

On automorphism groups of a curve as linear groups

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Introduction.

Let X be a complete non-singular curve over an algebraically closed field k . Assume that G is a finite group of automorphisms of X . Let χ_q ($q=1, 2, \dots$) denote the character $\text{Tr}(G|H^0(X, \Omega_X^q))$ of the natural representation of G on the space of q -differentials on X . G or χ_q 's have recently been studied with a new importance from their relation with the problems of moduli or Teichmüller space (cf. e. g., [5], [6], [11]).

In the present paper, we confine ourselves to the study of the characters χ_q in the case where G is cyclic and k is of characteristic zero or $k=\mathbb{C}$. We attempt to follow up some part of [5] and [8]. In fact, our aims are (i) to correct a "theorem" in [5] concerning the interrelation between the characters, (ii) to reveal a nature of the sequence $(\chi_q)_{q \geq 1}$, and (iii) to characterize $(\chi_q)_{q \geq 1}$ as a sequence of class functions of G by a special type of mapping $\lambda: G \rightarrow \text{Map}(\mathbb{Z}, \mathbb{Q})$.

We shall give a brief survey of this paper. In §1, we shall introduce for G a surjective group homomorphism $\phi_G: \Gamma \rightarrow G$, where Γ is a group characterized by the Riemann-Hurwitz relation for the covering: $X \rightarrow X/G$. Then we shall give an existence theorem of a cyclic automorphism group in a formulation including ϕ_G (Theorem 1.6). Our basic tool to investigate the characters is the trace formula which says that each χ_q (considered as an unknown) is determined by the information of the homomorphism ϕ_G (cf. (2.1)). In §2, as for (i) we shall show that ϕ_G and hence all of χ_q are determined by the first finite number of the χ_q 's (Theorem 2.2). In spite of the importance of χ_1 or χ_2 (for example, χ_2 determines the moduli space near the corresponding point of X), it will be shown (cf. (2.5)) that χ_1 and χ_2 do not necessarily determine other χ_q 's (cf. [5, p. 219 Corollary]). In §3, as for (ii) we shall prove:

THEOREM. *Let G (resp. G') be a cyclic group of automorphisms of a compact Riemann surface X (resp. X') of genus $\tilde{g} \geq 2$. Assume that $\prime: G \rightarrow G'$ ($\sigma \rightarrow \sigma'$) is an isomorphism. Then the following conditions are equivalent.*

(a) *There exists an orientation-preserving homeomorphism $h: X \rightarrow X'$ such that*

$h \circ \sigma = \sigma' \circ h$ for each $\sigma \in G$.

(b) $\text{Tr}(G|H^0(X, \Omega_X^{\otimes q}))(\sigma) = \text{Tr}(G'|H^0(X', \Omega_{X'}^{\otimes q}))(\sigma')$ for each $\sigma \in G$ and $q=1, 2, \dots$.

In §4, we shall settle (iii) as an interpretation of the existence theorem.

NOTATION. We denote by k our algebraically closed ground field of characteristic $p \geq 0$. A curve will always mean a complete non-singular curve defined over k . Throughout this paper n , G and ζ_n denote an integer (≥ 1) such that $p \nmid n$, a cyclic group of order n and a primitive n -th root of unity (in k), respectively. We set $\zeta_m = \zeta_n^{n/m}$ for $m|n$.

We write $\#X$ for the cardinality of a finite set X . However when σ is an element of a group, $\#\sigma$ means its order.

1. Existence theorem.

The purpose of this section is to give an elementary algebraic proof of the "existence theorem" (Theorem 1.6) of a cyclic covering having a given branch structure. In the case where $k = \mathbb{C}$, the theorem is classically obtained by using the theory of uniformization or covering spaces (cf. [7], [14, p. 116]).

(1.1) **Statements.** One of our motivation to introduce the homomorphism ϕ_G (defined below) is explained in Remark 1.1 which shall be used in §3, (cf. also (2.1)).

REMARK 1.1. Let X be a curve of genus \tilde{g} . Assume that our G is contained in $\text{Aut}(X)$, the automorphism group of X , i. e., assume that we have an injective homomorphism $\iota: G \rightarrow \text{Aut}(X)$ and identify G with its image via ι .

(a) *Definition.* (Cf. [14, p. 101].) Let P be a point on X . For $\sigma \in G$ we define $\zeta_P(\sigma) \in k$ as follows:

$$\zeta_P(\sigma) = \zeta \quad (\text{resp. } = 0) \quad \text{if } \sigma(P) = P \quad (\text{resp. } \neq P),$$

where ζ is a $\#\sigma$ -th root of unity satisfying the relation:

$$\sigma^*(t) = \zeta \cdot t \quad \text{mod } t^2 \cdot \mathcal{O}_P$$

for some local parameter t of the valuation ring \mathcal{O}_P at P (in the function field $\mathcal{K}(X)$ of X). It is noted that

(1) $\sigma \mapsto \zeta_P(\sigma)$ defines an injective homomorphism of $G(P) = \{\sigma \in G \mid \sigma(P) = P\}$ into k^\times , the group of the units in k .

(b) *Notation.* Let $Q_1, \dots, Q_l (\in X/G)$ be the branch points of the projection $\pi: X \rightarrow X/G$. For $P_j \in \pi^{-1}(Q_j)$ denote by m_j the ramification index e_{P_j} at P_j . We may assume that $m_1 \leq \dots \leq m_l$. By a theorem of Hurwitz we have the Riemann-Hurwitz relation:

$$(2) \quad 2 \cdot \tilde{g} - 2 = n(2 \cdot g - 2) + n \sum_{j=1}^l (1 - 1/m_j),$$

where g denotes the genus of X/G . The datum $(g, l; m_1, \dots, m_l)$ is called (in this paper) the signature of G .

(c) *Construction.* (Cf. [8, §4].) Here we confine ourselves to the case where $k = \mathbb{C}$ and $\tilde{g} \geq 2$. Let K be a Fuchsian group (acting on the unit disk U) uniformizing X , and let Γ denote the Fuchsian group generated by the elements of K and the liftings to U of the elements of G . Then $\pi: X \rightarrow X/G$ may be identified with the natural mapping: $U/K \rightarrow U/\Gamma$, and so the signature of Γ (cf. [10]) is $(g, l; m_1, \dots, m_l)$. $A \rightarrow$ (the induced mapping of A on U/K) defines a surjective homomorphism $\phi: \Gamma \rightarrow G$ of which kernel ($=K$) is torsion-free.

