymbol{C}), X)$) is an analytic submanifold of X, because there is an analytic Riemannian metric on X which is invariant under the restricted $\boldsymbol{Sp}(n)$ action (cf. [7], Remark 3.2). We want to show that each connected component of $F(L(n), X)$ is an analytic submanifold of X. By the last equation of Lemma 1.2, it is sufficient to show that for the natural $\boldsymbol{SL}(2, \boldsymbol{C})$ action on $Y=F(\boldsymbol{Sp}(n-1, \boldsymbol{C}), X)$, each connected component of $F(L(1), Y)$ is an analytic submanifold of Y. Let G be an isotropy group of the $\boldsymbol{SL}(2, \boldsymbol{C})$ action on Y. Notice that if $G \neq \boldsymbol{SL}(2, \boldsymbol{C})$ then $L(1) \subset hGh^{-1} \subset N(1)$ for an element $h \in \boldsymbol{SL}(2, \boldsymbol{C})$.

2. Infinitesimal transformations.

2.1. Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra of left invariant vector fields. Let $\phi: G\times M\to M$ be an analytic G action. Let $L(M)$ denote the Lie algebra of analytic vector fields on M. Then we can define a Lie algebra homomorphism $\phi^+: \mathfrak{g}\to L(M)$ as follows

$$\phi^+(X)_p(f) = \lim_{t\to 0}\frac{f(\phi(\exp(-tX),\ p))-f(p)}{t}$$

for $X\in\mathfrak{g}$, $p\in M$ and any analytic function f defined on a neighborhood of p (Palais [**6**], Chapter II, Theorem II). We shall show the above fact for completeness.

For each $p\in M$, we define $\phi^p: G\to M$ by $\phi^p(g)=\phi(g^{-1},\ p)$. If X and Y are analytic vector fields on G and M respectively, then we obtain an analytic vector field $X\oplus Y$ on $G\times M$ defined by $(X\oplus Y)_{(g,p)}=X_g\oplus Y_p$. For $X\in\mathfrak{g}$, $p\in M$ and an analytic function f defined on a neighborhood U of p, we see that

$$((d\phi^q)(X_e))(f) = X_e(f\circ\phi^q) = (X\oplus 0)_{(e,q)}(f\circ\phi\circ(\nu\times 1))$$

for $q\in U$, and hence the function $q\to((d\phi^q)(X_e))(f)$ is analytic on U. Here e is the identity element of G and $\nu: G\to G$ is defined by $\nu(g)=g^{-1}$. Therefore the correspondence $p\to(d\phi^p)(X_e)$ is an analytic vector field on M. Put $\phi^+(X)_p=(d\phi^p)(X_e)$. Then $\phi^+(X)\in L(M)$. Let $p\in M$, $h\in G$ and let $q=\phi^p(h)$. Define $L_h: G\to G$ by $L_h(g)=hg$. Then

$$(\phi^p\circ L_h)(g) = \phi^p(hg) = \phi(g^{-1}h^{-1},\ p) = \phi(g^{-1},\ q) = \phi^q(g)$$

and hence

$$\phi^+(X)_q = (d\phi^q)(X_e) = (d\phi^p)((dL_h)(X_e)) = (d\phi^p)(X_h), \qquad X\in\mathfrak{g}.$$

Therefore X and $\phi^+(X)$ are ϕ^p related. If $Y\in\mathfrak{g}$ then of course Y and $\phi^+(Y)$ are also ϕ^p related, and hence $[X,\ Y]$ and $[\phi^+(X),\ \phi^+(Y)]$ are ϕ^p related (Chevalley [1], Chapter III, § VI, Proposition 2), i. e.

$$\phi^+([X,\ Y])_p = (d\phi^p)([X,\ Y]_e) = [\phi^+(X),\ \phi^+(Y)]_p, \qquad p\in M.$$

Since ϕ^+ is obviously linear, this proves that $\phi^+: \mathfrak{g}\to L(M)$ is a Lie algebra homomorphism. By definition, we see that

$$\begin{aligned}\phi^+(X)_p(f) &= ((d\phi^p)(X_e))(f)=X_e(f\circ\phi^p)\\ &= \lim_{t\to 0}\frac{(f\circ\phi^p)(\exp(tX))-(f\circ\phi^p)(e)}{t}\\ &= \lim_{t\to 0}\frac{f(\phi(\exp(-tX),\ p))-f(p)}{t}.\end{aligned}$$

2.2. Put

$$X_1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad X_2=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\qquad X_3=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix},$$

$$Y_1=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\qquad Y_2=\begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix},\qquad Y_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Then $\{X_1, X_2, X_3, Y_1, Y_2, Y_3\}$, $\{X_1, X_2, X_3\}$ and $\{Y_1, Y_2\}$ are bases of the Lie algebras of $\boldsymbol{SL}(2, \boldsymbol{C})$, $\boldsymbol{SU}(2)$ and $L(1)$ respectively. We have the following relations:

$$[X_1, X_2]=2X_3,\qquad [X_2, X_3]=2X_1,\qquad [X_3, X_1]=2X_2,\qquad [Y_1, Y_2]=0,$$
$$[X_1, Y_1]=[X_2, Y_2]=Y_3,\qquad [X_1, Y_2]=[Y_1, X_2]=X_3,$$
$$[X_3, Y_1]=[Y_3, Y_2]=2Y_2,\qquad [Y_2, X_3]=[Y_3, Y_1]=2Y_1,$$
$$[X_1, Y_3]=2X_1-4Y_1,\qquad [X_2, Y_3]=2X_2-4Y_2,\qquad [X_3, Y_3]=0.$$

Let $\{x_1, x_2, x_3\}$ be a canonical coordinates of the second kind at the identity element of $\boldsymbol{SU}(2)$ with respect to the base $\{X_1, X_2, -X_3\}$ such that

$$x_i((\exp(-u_3X_3))(\exp(u_2X_2))(\exp(u_1X_1)))=u_i\qquad \text{for } i=1, 2, 3.$$

THEOREM 2.1. *Let $\phi:\boldsymbol{SL}(2, \boldsymbol{C})\times M\to M$ be an analytic $\boldsymbol{SL}(2, \boldsymbol{C})$ action on M. Under the following two conditions:*

(1) *the restricted $\boldsymbol{SU}(2)$ action is almost free,*

(2) *each isotropy group contains a subgroup conjugate to $L(1)$,*

the fixed point set $F(L(1), M)$ is an analytic submanifold of codimension two.

PROOF. We can find an analytic $\boldsymbol{SU}(2)$ invariant Riemannian metric on M. Then for each $p\in M$ there is an $\boldsymbol{SU}(2)$ equivariant analytic local isomorphism $f^p:\boldsymbol{SU}(2)\times \boldsymbol{D}^m\to M$ such that $f^p(e, 0)=p$, by the condition (1) and the differentiable slice theorem, where $m=\dim M-3$ and $\boldsymbol{D}^m$ is the unit m-disk. Hence we can find an analytic coordinate system $\{x_1, x_2, x_3, y_1, \cdots, y_m\}$ at $p\in M$ and a cubic neighborhood V of p with respect to this system which satisfy the following conditions:

(a) $x_i(p)=y_j(p)=0\qquad (1\leqq i\leqq 3,\ 1\leqq j\leqq m)$,

(b) $\phi^+(X_3)_q=\left(\dfrac{\partial}{\partial x_3}\right)_q\qquad (q\in V)$,

(c) $\phi^+(X_i)_q=\sum_{j=1}^{3}\lambda_{ij}(x_1(q), x_2(q), x_3(q))\left(\dfrac{\partial}{\partial x_j}\right)_q\qquad (q\in V;\ i=1, 2)$,

where λ_{ij}'s are analytic functions of three variables. In terms of these conditions, we see that the equalities

$$(\alpha)\qquad \begin{aligned} x_3(\phi(\exp(tX_3), q)) &= x_3(q)-t,\\ x_i(\phi(\exp(tX_3), q)) &= x_i(q)\qquad (i=1, 2),\\ y_j(\phi(\exp(tX_3), q)) &= y_j(q)\qquad (j=1, 2, \cdots, m)\end{aligned}$$

hold whenever $q \in V$ and $|t|$ is sufficiently small. Since the vectors $\psi^+(X_1)_q$, $\psi^+(X_2)_q$, $\psi^+(X_3)_q$, $(\partial/\partial y_1)_q, \cdots, (\partial/\partial y_m)_q$ form a base of the tangent space at $q \in V$, by the condition (1), we can express

$$\psi^+(Y_i)_q = \sum_{j=1}^{3} \alpha_{ij}(q)\psi^+(X_j)_q + \sum_{k=1}^{m} \beta_{ik}(q)\left(\frac{\partial}{\partial y_k}\right)_q \qquad (q \in V\,;\ 1 \leqq i \leqq 3),$$

where α_{ij}'s and β_{ik}'s are analytic functions on V.

To simplify the notation, we set $Z^+ = \psi^+(Z)$. We obtain the following equations:

$$(\beta) \qquad \begin{aligned} &X_1^+(\alpha_{13}) = \alpha_{33} - 2\alpha_{12}, \qquad && X_2^+(\alpha_{13}) = 2\alpha_{11} - 1, \qquad && X_3^+(\alpha_{13}) = 2\alpha_{23}, \\ &X_1^+(\alpha_{23}) = 1 - 2\alpha_{22}, && X_2^+(\alpha_{23}) = \alpha_{33} + 2\alpha_{21}, && X_3^+(\alpha_{23}) = -2\alpha_{13}. \end{aligned}$$

Since the computation is fairly similar, we shall show only the first equation. We have

$$[X_1^+, Y_1^+] = \sum_j X_1^+(\alpha_{1j})X_j^+ + \sum_j \alpha_{1j}[X_1^+, X_j^+] + \sum_k X_1^+(\beta_{1k})\frac{\partial}{\partial y_k} + \sum_k \beta_{1k}\left[X_1^+, \frac{\partial}{\partial y_k}\right],$$

where $[X_1^+, \partial/\partial y_k] = 0$ by the condition (c). Since ψ^+ is a Lie algebra homomorphism, we obtain $X_1^+(\alpha_{13}) = \alpha_{33} - 2\alpha_{12}$.

For an analytic function f on V, we can express

$$f(q) = f^*(x_1(q), x_2(q), x_3(q), y_1(q), \cdots, y_m(q)) \qquad (q \in V),$$

where f^* is an analytic function of $m+3$ variables. Let us assume $p \in F(L(1), M)$ in the following. We obtain from the equations (β)

$$\left(\frac{D(\alpha_{13}^*, \alpha_{23}^*, x_3^*)}{D(x_1, x_2, x_3)}\right)_0 \times \det\begin{pmatrix} \lambda_{11}(0) & \lambda_{12}(0) \\ \lambda_{21}(0) & \lambda_{22}(0) \end{pmatrix} = 1 + (\alpha_{33}(p))^2 \neq 0,$$

because the equalities $\alpha_{1j}(p) = \alpha_{2j}(p) = 0$ hold for $1 \leqq j \leqq 3$. The implicit function theorem gives the following: there exist two analytic functions h_1, h_2 of m variables defined on a neighborhood of the origin and there exists a small cubic neighborhood V_1 of p contained in V such that $h_1(0) = h_2(0) = 0$ and if $q \in V_1$, then

(d) $\quad -1 < \alpha_{ii}(q) < 1 \qquad$ for $i = 1, 2$,

and

(e) $\quad x_3(q) = \alpha_{13}(q) = \alpha_{23}(q) = 0 \qquad$ iff

$\qquad x_3(q) = 0$ and $x_i(q) = h_i(y_1(q), \cdots, y_m(q))$ for $i = 1, 2$.

Denote by W the set of points $q \in V_1$ whose coordinates satisfy the conditions:

$$x_i(q) = h_i(y_1(q), \cdots, y_m(q)) \qquad \text{for } i = 1, 2.$$

The set W is an $(m+1)$-dimensional analytic submanifold of V_1. By the equalities (α) and the condition (e), we see that W contains the intersection $V_1 \cap F(L(1), M)$.

We shall show conversely that W is contained in $V_1 \cap F(L(1), M)$. Let $q \in W$.

There exists a real number t such that for $q'=\psi(\exp(tX_3), q)$ the equations $x_3(q')=0$, $x_i(q')=x_i(q)$ for $i=1, 2$ and $y_j(q')=y_j(q)$ for $j=1, \cdots, m$ hold and $q'\in W$ from the equalities (α), and hence $\alpha_{13}(q')=\alpha_{23}(q')=0$ by the condition (e). If $q'\in F(L(1), M)$ then $q=\psi(\exp(-tX_3), q')$ is also contained in $F(L(1), M)$. So we can assume $\alpha_{13}(q)=\alpha_{23}(q)=0$ without loss of generality. Let $\mathfrak{g}$ be the isotropy algebra at q, that is, the Lie algebra of the isotropy group at q. By the condition (2), $\mathfrak{g}$ contains an abelian subalgebra $\mathfrak{g}_1$ conjugate to the Lie algebra of $L(1)$. We shall show that $\mathfrak{g}_1$ is equal to the Lie algebra of $L(1)$.

First we assume that $\mathfrak{g}_1$ is generated by Z_1, Z_2 of the form

$$Z_i = Y_i + \sum_{j=1}^{3} a_{ij}X_j \qquad (i=1, 2).$$

Since $\mathfrak{g}_1$ is abelian, we have $[Z_1, Z_2]=0$ and hence we obtain the following identities: $a_{13}=a_{23}=0$, $a_{12}=a_{21}$. Then, since $\mathfrak{g}_1$ is conjugate to the Lie algebra of $L(1)$, we obtain

$$a_{11} = a_{22} = a_{12} = a_{21} = 0 \qquad \text{or} \qquad 1+a_{11} = 1+a_{22} = 0.$$

Since $\mathfrak{g}_1$ is contained in the isotropy algebra at q, we have $\psi^+(Z_i)_q=0$ for $i=1, 2$ and hence $a_{ii}=-\alpha_{ii}(q)$ for $i=1, 2$. Hence the second case does not occur by the condition (d). The first case implies $Z_i=Y_i$ for $i=1, 2$ and hence $\mathfrak{g}_1$ is equal to the Lie algebra of $L(1)$.