To inquire the nature of ϕ we consider a standard fundamental region R for Γ (cf. [9]). It is a fundamental region satisfying the following conditions:

- 1) R is bounded by $4g + 2l$ Jordan arcs in U :

$$\widehat{z_0 z_1}, \widehat{z_1 z_2}, \dots, \widehat{z_{4g-1} z_{4g}}, \widehat{z_{4g} e_1}, \widehat{e_1 z_{4g+1}}, \dots, \widehat{e_l z_0};$$

forming a Jordan curve oriented so that the interior of R is on the left, where z_0 (resp. z_1) of $\widehat{z_0 z_1}$ denotes the initial (resp. end) point of the arc, and so force.

- 2) There exist hyperbolic elements A_i, B_i in Γ ($i=1, \dots, g$) such that $A_i(\widehat{z_{4i-2} z_{4i-1}}) = \widehat{z_{4i-3} z_{4i-4}}$, $B_i(\widehat{z_{4i-3} z_{4i-2}}) = \widehat{z_{4i} z_{4i-1}}$, and elliptic elements C_j of order m_j in Γ ($j=1, \dots, l$) such that $C_j(\widehat{e_j z_{4g+j}}) = \widehat{e_j z_{4g+j-1}}$, where $z_{4g+l} = z_0$. It is well-known (e.g., cf. [10, p. 227, p. 234]) that the elements A_i, B_i, C_j generate the group Γ and have

$$(3) \quad C_1^{m_1} = \dots = C_l^{m_l} = 1, \quad C_1 \dots C_l [A_1, B_1] \dots [A_g, B_g] = 1$$

as a basis for the relations in Γ . The point is that

- (4) if e_j projects onto the points $P_{k(j)}, Q_{k(j)}$ on $U/K = X$, $U/\Gamma = X/G$, respectively, then $\phi(C_j) = \sigma_{k(j)}$, where σ_j denotes the element of G such that $\zeta_{P_j}(\sigma_j) = \zeta_{m_j}$ ($j=1, \dots, l$) (cf. [8, Theorem 7]).

Before the definition of our ϕ_G we set:

NOTATION 1.2. For a given datum $(g, l; m_1, \dots, m_l)$, where $g, l, m_1, \dots, m_l \in \mathbb{Z}$, $g, l \geq 0$ and $2 \leq m_1 \leq \dots \leq m_l$, let Γ denote the group generated by A_i, B_i and C_j ($i=1, \dots, g; j=1, \dots, l$) having (3) as a basis for the relations in Γ .

DEFINITION 1.3. Let X be a curve and assume that $G \subseteq \text{Aut}(X)$. Let Γ, A_i, B_i and C_j be as in Notation 1.2 for the signature $(g, l; m_1, \dots, m_l)$ of G , and P_1, \dots, P_l be as in Remark 1.1 (b). We define a group homomorphism $\phi_G: \Gamma \rightarrow G$ as follows:

$$\phi_G(A_i) = \phi_G(B_i) = \tau \quad \text{and} \quad \phi_G(C_j) = \sigma_j,$$

where τ is a generator of G and σ_j denotes the element of G such that $\zeta_{P_j}(\sigma_j) = \zeta_{m_j}$.

Well-definedness of ϕ_G is non-trivial and shall be shown in the proof of Lemma 1.8.

To state our main result in this section, which (including a remark) shall be proved in (1.4), here we make:

DEFINITION 1.4. Let Γ and C_j ($j=1, \dots, l$) be as in Notation 1.2. Then we say that two homomorphisms ϕ_1, ϕ_2 of Γ into G are equivalent if there exists a permutation k of $\{1, \dots, l\}$ such that $\phi_1(C_j) = \phi_2(C_{k(j)})$ ($j=1, \dots, l$).

REMARK 1.5. Assume that G is an automorphism group of a curve. Then we have the followings:

- (a) ϕ_G is uniquely defined up to the equivalence;
- (b) ϕ_G is surjective and $\text{Ker}(\phi_G)$, the kernel of ϕ_G , is torsion-free.

THEOREM 1.6. Let Γ (resp. k and G) be as in Notation 1.2 (resp. Notation). Assume that $\phi: \Gamma \rightarrow G$ is a surjective homomorphism having the torsion-free kernel. Then there exists a curve X such that $G \subseteq \text{Aut}(X)$ and ϕ_G is equivalent to ϕ .

(1.2) A function-group theoretic lemma. To prove Theorem 1.6, first we prepare a preliminary result.

LEMMA 1.7. Let Γ, C_j and $(g, l; m_1, \dots, m_l)$ be as in Notation 1.2. Assume that $\phi: \Gamma \rightarrow G$ is a surjective homomorphism. Then $\text{Ker}(\phi)$ is torsion-free if and only if $\phi(C_j)$ has order m_j ($j=1, \dots, l$).

PROOF. The "only-if" part is obvious. To prove the converse we assume that $\phi(C_j)$ has order m_j . We shall examine three cases where $2 \cdot g - 2 + \sum(1 - 1/m_j)$ is (i) > 0 , (ii) $= 0$, (iii) < 0 .

Case (i). By [10, p. 239 Theorem] Γ is realizable as a Fuchsian group, so we obtain Lemma 1.7 using Theorem 3 in [7].

Case (ii). It is easy to see that the possibilities of $(g, l; m_1, \dots, m_l)$ are as follows:

$$\begin{aligned} (1, 0; \text{---}), & \quad (0, 4; 2, 2, 2, 2), & \quad (0, 3; 2, 3, 6), \\ (0, 3; 2, 4, 4), & \quad (0, 3; 3, 3, 3). \end{aligned}$$

In each case Γ is realizable as an elementary group acting on the plane \mathcal{C} (cf. [3, VI. 9.5]). So we obtain Lemma 1.7 as in the case (i).

Case (iii). The possibilities are as follows:

$$(0, 0; \text{---}), \quad (0, 1; m_1), \quad (0, 2; m_1, m_2), \quad (0, 3; 2, 2, m_3), \\ (0, 3; 2, 3, 3), \quad (0, 3; 2, 3, 4), \quad (0, 3; 2, 3, 5).$$

Then the group Γ is cyclic, cyclic, cyclic, dihedral, tetrahedral, octahedral, icosahedral, respectively. So it is easy to see Lemma 1.7. Q. E. D.

(1.3) Construction of cyclic coverings. The following lemma, which is Theorem 4 in [7] in case $k=C$, is essentially equivalent to our existence theorem.