By the condition (1), it remains only to consider the case that $\mathfrak{g}_1$ is generated by Z_1, Z_2 of the form

$$Z_1 = Y_3 + a_1Y_1 + a_2Y_2 + \sum_j b_jX_j, \qquad Z_2 = c_1Y_1 + c_2Y_2 + \sum_j d_jX_j.$$

We have $d_3=0$ by $\psi^+(Z_2)_q=0$ and $\alpha_{13}(q)=\alpha_{23}(q)=0$. Then by the relation $[Z_1, Z_2]=0$ we obtain the following identities:

$$d_1 = -b_3d_2, \qquad d_2 = b_3d_1, \qquad c_1 = c_2b_3-2d_1, \qquad c_2 = -c_1b_3-2d_2.$$

Then we obtin $d_1=d_2=0$ and $c_1=c_2=0$ which implies $Z_2=0$. This is a contradiction.

Consequently, we see that $W=V_1\cap F(L(1), M)$ and hence we see that $F(L(1), M)$ is an analytic submanifold of codimension two. q. e. d.

3. Certain $\boldsymbol{Sp}(n)$ actions.

3.1. In this section, we shall show the following result.

THEOREM 3.1. *Let Σ^k be an integral homology k-sphere with a non-trivial smooth $\boldsymbol{Sp}(n)$ action. Suppose $n\geqq 7$.* (i) *If $k<4n$, then $k=4n-1$ and the $\boldsymbol{Sp}(n)$ manifold Σ^{4n-1} is equivariantly diffeomorphic to the homogeneous space $\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-1)$.* (ii) *If $4n\leqq k\leqq 8n-2$, then there is an equivariant decomposition*

$\Sigma^k \cong \partial(\boldsymbol{D}^{4n}\times Y)$ as a smooth $\boldsymbol{Sp}(n)$ manifold, where Y is a compact orientable acyclic $(k-4n+1)$-manifold with a trivial $\boldsymbol{Sp}(n)$ action, and $\boldsymbol{D}^{4n}$ is the $4n$-disk with a standard $\boldsymbol{Sp}(n)$ action.

In the following, let X be a closed connected manifold with a non-trivial smooth $\boldsymbol{Sp}(n)$ action. Put

$$\begin{aligned} F_{(i)} &= \{x\in X : \boldsymbol{Sp}(n-i)\subset \boldsymbol{Sp}(n)_x \subset \boldsymbol{Sp}(n-i)\times \boldsymbol{Sp}(i)\},\\ F^0_{(i)} &= \{x\in X : \boldsymbol{Sp}(n)^0_x = \boldsymbol{Sp}(n-i)\},\\ X_{(i)} &= \boldsymbol{Sp}(n)\cdot F_{(i)}, \qquad X^0_{(i)} = \boldsymbol{Sp}(n)\cdot F^0_{(i)}. \end{aligned}$$

Here $\boldsymbol{Sp}(n)_x$ and $\boldsymbol{Sp}(n)^0_x$ denote the isotropy group at x and its identity component, respectively. Here we state the followings.

PROPOSITION 3.2 (Nakanishi-Uchida [**5**], § 1). *Suppose $n\geqq 7$ and* $\dim X<8n$. *Then $X=X_{(i)}\cup X_{(i+1)}$ for some $i=0, 1, 2$; and if $X_{(i)}$ and $X_{(i+1)}$ are both non-empty, then the codimension of each connected component of $F_{(i)}$ in X is equal to* $4(i+1)(n-i)$.

PROPOSITION 3.3 (Wada [**11**], § 1). *Suppose $X=X_{(0)}\cup X_{(1)}$. Then there is a compact connected $\boldsymbol{Sp}(1)$ manifold W such that the $\boldsymbol{Sp}(1)$ action is free on the boundary ∂W and the $\boldsymbol{Sp}(n)$ manifold X is equivariantly diffeomorphic to $\partial(\boldsymbol{D}^{4n}\times W)/\boldsymbol{Sp}(1)$. Here $\boldsymbol{Sp}(n)$ acts naturally on $\boldsymbol{D}^{4n}$ and trivially on W, and $\boldsymbol{Sp}(1)$ acts on $\boldsymbol{D}^{4n}$ as right scalar multiplication.*

PROPOSITION 3.4. *Let T_i be a maximal torus of $\boldsymbol{Sp}(n-i)$. If $2i<n$, then $F(T_i, X^0_{(i)})=F^0_{(i)}$ and $F(T_i, X^0_{(j)})$ is empty for $i<j$.*

LEMMA 3.5 (Hsiang-Hsiang [**4**], Proposition 2.3). *Suppose $2i<n$. Let K be a closed connected subgroup of $\boldsymbol{Sp}(i)$. Let ρ be a real representation of K and $\alpha_\xi(\rho)$ be the vector bundle associated with the principal bundle*

$$\xi : K \longrightarrow \boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-i) \longrightarrow \boldsymbol{Sp}(n)/(\boldsymbol{Sp}(n-i)\times K).$$

Then, $P_1(\boldsymbol{Sp}(n)/(\boldsymbol{Sp}(n-i)\times K))+P_1(\alpha_\xi(\rho))=0$ if and only if K consists of the identity element alone.

PROPOSITION 3.6 (Hsiang-Hsiang [**4**], Theorem 2.3). *If $P_1(X)=0$ for the $\boldsymbol{Sp}(n)$ manifold X, then $F_{(i)}=F^0_{(i)}$ and $X_{(i)}=X^0_{(i)}$ for each $i<n/2$.*

The proof of Proposition 3.4 is straightforward. The statements of Lemma 3.5 and Proposition 3.6 are simple modifications of the original results of Hsiang brothers.

3.2. Now we shall prove Theorem 3.1. In the remaining of this section, we suppose that $n\geqq 7$, $k\leqq 8n-2$ and the $\boldsymbol{Sp}(n)$ manifold X is an integral homology k-sphere.

(i) Consider the case $X=X_{(0)}\cup X_{(1)}$, such that $X_{(0)}$ is non-empty. By Proposition 3.3, there is an equivariant decomposition $X\cong\partial(\boldsymbol{D}^{4n}\times W)/\boldsymbol{Sp}(1)$, where W is a compact connected orientable $\boldsymbol{Sp}(1)$ manifold such that the $\boldsymbol{Sp}(1)$ action is free on the non-empty boundary ∂W. Put $B=(S^{4n-1}\times W)/\boldsymbol{Sp}(1)$. Since X is an integral homology sphere, we obtain $H_r(B;\boldsymbol{Z})=0$ for $0<r<4n-1$ by a standard method. Considering the Serre spectral sequence for the principal $\boldsymbol{Sp}(1)$ bundle $S^{4n-1}\times W$ over B, we obtain an isomorphism

$$f^x_* : H_3(\boldsymbol{Sp}(1);\boldsymbol{Z})\cong H_3(W;\boldsymbol{Z})$$

for each $x\in W$, where $f^x(g)=gx$. Hence we see that the $\boldsymbol{Sp}(1)$ action on W is free, and we can consider the sphere bundle

$$S^{4n-1}\longrightarrow B\longrightarrow W/\boldsymbol{Sp}(1).$$

Put $Y=W/\boldsymbol{Sp}(1)$. Then Y is a compact connected orientable manifold with non-empty boundary and $\dim Y\leqq 4n-1$. Considering the Gysin sequence for the above sphere bundle, we obtain $H_r(Y;\boldsymbol{Z})=0$ for $0<r<4n-1$, and hence Y is integrally acyclic. Now we see that the principal $\boldsymbol{Sp}(1)$ bundle W over Y has a cross-section by the obstruction theory. Hence W is equivariantly diffeomorphic to $\boldsymbol{Sp}(1)\times Y$. Then

$$X\cong\partial(\boldsymbol{D}^{4n}\times W)/\boldsymbol{Sp}(1)\cong\partial(\boldsymbol{D}^{4n}\times Y).$$

(ii) Consider the case $X=X_{(i)}$ $(i=1, 2, 3)$. Then there is an equivariant decomposition:

$$X=X_{(i)}\cong((\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-i))\times F_{(i)})/\boldsymbol{Sp}(i).$$

Since $P_1(X)=0$, we obtain $F_{(i)}=F^0_{(i)}$ and $X_{(i)}=X^0_{(i)}$ by Proposition 3.6. By Proposition 3.4 and Smith's theorem, we see that $F_{(i)}$ is an integral homology p-sphere, where $p=k-4i(n-i)$. Considering the Serre spectral sequence for the fibration

$$F_{(i)}\longrightarrow X\longrightarrow \boldsymbol{Sp}(n)/(\boldsymbol{Sp}(n-i)\times\boldsymbol{Sp}(i)),$$

we obtain $p=3$. Then we see that $i=1$ and $k=4n-1$, because $\boldsymbol{Sp}(i)$ acts almost freely on the 3-manifold $F_{(i)}$. Since

$$H_1(X;\boldsymbol{Z})\cong H_3(X;\boldsymbol{Z})\cong 0,$$

we obtain $X\cong\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-1)$.

(iii) Finally we shall show that if $X=X_{(j)}\cup X_{(j+1)}$ $(j=1, 2)$ and $X_{(j)}$ is non-empty then $X_{(j+1)}$ is empty. The result is obvious for $j=2$ by the second statement of Proposition 3.2. Now we assume that $X=X_{(1)}\cup X_{(2)}$ and both $X_{(1)}$ and $X_{(2)}$ are non-empty. Since $P_1(X)=0$, we obtain $F_{(i)}=F^0_{(i)}$ and $X_{(i)}=X^0_{(i)}$ for

$i=1, 2$ by Proposition 3.6. Let T_i denote the standard maximal torus of $\boldsymbol{Sp}(n-i)$. Then $F(T_i, X)$ is an integral homology n_i-sphere ($i=1, 2$) by Smith's theorem. By Proposition 3.4, we see

$$F(T_1, X) = F_{(1)}, \qquad F(T_2, X) = F(T_2, X_{(1)}) \cup F_{(2)}.$$

Put $F_1 = F(T_2, X_{(1)})$ and $F_2 = F(T_2, X)$. Considering the equivariant decomposition of $X_{(1)}$, we obtain a fibration

$$F_{(1)} \longrightarrow F_1 \longrightarrow S^4.$$

By the Serre spectral sequence for this fibration, there is an isomorphism

(a) $\quad H^*(F_1; \boldsymbol{Q}) \cong H^*(S^{n_1} \times S^4; \boldsymbol{Q}) \quad$ or $\quad H^*(F_1; \boldsymbol{Q}) \cong H^*(S^7; \boldsymbol{Q})$.

Since $\operatorname{codim} X_{(1)} = 4n-4$, we obtain an isomorphism $H^r(X; \boldsymbol{Z}) \cong H^r(X_{(2)}; \boldsymbol{Z})$ for $r < 4n-5$. Considering the equivariant decomposition of $X_{(2)}$, we obtain an isomorphism

(b) $\quad H^r(F_{(2)}; \boldsymbol{Z}) \cong H^r(\boldsymbol{Sp}(2); \boldsymbol{Z}) \quad$ for $r < 4n-5$.

On the other hand, we see

(c) $\quad n_2 = n_1 + 8, \quad 3 \leqq n_1 \leqq 6 \quad$ and $\quad \dim F_1 = n_1 + 4$.

Moreover, we obtain an isomorphism

(d) $\quad H^{r-3}(F_1; \boldsymbol{Z}) \cong H^{r+1}(F_2, F_{(2)}; \boldsymbol{Z}) \cong H^r(F_{(2)}; \boldsymbol{Z})$

for $0 < r < n_2 - 1$, by the Thom isomorphism and the fact that F_2 is an integral homology n_2-sphere. Combining (a), (b), (c) and (d), we obtain a contradiction.

Here we complete the proof of Theorem 3.1.

4. Analytic $\boldsymbol{Sp}(n, \boldsymbol{C})$ actions.

4.1. Let $\psi: \boldsymbol{Sp}(n, \boldsymbol{C}) \times M \to M$ be a non-trivial analytic action on a connected paracompact m-manifold. Suppose that $(*)$ each isotropy group of the restricted $\boldsymbol{Sp}(n)$ action contains a subgroup conjugate to $\boldsymbol{Sp}(n-1)$ and $n \geqq 4$. Put $F = F(\boldsymbol{Sp}(n, \boldsymbol{C}), M)$ and let $p \in F$.

By a theorem of Guillemin and Sternberg [3], there exists an analytic system of coordinates $(U; u_1, \cdots, u_m)$, with origin at p, and there exists $a_{ij} \in \mathfrak{sp}(n, \boldsymbol{C})^*$ such that

$$\psi^+(X)_q = -\sum_{i,j} a_{ij}(X) u_j(q) \frac{\partial}{\partial u_i} \qquad \text{for } X \in \mathfrak{sp}(n, \boldsymbol{C}),\ q \in U.$$

Here the correspondence $X \to (a_{ij}(X))$ defines a Lie algebra homomorphism of $\mathfrak{sp}(n, \boldsymbol{C})$ into $\mathfrak{gl}(m, \boldsymbol{R})$. Since $\boldsymbol{Sp}(n, \boldsymbol{C})$ is simply connected, there is an analytic homomorphism $\rho: \boldsymbol{Sp}(n, \boldsymbol{C}) \to \boldsymbol{GL}(m, \boldsymbol{R})$ such that $(d\rho)(X) = (a_{ij}(X))$ for $X \in \mathfrak{sp}(n, \boldsymbol{C})$.