LEMMA 1.8. *Let Γ and $(g, l; m_1, \dots, m_l)$ (resp. k and G) be as in Notation 1.2 (resp. Notation). Let Y be a curve of genus g and Q_1, \dots, Q_l be l distinct points on Y . Put $m=\text{l.c.m.}\{m_1, \dots, m_l\}$. Then the following conditions (a)~(c) are equivalent:*

- (a) $G=\text{Aut}(X/Y)$ ($=\{\sigma \in \text{Aut}(X) \mid \pi \circ \sigma = \pi\}$) for a Galois covering $\pi: X \rightarrow Y$ of curves such that $e_P=m_j$ (resp. $=1$) whenever $\pi(P)=Q_j$ (resp. $\neq Q_j, j=1, \dots, l$).
- (b) There exists a surjective homomorphism $\phi: \Gamma \rightarrow G$ such that $\text{Ker}(\phi)$ is torsion-free.
- (c) (i) $\text{l.c.m.}\{m_1, \dots, \hat{m}_i, \dots, m_l\}=m$ for all i , where \hat{m}_i denotes the omission of m_i ;
 (ii) m divides n , and if $g=0$, $m=n$;
 (iii) If $2^e \mid m$ ($e \geq 1$) and $2^{e+1} \nmid m$, $\#\{j \mid 2^e \mid m_j\}$ is even.

PROOF. We shall establish the following chain of implications: (c) \Rightarrow (b) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c).

(c) \Rightarrow (b). This can be proved in the same way with some slight modification as the latter part of the proof of Theorem 4 in [7], so we omit the details.

(b) \Rightarrow (a). Let τ denote a generator of G and Q be a point on Y . Assume that $\phi(C_j)=(\tau^{n/m_j})^{s_j}$ ($j=1, \dots, l$). We denote by E the divisor $\sum_j (s_j n/m_j) \cdot Q_j$ on Y . Then by the assumption the degree of E is equal to $n \cdot c$ for some integer c . Using the fact that the group of n -division points on the Jacobian variety of Y is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{2g}$, we shall see

- (5) there exists a divisor D on Y with $E \sim n \cdot D$ such that $(1/s) \cdot E \not\sim (n/s) \cdot D$ for any $s \mid (n/m)$ with $s \neq 1$,

where \sim means the relation of linear equivalence. In fact, in the case where $g=0$, we may choose $c \cdot Q$ as D , since we have $n=m$ by the surjectivity of ϕ . Now assume that $g>0$. First take a divisor D_1 on Y such that $E - nc \cdot Q \sim n \cdot D_1$. Next let $\{p_i\}$ be the set consisting of the prime divisors of n/m . Then we can choose a divisor D_2 on Y with $n \cdot D_2 \sim O$ such that

$$(n/p_i) \cdot D_2 \sim O \quad \text{if and only if} \quad (1/p_i)(E - nc \cdot Q) \not\sim (n/p_i) \cdot D_1.$$

Then it is obvious that $D=D_1+D_2+c\cdot Q$ serves our need.

Take an element z of $K=\mathcal{K}(Y)$ such that

$$(6) \quad \operatorname{div}_Y(z) = E - n \cdot D,$$

and let $\pi: X \rightarrow Y$ denote the covering (of curves) defined by the inclusion of function fields: $\mathcal{K}(X)=K(y) \cong K$, where $y=z^{1/n}$. Here, using the theory of Kummer extensions, we note that

$$(7) \quad [K(y):K] = n.$$

In fact, if we put $s=n/[K(y):K]$, then, observing that $y^{n/s}$ belongs to K , we have that $\operatorname{div}_Y(z)=s \cdot \operatorname{div}_Y(y^{n/s})$. Hence it follows from (6) that $s|(n/m)$. By (5) this means that $s=1$, as desired.

By (7) we may assume that $G=\operatorname{Aut}(X/Y)$ and $\tau^*(y)=\zeta_n \cdot y$. Here we show that

$$(8) \quad \begin{array}{l} \text{if } \pi(P) \neq Q_j, \ j=1, \dots, l \text{ (resp. } \pi(P)=Q_j), \text{ then} \\ e_P=1 \text{ (resp. } e_P=m_j \text{ and } \zeta_P(\tau^{n/m_j})^{s_j}=\zeta_{m_j}). \end{array}$$

To observe (8), let $(g, l'; m'_1, \dots, m'_{l'})$ denote the signature of $\operatorname{Aut}(X/Y)$ and let $Q'_1, \dots, Q'_{l'}$ (resp. $P'_1, \dots, P'_{l'}$) be the points on Y (resp. X) as in Remark 1.1 (b). Since $\tau^{n/m'_i}(y)=\zeta_{m'_i} \cdot y$, if $\zeta_{P'_i}(\tau^{n/m'_i})^{s'_i}=\zeta_{m'_i}$ then we have that

$$(9) \quad (\text{order of } y \text{ at } P'_i) \equiv s'_i \pmod{m'_i}.$$

Moreover since

$$\pi^*(Q'_i) = m'_i \{ \tau(P'_i) + \tau^2(P'_i) + \dots + \tau^{n/m'_i}(P'_i) \}$$

(as divisors), and since $\operatorname{div}_X(y)$ is τ -invariant, it follows from (9) that

$$\operatorname{div}_X(y) = \sum_i (s'_i/m'_i) \cdot \pi^*(Q'_i) + \pi^*(D'), \quad \text{i. e.,}$$

$$(10) \quad \operatorname{div}_Y(y^n) = \sum_i (s'_i n/m'_i) \cdot Q'_i + n \cdot D'$$

for some divisor D' on Y . Comparing the order of $z=y^n$ at Q_j in (6) and (10), we obtain that

$$s_j n/m_j \equiv \begin{cases} s'_i n/m'_i \pmod{n} & \text{if } Q_j = Q'_i, \\ 0 \pmod{n} & \text{if } Q_j \neq Q'_i \ (i=1, \dots, l'). \end{cases}$$

Since $(s_j n/m_j, n)=n/m_j$ and $(s'_i n/m'_i, n)=n/m'_i$, it follows that for each j there exists a unique integer $k(j)$ ($1 \leq k(j) \leq l'$) such that $Q_j = Q'_{k(j)}$. Then we have that

$$(11) \quad m_j = m'_{k(j)} \quad \text{and} \quad s_j \equiv s'_{k(j)} \pmod{m_j}.$$

Comparing the order of z at Q'_i in the same way, we see moreover that $l=l'$

and k is a permutation. Therefore by (11) we obtain (8) and hence (a).

(a) \Rightarrow (b). Let τ be a generator of G and P_j be a point in $\pi^{-1}(Q_j)$ ($j=1, \dots, l$). Assume that $\zeta_{P_j}(\tau^{n/m_j})^{s_j} = \zeta_{m_j}$, $j=1, \dots, l$. By the theory of Kummer extensions we can find an element y of $\mathcal{K}(X)$ such that $\tau^*(y) = \zeta_n \cdot y$. Then it follows as in the proof of (b) \Rightarrow (a) that

$$\operatorname{div}_Y(y^n) = \sum_j (s_j n / m_j) \cdot Q_j - n \cdot D$$

for some divisor D on Y . Considering the degree, we obtain

$$\sum_j s_j n / m_j \equiv 0 \pmod{n}.$$

This implies the well-definedness of ϕ_G as a group homomorphism.