Since $\boldsymbol{Sp}(n, \boldsymbol{C})$ is semi-simple, we see that ρ is completely reducible. Let V be a representation space of a non-trivial irreducible factor of ρ. From the assumption (*), we obtain the following decomposition:

$$V-\{0\} \cong (\boldsymbol{Sp}(n, \boldsymbol{C})\times(F(L(n), V)-\{0\}))/N(n),$$

by Corollary 1.3. Then we obtain $\dim_{\boldsymbol{R}}V=4n$ by considering fundamental groups, and hence V is the representation space of the standard representation ν_n by Weyl's formula. Therefore we see that $\rho\cong\nu_n\oplus\theta^{m-4n}$ by the assumption (*), where θ^t is a trivial real representation of degree t. Consequently, we see that there exists an analytic system of coordinates $(U; v_1, \cdots, v_m)$, with origin at p, such that

$$\psi^+(X)_q = -\sum_{i,j=1}^{2n}\left\{(\alpha_{ij}v_j-\beta_{ij}v_{2n+j})\frac{\partial}{\partial v_i}+(\alpha_{ij}v_{2n+j}+\beta_{ij}v_j)\frac{\partial}{\partial v_{2n+i}}\right\}$$

for $X\in\mathfrak{sp}(n, \boldsymbol{C})$ and $q\in U$, where $v_k=v_k(q)$ and $\alpha_{ij}+\sqrt{-1}\beta_{ij}$ is the (i, j)-component of X. Let k be an analytic isomorphism of U onto an open set of $\boldsymbol{R}^m$ defined by $k(q)=(v_1(q), \cdots, v_m(q))$. There is a positive real number r such that $\boldsymbol{D}_r^{4n}\times\boldsymbol{D}_r^{m-4n}$ is contained in $k(U)$, where $\boldsymbol{D}_r^t=\{x\in\boldsymbol{R}^t: \|x\|<r\}$. Then we see that (cf. [7], Lemma 3.1) $k^{-1}: \boldsymbol{D}_r^{4n}\times\boldsymbol{D}_r^{m-4n}\to U$ is extendable uniquely to an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism h' of $\boldsymbol{R}^{4n}\times\boldsymbol{D}_r^{m-4n}$ onto an open set of M, because the standard $\boldsymbol{Sp}(n, \boldsymbol{C})$ action on $\boldsymbol{R}^{4n}-\{0\}$ is transitive and its isotropy group is $L(n)$. Then $W=h'(0\times\boldsymbol{D}_r^{m-4n})$ is an open neighborhood of p in F. Define $h: \boldsymbol{C}^{2n}\times W\to M$ by

$$h(u_1+\sqrt{-1}v_1, \cdots, u_{2n}+\sqrt{-1}v_{2n}, h'(0, x)) = h'(u_1, \cdots, u_{2n}, v_1, \cdots, v_{2n}, x)$$

for $x\in\boldsymbol{D}_r^{m-4n}$. Then h is an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism of $\boldsymbol{C}^{2n}\times W$ onto an open set of M such that $h(0, q)=q$ for $q\in W$.

Consequently, we obtain a family $\{(W_\alpha, h_\alpha), \alpha\in\Lambda\}$ such that $\{W_\alpha, \alpha\in\Lambda\}$ is an open covering of F, and each h_α is an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism of $\boldsymbol{C}^{2n}\times W_\alpha$ onto an open set of M such that $h_\alpha(0, q)=q$ for $q\in W_\alpha$. Put

$$N=\bigcup_\alpha h_\alpha(\boldsymbol{C}^{2n}\times W_\alpha), \qquad E=F(L(n), N-F).$$

Then N is the smallest $\boldsymbol{Sp}(n, \boldsymbol{C})$ invariant open neighborhood of F in M, E is an analytic submanifold of N and the multiplicative group $\boldsymbol{C}_0$ of non-zero complex numbers acts analytically on E via the natural isomorphism $\boldsymbol{C}_0\cong N(n)/L(n)$. Let k_α be a $\boldsymbol{C}_0$ equivariant analytic isomorphism of $\boldsymbol{C}_0\times W_\alpha$ onto an open set of E defined by

$$k_\alpha(\lambda, q) = h_\alpha(\lambda e_1, q) \qquad \text{for } \lambda\in\boldsymbol{C}_0,\ q\in W_\alpha,$$

where e_1 is the first vector of the standard base of $\boldsymbol{C}^{2n}$. Define $\pi: E\to F$ by

$\pi k_\alpha(\lambda, q)=q$ for $\lambda\in \boldsymbol{C}_0$ and $q\in W_\alpha$. We see that (cf. [7], Theorem 3.7) π is an analytic principal $\boldsymbol{C}_0$ bundle, and we can define an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism f of $(\boldsymbol{C}^{2n}\times E)/\boldsymbol{C}_0$ onto N by

$$f([u, k_\alpha(\lambda, q)]) = h_\alpha(\lambda u, q) \qquad \text{for } u\in \boldsymbol{C}^{2n},\ \lambda\in \boldsymbol{C}_0,\ q\in W_\alpha.$$

In particular, we see that $f([0, x])=\pi(x)$ for $x\in E$.

Summing up the above discussion, we obtain the following.

THEOREM 4.1. *Let $\psi: \boldsymbol{Sp}(n, \boldsymbol{C})\times M\to M$ be a non-trivial analytic action on a connected paracompact m-manifold. Suppose that each isotropy group of the restricted $\boldsymbol{Sp}(n)$ action contains a subgroup conjugate to $\boldsymbol{Sp}(n-1)$ and $n\geqq 4$. Put $F=F(\boldsymbol{Sp}(n, \boldsymbol{C}), M)$. Then F is an $(m-4n)$-dimensional analytic submanifold of M, and there exist an analytic left principal $\boldsymbol{C}_0$ bundle $\pi: E\to F$ and an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism f of $(\boldsymbol{C}^{2n}\times E)/\boldsymbol{C}_0$ onto an open set of M such that $f([0, x])=\pi(x)$ for $x\in E$. In addition, the image of f is the smallest $\boldsymbol{Sp}(n, \boldsymbol{C})$ invariant open neighborhood of F in M.*

4.2. Let V be an analytic vector bundle over a connected paracompact analytic manifold X. Let $i: X\to V$ be the zero section. Then it follows from a calculation of transition functions that there is an isomorphism $i^*\tau(V)\cong V\oplus\tau(X)$ as analytic vector bundles. Here $\tau(\ )$ denotes the tangent bundle. Since V is a connected paracompact analytic manifold, there exists an analytic embedding f of V into $\boldsymbol{R}^N$ such that $f(V)$ is a closed analytic submanifold of $\boldsymbol{R}^N$ (Grauert [**2**], Theorem 3). It follows that there is an isomorphism $\tau(V)\oplus\nu\cong \boldsymbol{R}^N\times V$ as analytic vector bundles. Here ν denotes the normal bundle. Therefore there is an isomorphism

$$V\oplus\tau(X)\oplus i^*\nu \cong \boldsymbol{R}^N\times X$$

as analytic vector bundles. Hence we obtain the following.