To obtain (b) it suffices by Lemma 1.7 to verify the surjectivity of ϕ_G . The case where $g > 0$ is trivial, so we assume $g = 0$. Since then the image of ϕ_G is the subgroup generated by τ^{n/m_j} ($j=1, \dots, l$), which is of order m , it suffices to show that $m = n$. In fact, to see this we note that the group G/G' is naturally identified with $\operatorname{Aut}((X/G')/(X/G))$, where $G' = \langle \tau^{n/m} \rangle$. Then it follows from the choice of m that the covering $X/G' \rightarrow X/G$ is unramified. Hence the Riemann-Hurwitz relation for this covering reads: $2 \cdot g' - 2 = (n/m)(2 \cdot g - 2)$, i. e., $g' + (n/m) = 1$, where g' denotes the genus of X/G' . Since $g' \geq 0$, this implies that $n/m = 1$, as desired.

(b) \Rightarrow (c). Using the implication: (b) \Rightarrow (a), we can prove in the same way as the former part of the proof of Theorem 4 in [7], so we omit the details. Q.E.D.

(1.4) Proofs. PROOF OF REMARK 1.5. The statement (a) is obvious and (b) has been already proved in the proof of Lemma 1.8. Q. E. D.

PROOF OF THEOREM 1.6. This can be proved easily by using (a) \Rightarrow (b), in particular (8), in the proof of Lemma 1.8. Q. E. D.

2. Determinability.

Throughout this section we suppose that

- (1) The characteristic of our k is zero, and our G is contained in $\operatorname{Aut}(X)$ with X being a curve of genus \tilde{g} ,

and for $q=1, 2, \dots$, we put

- (2) $\chi_q = \operatorname{Tr}(G | H^0(X, \Omega_X^{\otimes q}))$,

the character of the natural representation of G on the space of q -differentials on X .

In this section we shall see another nature of the homomorphism ϕ_G , the close relation to our characters. Using it we shall obtain an interrelation between the characters.

(2.1) Statements. Our basic tool is the following “Trace formula”:

LEMMA 2.1. *Let G , \tilde{g} and χ_q be as in (1) and (2). Then:*

(a) *If $\sigma \in G$ with $\sigma \neq 1$, then*

$$\sum_{P \in X} \zeta_P(\sigma)^q / (1 - \zeta_P(\sigma)) = \chi_q(\sigma) \quad (\text{resp. } = \chi_1(\sigma) - 1)$$

for $q \geq 2$ and $\tilde{g} \geq 2$ (resp. for $q=1$).

(b) $(2 \cdot q - 1)(\tilde{g} - 1) = \chi_q(1)$ (resp. $= \chi_1(1) - 1$)

for $q \geq 2$ and $\tilde{g} \geq 2$ (resp. for $q=1$).

This has been proved in several ways, cf. e. g., [3, V.2.9].

We note the summation in Lemma 2.1 (a) may be rewritten as follows;

$$\sum_{\substack{\#\sigma \\ (r, \#\sigma)=1}} \#\{P \mid \zeta_P(\sigma) = \zeta_{\#\sigma}^r\} \cdot \zeta_{\#\sigma}^{qr} / (1 - \zeta_{\#\sigma}^r).$$

Since these $\#\{\dots\}$ and \tilde{g} are recovered from the information of ϕ_G (cf. Remark 2.8 below), Lemma 2.1 implies that

- (i) we have a (explicit) relation between χ_q and ϕ_G , and
- (ii) considering χ_q as an unknown and ϕ_G as a variable, we can solve the relation (uniquely).

In such a meaning we say that χ_q is “determined” by ϕ_G .

As for the converse, we have the followings, which shall be proved in (2.4).

THEOREM 2.2. *Let G , X , \tilde{g} and χ_q be as in (1) and (2). If $\tilde{g} \geq 2$ then the characters $\chi_1, \chi_2, \dots, \chi_{n'+1}$ “determine” the homomorphism ϕ_G up to the equivalence, where $n' = n / \prod p_i$ with $\{p_i\}$ being the set of prime factors of n .*

COROLLARY 2.3. *Let G , X , \tilde{g} and χ_q be as in (1) and (2). Assume moreover that our G is also contained in $\text{Aut}(X')$ with X' being a curve (cf. Remark 1.1). Let ϕ (resp. ϕ') denote a ϕ_G defined for G considered as a subgroup of $\text{Aut}(X)$ (resp. $\text{Aut}(X')$). Put $\chi'_q = \text{Tr}(G \mid H^0(X', \Omega_{X'}^{\otimes q}))$. If $\tilde{g} \geq 2$ then the following three conditions are equivalent:*

- (a) $\phi = \phi'$ up to the equivalence.
- (b) $\chi_q = \chi'_q$ for $q=1, 2, \dots$.
- (c) $\chi_q = \chi'_q$ for $q=1, \dots, n'+1$, where $n' = n / \prod p_i$ with $\{p_i\}$ as above.

COROLLARY 2.4. *Let X , \tilde{g} and χ_q be as in (1) and (2). Then the characters $\chi_1, \dots, \chi_{4\tilde{g}+2}$ “determine” the other χ_q 's.*

(2.2) **Basics on the rotation datum.** In this numero we introduce an object which describe the essential part of the homomorphism ϕ_G .

NOTATION 2.5. (a) Let m be a given integer ($m \geq 1$).

(i) For $r \in \mathbf{Z}$ we define a mapping $\lambda_{mr} : (\mathbf{Z}/m\mathbf{Z})^\times \rightarrow \mathbf{Q}$ by $(i \bmod m\mathbf{Z}) \mapsto 1$ (resp. 0) if $i \equiv r \pmod m$ (resp. otherwise).

(ii) We view $\text{Map}((\mathbf{Z}/m\mathbf{Z})^\times, \mathbf{Q})$ and $\text{Map}(\mathbf{Z}, \mathbf{Q})$ as \mathbf{Q} -vector spaces in a natural manner and consider as $\text{Map}((\mathbf{Z}/m\mathbf{Z})^\times, \mathbf{Q}) \subseteq \text{Map}(\mathbf{Z}, \mathbf{Q})$ via $\lambda_{mr} \mapsto \tilde{\lambda}_{mr}$, where $\tilde{\lambda}_{mr}$ is the mapping defined by $i \mapsto 1$ (resp. $\mapsto 0$) if $i \equiv r \pmod m$ (resp. otherwise).

(iii) We denote by $\text{deg} : \text{Map}((\mathbf{Z}/m\mathbf{Z})^\times, \mathbf{Q}) \rightarrow \mathbf{Q}$ the \mathbf{Q} -homomorphism: $\sum_r a_r \lambda_{mr} \mapsto \sum_r a_r$.

(b) In case $m' | m$, we denote by $\pi_{m'} : \text{Map}((\mathbf{Z}/m\mathbf{Z})^\times, \mathbf{Q}) \rightarrow \text{Map}((\mathbf{Z}/m'\mathbf{Z})^\times, \mathbf{Q})$ the \mathbf{Q} -homomorphism given by $\lambda_{mr} \mapsto \lambda_{m'r}$.

DEFINITION 2.6. Let G, X, \tilde{g} (resp. $\phi_G : \Gamma \rightarrow G, (g, l; m_1, \dots, m_l)$ and C_j) be as in (1) (resp. in Definition 1.3).

(a) We define the rotation datum $\lambda_G : G \rightarrow \text{Map}(\mathbf{Z}, \mathbf{Q})$ of G by

$$\sigma \mapsto \begin{cases} (2-2 \cdot \tilde{g}) \cdot \lambda_{11} & \text{if } \sigma=1, \\ \sum_{\substack{\# \sigma \\ (\tau, \# \sigma)=1}}^{\# \sigma} \#\{P \mid \zeta_P(\sigma) = \zeta_{\# \sigma}^r\} \cdot \lambda_{\# \sigma r} & \text{otherwise.} \end{cases}$$

(b) We also define a mapping $\text{Red } \lambda_G : G \setminus \{1\} \rightarrow \text{Map}(\mathbf{Z}, \mathbf{Q})$ by using ϕ_G as follows:

$$(3) \quad \sigma \mapsto \sum_{\# \sigma = m_j} (n/m_j) \cdot \lambda_{m_j u_j},$$

where $u_j \in \mathbf{Z}$ such that $\phi_G(C_j^{u_j}) = \sigma$.

The followings are basic for later use:

LEMMA 2.7. Let G be as in (1).

(a) Assume that $u \in \mathbf{Z}, (u, n)=1$ and $\sigma \in G, \sigma \neq 1$. The \mathbf{Q} -automorphism of $\text{Map}((\mathbf{Z}/\# \sigma \cdot \mathbf{Z})^\times, \mathbf{Q})$ defined by $\lambda_{\# \sigma r} \mapsto \lambda_{\# \sigma ru}$ sends $\lambda_G(\sigma)$ (resp. $\text{Red } \lambda_G(\sigma)$) to $\lambda_G(\sigma^u)$ (resp. $\text{Red } \lambda_G(\sigma^u)$).

(b) Assume that $G = \langle \tau \rangle$ and $d | n, d \neq n$. Then

$$(i) \quad \lambda_G(\tau^d) = \sum_{e|d} \pi_{n/d}(\text{Red } \lambda_G(\tau^e)),$$

where the symbol $\sum_{e|d}$ is meant a summation over all e with $e | d$.

$$(ii) \quad \text{Red } \lambda_G(\tau^d) = \sum_{e|d} \mu(d/e) \cdot \pi_{n/d}(\lambda_G(\tau^e)),$$

where μ denotes the Möbius function.

PROOF. (a) This follows from (1) of §1 and the fact that ϕ_G is a homomorphism.

(b) To prove (b) we use the notation in Definition 1.3. For $\sigma \in G$, $\sigma \neq 1$ we have that

$$(4) \quad \phi_G(C_j^u) = \sigma \quad \text{if and only if} \quad \zeta_{P_j}(\sigma) = \zeta_{m_j}^u,$$

and hence that

$$(5) \quad \text{Red } \lambda_G(\sigma) = \sum_{\tau=1}^{\#\sigma} \#\{P \mid G(P) = \langle \sigma \rangle, \zeta_P(\sigma) = \zeta_{\#\sigma}^r\} \cdot \lambda_{\#\sigma \tau}.$$

This implies (i), in fact, we have that

$$\begin{aligned} \lambda_G(\tau^d) &= \sum_{e \mid d} \sum_{r=1}^{n/d} \#\{P \mid G(P) = \langle \tau^e \rangle, \zeta_P(\tau^d) = \zeta_{n/d}^r\} \cdot \lambda_{n/d r} \\ &= \sum_{e \mid d} \sum_{r=1}^{n/e} \#\{P \mid G(P) = \langle \tau^e \rangle, \zeta_P(\tau^e) = \zeta_{n/e}^r\} \cdot \pi_{n/d}(\lambda_{n/e r}) \\ &= \sum_{e \mid d} \pi_{n/d}(\text{Red } \lambda_G(\tau^e)). \end{aligned}$$

To prove (ii) we recall the well-known fact:

$$(6) \quad \sum_{e \mid d} \mu(d/e) = 0 \quad (\text{resp.} = 1) \quad \text{if } d \neq 1 \quad (\text{resp.} = 1).$$

Using (i) and (6) we have that

$$\begin{aligned} \sum_{e \mid d} \mu(d/e) \cdot \pi_{n/d}(\lambda_G(\tau^e)) &= \sum_{e \mid d} \mu(d/e) \cdot \pi_{n/d} \left\{ \sum_{f \mid e} \pi_{n/e}(\text{Red } \lambda_G(\tau^f)) \right\} \\ &= \sum_{f \mid d} \left\{ \sum_{m \mid (d/f)} \mu(m) \right\} \cdot \pi_{n/d}(\text{Red } \lambda_G(\tau^f)) = \text{Red } \lambda_G(\tau^d), \end{aligned}$$

as desired. Q. E. D.

REMARK 2.8. Let G and X be as in (1). Then together with the Riemann-Hurwitz relation (cf. (2) in §1), Lemma 2.7 with (4) and (5) implies the following:

- (i) λ_G is “determined” by ϕ_G ; and
- (ii) λ_G “determines” ϕ_G up to the equivalence.

LEMMA 2.9. Let $G = \langle \tau \rangle$ and X be as in (1). Let $G' = \langle \tau^{\bar{n}} \rangle$ where $\bar{n} \mid n$, and put $\tau' = \tau^{\bar{n}}$ with $n' = n/\bar{n}$. Assume that $(g, l; m_1, \dots, m_l)$ (resp. $(g', l'; m'_1, \dots, m'_l)$) denotes the signature of G (resp. G'). For $s \mid n'$, $s \neq n'$ we have that

- (i) $\text{Red } \lambda_{G'}(\tau'^s) = \sum_d \pi_{n'/s}(\text{Red } \lambda_G(\tau^d))$; and
- (ii) $\#\{j' \mid m'_j = n'/s\} = \sum_d (d, \bar{n}) \cdot \#\{j \mid m_j = n/d\}$,

where d runs over the set $\{d \mid d \mid n, d = (d, \bar{n})s\}$.

$$\begin{aligned}
\text{PROOF. } \text{Red } \lambda_{G'}(\tau'^s) &= \sum_{f|s} \mu(s/f) \cdot \pi_{n'/s}(\lambda_{G'}(\tau'^f)) \\
&= \sum_{f|s} \sum_{d|\bar{n}_f} \mu(s/f) \cdot \pi_{n'/s}(\text{Red } \lambda_G(\tau^d)) \\
&= \sum_{d|\bar{n}_s} \left(\sum_{\substack{f|s \\ d|\bar{n}_f}} \mu(s/f) \right) \cdot \pi_{n'/s}(\text{Red } \lambda_G(\tau^d)) \\
&= \sum_{d|\bar{n}_s} \left\{ \sum_{e|(s/(d/(d, \bar{n})))} \mu(s/(d/(d, \bar{n}))e) \right\} \cdot \pi_{n'/s}(\text{Red } \lambda_G(\tau^d)) \\
&= \sum_d \pi_{n'/s}(\text{Red } \lambda_G(\tau^d)),
\end{aligned}$$

where d runs as above. The rest part follows from a modification of the above relation. Q. E. D.