LEMMA 4.2. *Let V be an analytic vector bundle over a connected paracompact analytic manifold X. Then V is embedded in a product vector bundle as an analytic subbundle.*

COROLLARY 4.3. *Let V be an analytic vector bundle over a connected paracompact analytic manifold X. If V has a C^∞ cross-section which is everywhere non-zero, then V has an analytic cross-section which is everywhere non-zero.*

PROOF. By Lemma 4.2, there exist an analytic vector bundle V' over X and an isomorphism $V\oplus V'\cong \boldsymbol{R}^N\times X$ as analytic vector bundles. Let $\sigma: X\to V$ be a C^∞ cross-section which is everywhere non-zero. Since $C^\omega(X, \boldsymbol{R}^N)$ is dense in $C^\infty(X, \boldsymbol{R}^N)$ with respect to C^∞-topology (Whitney [**12**], Part III), we can approximate σ by an analytic cross-section which is everywhere non-zero by a standard method. Here $C^r(X, \boldsymbol{R}^N)$ denotes the set of C^r-mappings from X into $\boldsymbol{R}^N$. q. e. d.

5. Analytic $\boldsymbol{Sp}(n, \boldsymbol{C})$ actions on spheres.

5.1. Let $\psi: \boldsymbol{Sp}(n, \boldsymbol{C})\times\Sigma\to\Sigma$ be an analytic action on a closed orientable analytic manifold Σ which is an integral homology k-sphere. Suppose that $n\geqq 4$ and there is an $\boldsymbol{Sp}(n)$ equivariant smooth decomposition $(*)$ $\Sigma\cong\partial(\boldsymbol{D}^{4n}\times Y)$, with respect to the restricted $\boldsymbol{Sp}(n)$ action (see Theorem 3.1).

Put $F=F(\boldsymbol{Sp}(n, \boldsymbol{C}), \Sigma)$ and denote by N the smallest $\boldsymbol{Sp}(n, \boldsymbol{C})$ invariant open neighborhood of F in Σ. By Theorem 4.1, F is a $(k-4n)$-dimensional closed analytic submanifold of Σ, and there exist an analytic left principal $\boldsymbol{C}_0$ bundle π_1: $E\to F$ and an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism f' of $(\boldsymbol{C}^{2n}\times E)/\boldsymbol{C}_0$ onto N such that $f'([0, x])=\pi_1(x)$ for $x\in E$.

Put $U=F(\boldsymbol{Sp}(n-1, \boldsymbol{C}), \Sigma-F)$ and $U_1=F(L(n), \Sigma-F)$. We see that U is an analytic submanifold of codimension $4n-4$ in Σ, the identity component $\boldsymbol{MSL}(2, \boldsymbol{C})$ of the centralizer of $\boldsymbol{Sp}(n-1, \boldsymbol{C})$ acts naturally on U and the restricted $\boldsymbol{MSU}(2)$ action on U is free, by Lemma 1.2 and the decomposition $(*)$. Denote by U^* the orbit space of the free $\boldsymbol{MSU}(2)$ action on U and $\pi': U\to U^*$ the natural projection. By Theorem 2.1 and a discussion in § 1.3, we see that U_1 is an analytic submanifold of codimension two in U. By Corollary 1.3, there is an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism $\Sigma-F\cong(\boldsymbol{C}_0^{2n}\times U_1)/\boldsymbol{C}_0$, where $\boldsymbol{C}_0^{2n}=\boldsymbol{C}^{2n}-\{0\}\cong\boldsymbol{Sp}(n, \boldsymbol{C})/L(n)$.

By Theorem 1.1, for each $x\in U$ there exists $g\in\boldsymbol{MSU}(2)$ such that $gx\in U_1$; if $x\in U_1$ and $gx\in U_1$ for some $g\in\boldsymbol{MSU}(2)$ then $g\in\boldsymbol{MSU}(2)\cap N(n)$. Put $\pi_2=\pi'|U_1$. By the above discussion, we see that $\pi_2: U_1\to U^*$ is a projection of a principal $\boldsymbol{U}(1)$ bundle, where $\boldsymbol{U}(1)$ acts on U_1 via the natural isomorphism $\boldsymbol{U}(1)\cong\boldsymbol{MSU}(2)\cap N(n)$. By the decomposition $(*)$, we see that U^* is homotopy equivalent to Y which is acyclic, and hence U^* is acyclic. Therefore $U_1\cong\boldsymbol{U}(1)\times U^*$ as a smooth $\boldsymbol{U}(1)$ manifold. On the other hand, $F(L(n), N-F)$ is $\boldsymbol{C}_0$ equivariantly analytically isomorphic to E via f'. Since $F(L(n), N-F)$ is an open set of U_1, we see that $E\cong\boldsymbol{U}(1)\times(E/\boldsymbol{U}(1))$ as a smooth $\boldsymbol{U}(1)$ manifold, and $E/\boldsymbol{U}(1)$ is a smooth principal $\boldsymbol{C}_0/\boldsymbol{U}(1)$ bundle over F. Since $\boldsymbol{C}_0/\boldsymbol{U}(1)$ is contractible, we see that the projection π_1 has a smooth cross-section, and hence π_1 has an analytic cross-section by Corollary 4.3. Therefore $E\cong\boldsymbol{C}_0\times F$ as an analytic $\boldsymbol{C}_0$ manifold, and there is an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism f of $\boldsymbol{C}^{2n}\times F$ onto N such that $f(0, x)=x$ for $x\in F$.

Considering the Mayer-Vietoris sequence for the couple $\{U_1, F(L(n), N)\}$, we see that $F(L(n), \Sigma)$ is an integral homology $(k-4n+2)$-sphere, because F is diffeomorphic to ∂Y which is an integral homology $(k-4n)$-sphere.

Summing up the above discussion, we obtain the following.

THEOREM 5.1. *Let $\phi: \boldsymbol{Sp}(n, \boldsymbol{C})\times\Sigma\to\Sigma$ be an analytic action on a closed orientable analytic manifold Σ which is an integral homology k-sphere. Suppose that $n\geqq 4$ and there is an $\boldsymbol{Sp}(n)$ equivariant smooth decomposition $\Sigma\cong\partial(\boldsymbol{D}^{4n}\times Y)$ with respect to the restricted $\boldsymbol{Sp}(n)$ action. Put $F=F(\boldsymbol{Sp}(n, \boldsymbol{C}), \Sigma)$. Then F is a $(k-4n)$-dimensional closed analytic submanifold of Σ which is an integral homology sphere, and there is an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic isomorphism f of $\boldsymbol{C}^{2n}\times F$ onto an open set of Σ such that $f(0, x)=x$ for $x\in F$. Moreover, $F(L(n), \Sigma)$ is a $(k-4n+2)$-dimensional closed analytic submanifold of Σ which is an integral homology sphere, $\boldsymbol{C}_0$ acts on $F(L(n), \Sigma)$ via the natural isomorphism $\boldsymbol{C}_0\cong N(n)/L(n)$, and there is an $\boldsymbol{Sp}(n, \boldsymbol{C})$ equivariant analytic decomposition*

$$\Sigma \cong \boldsymbol{C}^{2n}\times F\bigcup_{\alpha}(\boldsymbol{C}_0^{2n}\times F(L(n), \Sigma-F))/\boldsymbol{C}_0,$$

where α is an equivariant analytic isomorphism of $\boldsymbol{C}_0^{2n}\times F$ onto an open set of $(\boldsymbol{C}_0^{2n}\times F(L(n), \Sigma-F))/\boldsymbol{C}_0$ defined by

$$\alpha(u, x) = [u, f(e_1, x)] \qquad \textit{for } u\in\boldsymbol{C}_0^{2n},\ x\in F.$$

In addition, the restricted $\boldsymbol{U}(1)$ action on $F(L(n), \Sigma-F)$ is free.