LEMMA 2.10. Let $G = \langle \tau \rangle$ and X be as in (1). Let $G' = \langle \tau^{\bar{n}} \rangle$ where $\bar{n} | n$. Put $\bar{G} = G/G'$ and $\bar{\tau} = (\tau \bmod G') \in \bar{G}$. In this case \bar{G} is naturally identified with $\text{Aut}((X/G')/(X/G))$. Let $(g, l; m_1, \dots, m_l)$ (resp. $(g, \bar{l}; \bar{m}_1, \dots, \bar{m}_{\bar{l}})$) be the signature of G (resp. \bar{G}). For $t | \bar{n}$, $t \neq \bar{n}$ we have:

$$(i) \quad \text{Red } \lambda_{\bar{G}}(\bar{\tau}^t) = t \cdot \sum_{\substack{d|\bar{n} \\ (d, \bar{n})=t}} \pi_{\bar{n}/t}((1/d) \cdot \text{Red } \lambda_G(\tau^{d u_d})),$$

where $u_d \in \mathbf{Z}$ such that $(u_d, n) = 1$ and $d \cdot u_d \equiv (d, \bar{n}) \pmod{\bar{n}}$;

$$(ii) \quad \#\{j \mid \bar{m}_j = \bar{n}/t\} = \sum_{\substack{d|\bar{n} \\ (d, \bar{n})=t}} \#\{j \mid m_j = n/d\}.$$

PROOF. Using the fact that

$$\langle \tau^d \rangle \cap G' = \langle \tau^{\bar{n}d/(d, \bar{n})} \rangle \quad \text{for } d | n,$$

first we note that

$$(7) \quad \text{for } t | \bar{n}, \{\langle \tau^d \rangle \mid d | n, (d, \bar{n}) = t\} \text{ is the set of cyclic subgroups } H \text{ of } G \text{ such that } (H \bmod G') = \langle \bar{\tau}^t \rangle.$$

Next let $\bar{P} \in X/G'$ with $\bar{G}(\bar{P}) = \langle \bar{\tau}^t \rangle$, $t | \bar{n}$, and let $P \in \pi'^{-1}(\bar{P})$, where π' denotes the projection: $X \rightarrow X/G'$. Since $(G(P) \bmod G') = \bar{G}(\bar{P})$, it follows from (7) that $G(P) = \langle \tau^d \rangle$ for some $d | n$ with $(d, \bar{n}) = t$. Let $\bar{r}(\bar{P})$ and $r(P)$ be such that

$$\zeta_{\bar{P}}(\bar{\tau}^t) = \zeta_{\bar{n}/t}^{\bar{r}(\bar{P})} \quad \text{and} \quad \zeta_P(\tau^{d u_d}) = \zeta_{n/d}^{r(P)}.$$

Then it is easy to see that

$$(8) \quad r(P) \equiv \bar{r}(\bar{P}) \pmod{\bar{n}/t}.$$

Since $\#\pi'^{-1}(\bar{P}) = [G' : G'(P)] = d/t$, from (8) it follows that

$$(9) \quad \lambda_{\bar{n}/t, \bar{r}(\bar{P})} = \sum_{P \in \pi'^{-1}(\bar{P})} (t/d) \cdot \pi_{\bar{n}/t}(\lambda_{n/d, r(P)}).$$

Finally for $t | \bar{n}$, $t \neq \bar{n}$, since

$$\pi'^{-1}(\{\bar{P} \mid \bar{G}(\bar{P}) = \langle \bar{\tau}^t \rangle\}) = \bigcup_{\substack{d \mid n \\ (d, \bar{n}) = t}} \{P \mid G(P) = \langle \tau^d \rangle\}.$$

we have by (9) that

$$\begin{aligned} \text{Red } \lambda_{\bar{G}}(\bar{\tau}^t) &= \sum_{\substack{\bar{P} \\ \bar{G}(\bar{P}) = \langle \bar{\tau}^t \rangle}} \lambda_{\bar{n}/t, \bar{\tau}(\bar{P})} \\ &= \sum_{\substack{d \mid n \\ (d, \bar{n}) = t}} \sum_{\substack{P \\ G(P) = \langle \tau^d \rangle}} (t/d) \cdot \pi_{\bar{n}/t}(\lambda_{n/d, \tau(P)}) \\ &= t \cdot \sum_{\substack{d \mid n \\ (d, \bar{n}) = t}} \pi_{\bar{n}/t}((1/d) \cdot \text{Red } \lambda_G(\tau^{du_d})), \end{aligned}$$

which completes the proof of (i) and hence of (ii).

Q. E. D.

(2.3) Basics on class functions. In this numero we shall introduce a special type of class function and prepare a lemma on it, which is crucial for later use.

NOTATION 2.11. Let G and X be as in (1). For the rotation datum λ_G , we define a class function $\chi(\lambda_G)_q: G \rightarrow k$ ($q \in \mathbf{Z}$) as follows:

$$\sigma \longmapsto \begin{cases} (1/2)(1-2q) \cdot a_{11} & \text{if } \sigma = 1, \\ \sum_{\substack{r=1 \\ (r, \#\sigma) = 1}}^{\#\sigma} a_{\#\sigma r} \cdot \zeta_{\#\sigma}^{qr} / (1 - \zeta_{\#\sigma}^r) & \text{otherwise,} \end{cases}$$

where $\lambda_G(\sigma) = \sum_r a_{\#\sigma r} \cdot \lambda_{\#\sigma r}$.

By using $\chi(\lambda_G)_q$ the trace formula (in (2.1)) is rewritten as follows:

LEMMA 2.1'. Let G , X , \tilde{g} and χ_q be as in (1) and (2). Then:

- (i) $\chi_1 = 1 + \chi(\lambda_G)_1$;
- (ii) $\chi_q = \chi(\lambda_G)_q$ ($q \geq 2$) if $\tilde{g} \geq 2$.

We sometimes consider our theory in the following:

SITUATION 2.12. Let $G = \langle \tau \rangle$, X and \tilde{g} be as in (1).

- (a) (i) Let $(g, l; m_1, \dots, m_l)$ denote the signature of G . Assume that

$$\text{Red } \lambda_G(\tau^d) = d \cdot \sum_{j=1}^{l_d} \lambda_{n/d, r_{dj}} \quad (d \mid n, d \neq n),$$

where $l_d = \#\{j \mid m_j = n/d\}$ (we put: $l_n = 0$), and $r_{dj} \in I_{n/d}$ (in general we put: $I_m = \{r \in \mathbf{Z} \mid 0 < r \leq m, (r, m) = 1\}$).

- (ii) For $a \in \mathbf{Z}$, we denote by $\underline{a} \cdot s_{dj}$ ($d \mid n, j = 1, \dots, l_d$) the integer r' ($0 < r' \leq n/d$) such that $r' \cdot r_{dj} \equiv a \pmod{n/d}$.

- (b) *Notation.* For $a \in \mathbf{Z}$ we denote by Δ^a or Δ_τ^a the homomorphism of G into k^\times defined by: $\tau \mapsto \zeta_n^a$.

The following lemma is already well-known, however we shall give a proof

to it as a lemma for the class function of the form $\chi(\lambda_G)_q$, which shall be used in § 4.

LEMMA 2.13. *In Situation 2.12, we have for $a, q \in \mathbf{Z}$ that*

$$\begin{aligned} \langle \chi(\lambda_G)_q, \Delta^a \rangle &= (1/\#G) \cdot \sum_{\sigma \in G} \chi(\lambda_G)_q(\sigma) \cdot \Delta^a(\sigma^{-1}) \\ &= (2q-1)(g-1) + \sum_{d|n} \sum_{j=1}^{l_d} \{ [(d/n)(\underline{a} \cdot s_{dj} - q)] - ((d/n) \cdot \underline{a} \cdot s_{dj} - q) \}, \end{aligned}$$

where $[\]$ means the Gauss symbol.

To prove this lemma we need the following sublemma on the trace mapping $S_m: \mathbf{Q}(\zeta_m) \rightarrow \mathbf{Q}$.

SUBLEMMA 2.14. *For $r \in \mathbf{Z}$, $m|n$ with $0 < r \leq m$, we have that*

$$\sum'_{d|m} S_d(\zeta_d^r/(1-\zeta_d)) = r - (1/2)(m+1),$$

where the symbol $\sum'_{d|m}$ means a summation over all d with $d|m$, $d \neq 1$.

PROOF. First we note that

$$(10) \quad \sum'_{d|m} S_d(\zeta_d^r) = m \quad (\text{resp. } = 0) \quad \text{if } r = m \quad (\text{resp. } \neq m).$$

Next, from $1/(1-\zeta_d) + 1/(1-\zeta_d^{-1}) = 1$ it follows that

$$(11) \quad 2 \cdot S_d(1/(1-\zeta_d)) = S_d(1) = \phi(d) \quad \text{if } d|m, d \neq 1,$$

where ϕ denotes the Euler phi-function. On the other hand,

$$S_d(\zeta_d^r/(1-\zeta_d)) = S_d(1/(1-\zeta_d)) - \sum_{u=0}^{r-1} S_d(\zeta_d^u),$$

since $\zeta_d^r/(\zeta_d-1) - 1/(\zeta_d-1) = \zeta_d^{r-1} + \zeta_d^{r-2} + \dots + 1$. Hence it follows from (10) and (11) that

$$\begin{aligned} \sum'_{d|m} S_d(\zeta_d^r/(1-\zeta_d)) &= \sum'_{d|m} S_d(1/(1-\zeta_d)) - \sum'_{d|m} \sum_{u=0}^{r-1} S_d(\zeta_d^u) \\ &= \sum'_{d|m} \phi(d)/2 - \sum_{u=0}^{r-1} \left\{ \sum'_{d|m} S_d(\zeta_d^u) - 1 \right\} = (m-1)/2 - (m-r), \end{aligned}$$

as asserted. Q. E. D.

PROOF OF LEMMA 2.13. Let $s|n$. Since

$$\chi(\lambda_G)_q(\tau^s) = \begin{cases} (2q-1)(\check{g}-1) & \text{if } s=n; \\ \sum_{d|s} \sum_j d \{ \zeta_n^{qs dj} / (1-\zeta_n^{r dj}) \} & \text{otherwise,} \end{cases}$$

it follows from Lemma 2.7 (a) that

the \mathbf{Q} -automorphism of the field $\mathbf{Q}(\zeta_{n/s})$ defined by $\zeta_{n/s} \mapsto \zeta_{n/s}^u$, sends $\chi(\lambda_G)_q(\tau^s)$ to $\chi(\lambda_G)_q(\tau^{su})$ for $u \in \mathbf{Z}$ such that $(u, n)=1$.

This implies that

$$\begin{aligned} n \cdot \langle \chi(\lambda_G)_q, \Delta^a \rangle &= \sum_{s|n} \sum_{u \in I_{n/s}} \{ \chi(\lambda_G)_q(\tau^{su}) / \zeta_{n/s}^{au} \} \\ &= \sum_{s|n} S_{n/s}(\chi(\lambda_G)_q(\tau^s) / \zeta_{n/s}^a) \end{aligned}$$

and hence that

$$\begin{aligned} (12) \quad n \cdot \langle \chi(\lambda_G)_q, \Delta^a \rangle &- (2q-1)(\tilde{g}-1) \\ &= \sum_{\substack{s|n \\ s \neq n}} S_{n/s} \left(\sum_{d|s} \sum_j d \{ \zeta_{n/s}^{q \cdot r \cdot dj - a} / (1 - \zeta_{n/s}^{r \cdot dj}) \} \right) \\ &= \sum'_{e|n} \sum_{d|(n/e)} \sum_j d \cdot S_e(\zeta_e^{q \cdot r \cdot dj - a} / (1 - \zeta_e^{r \cdot dj})) \\ &= \sum_{d|n} \sum_j d \{ \sum'_{e|(n/d)} S_e(\zeta_e^{q \cdot r \cdot dj - a} / (1 - \zeta_e^{r \cdot dj})) \} \\ &= \sum_{d|n} \sum_j d \{ \sum'_{e|(n/d)} S_e(\zeta_e^{q \cdot a \cdot s \cdot dj} / (1 - \zeta_e)) \}. \end{aligned}$$

On the other hand we note that

$$q - \underline{a \cdot s}_{dj} \equiv r \pmod{n/d} \quad \text{with } 0 < r \leq n/d,$$

where r denotes $(n/d)\{1 + [(d/n)(\underline{a \cdot s}_{dj} - q)] - (d/n)(\underline{a \cdot s}_{dj} - q)\}$. Hence, applying Sublemma 2.14 to (12), and using the Riemann-Hurwitz relation for the covering: $X \rightarrow X/G$, we obtain that

$$\begin{aligned} n \cdot \langle \chi(\lambda_G)_q, \Delta^a \rangle &- n(2q-1)(g-1) \\ &= \sum_{d|n} \sum_j \{ (2q-1)(n-d)/2 + dr - (n+d)/2 \} \\ &= n \cdot \sum_{d|n} \sum_j \{ [(d/n)(\underline{a \cdot s}_{dj} - q)] - ((d/n) \cdot \underline{a \cdot s}_{dj} - q) \}. \end{aligned}$$

This completes the proof of Lemma 2.13. Q. E. D.