5.2. Let $\mu: \boldsymbol{C}_0\times\Sigma_1\to\Sigma_1$ be an analytic action on a closed orientable analytic manifold Σ_1 which is an integral homology m-sphere. Put $F=F(\boldsymbol{C}_0, \Sigma_1)$. We say that (Σ_1, μ) satisfies a *condition* (P) iff F is an $(m-2)$-dimensional analytic submanifold which is an integral homology sphere and there exists a $\boldsymbol{C}_0$ equivariant analytic isomorphism j of $\boldsymbol{C}\times F$ onto an open set of Σ_1 such that $j(0, x)=x$ for $x\in F$. Such an action has been studied by Uchida ([8], § 6).

Construct an analytic manifold Σ by

$$\Sigma = \boldsymbol{C}^{2n}\times F\bigcup_{\alpha}(\boldsymbol{C}_0^{2n}\times(\Sigma_1-F))/\boldsymbol{C}_0,$$

where α is an analytic isomorphism of $\boldsymbol{C}_0^{2n}\times F$ onto an open set of $(\boldsymbol{C}_0^{2n}\times(\Sigma_1-F))/\boldsymbol{C}_0$ defined by $\alpha(u, x)=[u, j(1, x)]$ for $u\in\boldsymbol{C}_0^{2n}$, $x\in F$. We see that Σ is an integral homology $(m+4n-2)$-sphere by the Mayer-Vietoris sequence, because the restricted $\boldsymbol{U}(1)$ action on Σ_1-F is free by the Smith theory and its orbit manifold is acyclic by the Gysin sequence. Considering the natural $\boldsymbol{Sp}(n, \boldsymbol{C})$ action on Σ, we see that the induced $\boldsymbol{C}_0$ action on $F(L(n), \Sigma)$ is naturally isomorphic to the action μ on Σ_1. Combining Theorem 3.1 and Theorem 5.1, we obtain the following.

COROLLARY 5.2. *For $n\geqq 7$ and $2\leqq m\leqq 4n$, each non-trivial analytic $\boldsymbol{Sp}(n, \boldsymbol{C})$ action on an integral homology $(m+4n-2)$-sphere Σ is characterized by the induced $\boldsymbol{C}_0$ action on $F(L(n), \Sigma)$ satisfying the condition* (P), *where $F(L(n), \Sigma)$ is an integral homology m-sphere.*

5.3. For each real number y, we can define an analytic $\boldsymbol{GL}(k, \boldsymbol{C})$ action

ξ_y on the unit $(2k-1)$-sphere S^{2k-1} of $\boldsymbol{C}^k$ by

$$\xi_y(A, u) = \|Au\|^{-1-iy}Au \qquad \text{for } A\in \boldsymbol{GL}(k, \boldsymbol{C}),\ u\in S^{2k-1}.$$

Considering the restricted $\boldsymbol{Sp}(n, \boldsymbol{C})$ action for $k=2n$, we obtain an analytic transitive $\boldsymbol{Sp}(n, \boldsymbol{C})$ action χ_y on $(4n-1)$-sphere. Denote by G_y its isotropy group at e_1. Then $L(n)\subset G_y\subset N(n)$ and the factor group $G_y/L(n)$ is isomorphic to the subgroup $\{e^{t(1+iy)}, t\in \boldsymbol{R}\}$ of $C_0\cong N(n)/L(n)$. We see that any transitive $\boldsymbol{Sp}(n, \boldsymbol{C})$ action on the $(4n-1)$-sphere is one of the actions χ_y for some real number y.

Similarly, if $k>2n$, we obtain an analytic $\boldsymbol{Sp}(n, \boldsymbol{C})$ action ψ_y^k on the $(2k-1)$-sphere. We see that the complement of the smallest $\boldsymbol{Sp}(n, \boldsymbol{C})$ invariant open neighborhood of the fixed point set of the action ψ_y^k is equivariantly isomorphic to the homogeneous space $\boldsymbol{Sp}(n, \boldsymbol{C})/G_y$. Therefore we see that if $y\neq y'$ then ψ_y^k and $\psi_{y'}^k$ are still not equivalent as continuous $\boldsymbol{Sp}(n, \boldsymbol{C})$ actions, that is, there is not any equivariant homeomorphism between the actions ψ_y^k and $\psi_{y'}^k$.

6. Analytic actions of $SO(n, C)$ and $SL(n, R)$ on spheres.

6.1. Denote by $\boldsymbol{SO}(n)$ and $\boldsymbol{SO}(n, \boldsymbol{C})$ the group of special orthogonal matrices of degree n, and the group of complex special orthogonal matrices of degree n, respectively, that is

$$\boldsymbol{SO}(n) = \{A\in \boldsymbol{GL}(n, \boldsymbol{R}) : {}^tAA=I_n,\ \det A=1\},$$
$$\boldsymbol{SO}(n, \boldsymbol{C}) = \{A\in \boldsymbol{GL}(n, \boldsymbol{C}) : {}^tAA=I_n,\ \det A=1\}.$$

By the similar way as the proof of Theorem 1.1, we can prove the following: Let G be a connected closed proper subgroup of $\boldsymbol{SO}(n, \boldsymbol{C})$ which contains $\boldsymbol{SO}(n-1)$ for $n\geqq 6$. Then G is one of the following: $\boldsymbol{SO}(n-1)$, $\boldsymbol{SO}(n)$, $\boldsymbol{SO}(n-1, \boldsymbol{C})$ or $h\boldsymbol{GL}(n, \boldsymbol{R})h^{-1}\cap \boldsymbol{SO}(n, \boldsymbol{C})$, where h is the diagonal matrix with diagonal entries $i, 1, \cdots, 1$. Moreover, for each such group G, there exists an isotropy group of the restricted $\boldsymbol{SO}(n)$ action on the homogeneous space $\boldsymbol{SO}(n, \boldsymbol{C})/G$ which does not contain a subgroup conjugate to $\boldsymbol{SO}(n-1)$.

On the other hand, we see that any non-trivial smooth $\boldsymbol{SO}(n)$ action on an integral homology k-sphere has $\boldsymbol{SO}(n-1)$ as a principal isotropy group for $k\leqq 2n-2$ and $n\geqq 10$ (cf. [4], Theorem 3.1; [7], Theorem 4.11). Hence we obtain the following: If $k\leqq 2n-2$ and $n\geqq 10$, then $\boldsymbol{SO}(n, \boldsymbol{C})$ does not act smoothly and non-trivially on any integral homology k-sphere.

6.2. For each real number y, we can define an analytic $\boldsymbol{SO}(n, \boldsymbol{C})$ action ζ_y on the unit $(2n-1)$-sphere S^{2n-1} of $\boldsymbol{C}^n$ by the restriction of the $\boldsymbol{GL}(n, \boldsymbol{C})$ action ξ_y. Let $e_1, \cdots, e_n$ be the standard base of $\boldsymbol{C}^n$. Denote by H_y and $\mathfrak{h}_y$ the isotropy group of the action ζ_y at $(e_1+ie_2)/\sqrt{2}$ and its Lie algebra, respectively. By the definition of ζ_y, we see that $X\in \mathfrak{h}_y$ iff

$$X(e_1+ie_2)=a(X)(1+iy)(e_1+ie_2)$$

for certain real number $a(X)$. We see that $\dim \boldsymbol{SO}(n, \boldsymbol{C})/H_y=2n-3$ and each isotropy group at u is conjugate to $\boldsymbol{SO}(n-1, \boldsymbol{C})$ if u does not belong to the orbit of $(e_1+ie_2)/\sqrt{2}$. With respect to the natural $\mathfrak{h}_y$ action on $\boldsymbol{C}^n$, the complex line generated by e_1+ie_2 is only $\mathfrak{h}_y$ invariant 1-dimensional linear subspace for $n\geqq 4$. Therefore we see that if $y\neq y'$ then ζ_y and $\zeta_{y'}$ are still not equivalent as continuous $\boldsymbol{SO}(n, \boldsymbol{C})$ actions for $n\geqq 4$.

6.3. Denote by $W_n(d)$ the complex hypersurface of $\boldsymbol{C}^{n+1}-\{0\}$ determined by the equation $z_0^d+z_1^2+\cdots+z_n^2=0$ for each positive integer d. Since the natural action of $\boldsymbol{SO}(n, \boldsymbol{C})$ on $\boldsymbol{C}^n$ leaves invariant the quadratic form $z_1^2+\cdots+z_n^2$, we can define naturally an action of $\boldsymbol{SO}(n, \boldsymbol{C})$ on $W_n(d)$.

For each real number y, we can define an analytic one-parameter group ν_y on $W_n(d)$ by

$$\nu_y(t, (z_0, \cdots, z_n))=(e^{2t(1+iy)}z_0,\ e^{dt(1+iy)}z_1,\ \cdots,\ e^{dt(1+iy)}z_n).$$

Denote by $W_y^{2n-1}(d)$ the orbit manifold of the free $\boldsymbol{R}$ action ν_y on $W_n(d)$. We see that $W_y^{2n-1}(d)$ is naturally isomorphic to the Brieskorn variety $W^{2n-1}(d)$. Since the $\boldsymbol{R}$ action ν_y and the $\boldsymbol{SO}(n, \boldsymbol{C})$ action on $W_n(d)$ are commutative, we can define naturally an analytic action of $\boldsymbol{SO}(n, \boldsymbol{C})$ on $W_y^{2n-1}(d)$. We see that if $y\neq y'$ then the $\boldsymbol{SO}(n, \boldsymbol{C})$ actions on $W_y^{2n-1}(d)$ and on $W_{y'}^{2n-1}(d)$ are still not equivalent as continuous actions for $n\geqq 4$.

6.4. We have studied analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on the k-sphere for $k\leqq 2n-2$ in the previous papers [7], [9]. Here we study analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on the $(2n-1)$-sphere.

For each real number y, we can define an analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ action σ_y on the unit $(2n-1)$-sphere S^{2n-1} of $\boldsymbol{C}^n$ by the restriction of the $\boldsymbol{GL}(n, \boldsymbol{C})$ action ξ_y. Let $e_1, \cdots, e_n$ be the standard base of $\boldsymbol{C}^n$ and suppose $n\geqq 3$. Denote by K_y and $\mathfrak{k}_y$ the isotropy group of the action σ_y at $(e_1+ie_2)/\sqrt{2}$ and its Lie algebra, respectively. By the definition of σ_y, we see that $X\in\mathfrak{k}_y$ iff

$$X(e_1+ie_2)=a(X)(1+iy)(e_1+ie_2)$$

for certain real number $a(X)$. With respect to the natural $\mathfrak{k}_y$ action on $\boldsymbol{R}^n$, the subspace spanned by $\{e_1, e_2\}$ is only $\mathfrak{k}_y$ invariant 2-dimensional linear subspace. We see that the orbit of $(e_1+ie_2)/\sqrt{2}$ is open and dense in S^{2n-1}. Hence we see that if $|y|\neq|y'|$, then σ_y and $\sigma_{y'}$ are still not equivalent as continuous $\boldsymbol{SL}(n, \boldsymbol{R})$ actions. On the other hand, we see that σ_y and σ_{-y} are equivalent as analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions, because the equation $\sigma_{-y}(A, u)=\sigma_y(A, \bar{u})$ holds for $A\in\boldsymbol{SL}(n, \boldsymbol{R})$, $u\in S^{2n-1}$, where $\bar{u}=(\bar{u}_1, \cdots, \bar{u}_n)$ for $u=(u_1, \cdots, u_n)$.

References

[1] C. Chevalley, Theory of Lie groups I, Princeton Univ. Press, 1946.
[2] H. Grauert, On Levi's problem and the imbedding of real analytic manifolds, Ann. of Math., **68** (1958), 460-472.
[3] V. W. Guillemin and S. Sternberg, Remarks on a paper of Hermann, Trans. Amer. Math. Soc., **130** (1968), 110-116.
[4] W. C. Hsiang and W. Y. Hsiang, Differentiable actions of compact connected classical groups I, Amer. J. Math., **89** (1967), 705-786.
[5] A. Nakanishi and F. Uchida, Actions of symplectic groups on certain manifolds, Tôhoku Math. J., **36** (1984), 81-98.
[6] R. S. Palais, A global formulation of the Lie theory of transformation groups, Mem. Amer. Math. Soc., 22, 1957.
[7] F. Uchida, Real analytic $SL(n, \boldsymbol{R})$ actions on spheres, Tôhoku Math. J., **33** (1981), 145-175.
[8] F. Uchida, Real analytic $SL(n, \boldsymbol{C})$ actions, Bull. Yamagata Univ. Nat. Sci., **10** (1980), 1-14.
[9] F. Uchida, Actions of special linear groups on a product manifold, Bull. Yamagata Univ. Nat. Sci., **10** (1982), 227-233.
[10] F. Uchida, On the non-existence of smooth actions of complex symplectic group on cohomology quaternion projective spaces, Hokkaido Math. J., **12** (1983), 226-236.
[11] T. Wada, Smooth $Sp(n)$ actions on cohomology quaternion projective spaces, Tôhoku Math. J., **29** (1977), 375-384.
[12] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc., **36** (1934), 63-89.

Fuichi UCHIDA
Department of Mathematics
Faculty of Science
Yamagata University
Yamagata 990
Japan