(2.4) **Proof of Theorem 2.2.** We need two preliminary results:

LEMMA 2.15. *Let G and X be as in (1). Then for $\sigma \in G$,*

$$(13) \quad \deg(\lambda_G(\sigma)) = -\{ \chi(\lambda_G)_1(\sigma) + \chi(\lambda_G)_1(\sigma^{-1}) \}.$$

Hence the signature of G and the genus of X are "determined" by the function $\chi(\lambda_G)_1$.

PROOF. (13) follows from a direct calculation. The rest part follows from Lemma 2.7 (cf. also, Remark 2.8). Q. E. D.

COROLLARY 2.16 (to Lemma 2.13). *In Situation 2.12, for $a, q \in \mathbf{Z}$ we have:*

$$\sum_{d|n} \#\{j \mid \underline{a} \cdot s_{d_j} \equiv q \pmod{n/d}\} = \langle \chi(\lambda_G)_q - \chi(\lambda_G)_{q+1}, \Delta^a \rangle + 2g - 2 + l.$$

PROOF. This follows from the fact that

$$\begin{aligned} & [(d/n)(\underline{a} \cdot s_{d_j} - q)] - [(d/n)(\underline{a} \cdot s_{d_j} - q - 1)] = 1 \quad (\text{resp.} = 0) \\ & \text{if } \underline{a} \cdot s_{d_j} \equiv q \pmod{n/d} \quad (\text{resp. otherwise}). \end{aligned} \quad \text{Q. E. D.}$$

Instead of Theorem 2.2 we shall prove Theorem 2.2' below by the technical reason to use an induction. That Theorem 2.2' implies Theorem 2.2 is obvious by Lemma 2.1' and Remark 2.8.

THEOREM 2.2'. *Let G be as in (1). Then the functions $\chi(\lambda_G)_1, \dots, \chi(\lambda_G)_{n'+1}$ "determine" the rotation datum λ_G of G , where $n' = n/\prod p_i$ is as in Theorem 2.2.*

To prove Theorem 2.2' we use the induction on $n = \#G$. It is obvious that Theorem 2.2' is true when $n=1$.

Let $\tilde{n} > 1$. Assuming as the induction hypothesis that

$$(14) \quad \text{Theorem 2.2' is true when } n < \tilde{n},$$

we shall show that

$$(15) \quad \text{Theorem 2.2' is true when } n = \tilde{n}.$$

First we notice, applying (14) to the group $\langle \tau^e \rangle$, that

$$(16) \quad \lambda_G(\tau^e) \ (e|n, e \neq 1) \text{ are "determined",}$$

and hence, by Lemma 2.7 (b), that

$$(17) \quad \text{Red } \lambda_G(\tau^e) \ (e|n, e \neq 1, n \text{ with } \mu(e)=0) \text{ are "determined",}$$

where we suppress the phrase: by $\chi(\lambda_G)_1, \dots, \chi(\lambda_G)_{n'+1}$.

Next we shall show that

$$(18) \quad \text{Red } \lambda_G(\tau^e) \ (e|n, e \neq n \text{ with } e \nmid n_0) \text{ is "determined",}$$

where $n_0 = \prod_{e_i=1} p_i$ for the prime decomposition $n = \prod p_i^{e_i}$. To prove (18) first we make a reduction. Since $e \nmid n_0$, there exists a prime factor p of e such that $p^2 | n$. Then, put $G' = \langle \tau^{\bar{n}} \rangle$ with $\bar{n} = n/p$ and view $\bar{G} = G/G'$ as $\text{Aut}((X/G')/(X/G))$. In our case we note by Lemma 2.10 that

$$(19) \quad \lambda_{\bar{G}}(\bar{\tau}) = \pi_{\bar{n}}(\lambda_G(\tau)), \quad \text{where } \bar{\tau} = (\tau \pmod{G'}),$$

and have by Lemma 2.7 (b) the relation:

$$(20) \quad \text{Red } \lambda_G(\tau^e) = \mu(e) \cdot \pi_{n/e}(\pi_{\bar{n}}(\lambda_G(\tau))) + \sum_{\substack{f|e \\ f \neq 1}} \mu(e/f) \cdot \pi_{n/e}(\lambda_G(\tau^f)).$$

Hence, applying (14) to \bar{G} and using (16), (19) and (20), we see that it suffices to show:

$$(21) \quad \chi(\lambda_{\bar{G}})_{\bar{q}} \quad (\bar{q}=1, \dots, n'/p+1) \text{ are "determined".}$$

To verify (21) we use the notation in Situation 2.12 and prove the following:

CLAIM 1. $\chi(\lambda_{\bar{G}})_1$ is equal to the induced function of $\chi(\lambda_G)_1$ via the natural homomorphism: $G \rightarrow \bar{G}$.

CLAIM 2. For $\bar{q}, \bar{a} \in \mathbf{Z}$ we have the relation:

$$(22) \quad \langle \chi(\lambda_G)_{p\bar{q}} - \chi(\lambda_G)_{p\bar{q}+1}, \Delta_{\bar{r}}^{p\bar{a}} \rangle - \langle \chi(\lambda_{\bar{G}})_{\bar{q}} - \chi(\lambda_{\bar{G}})_{\bar{q}+1}, \Delta_{\bar{r}}^{\bar{a}} \rangle \\ = \sum_{\substack{d|n \\ d \neq (d, \bar{n})}} (\#\{j \mid p\bar{a} \cdot s_{dj} \equiv p \cdot \bar{q} \pmod{n/d}\} - \#\{j \mid p\bar{a} \cdot s_{dj} \equiv \bar{q} \pmod{n/d}\}).$$

Before giving the proof, we set the notation. By Lemma 2.10 we have that

$$\text{Red } \lambda_{\bar{G}}(\bar{\tau}^t) = t \cdot \sum_{d,j} \lambda_{\bar{n}/t} \bar{r}_{t \ a_j} \quad (t \mid \bar{n}, t \neq \bar{n}),$$

where $\bar{r}_{t \ a_j} = u_d \cdot r_{dj}$ (with u_d being as in Lemma 2.10), and d, j run through the sets $\{d \mid d \mid n, (d, \bar{n})=t\}$, $\{1, \dots, l_d\}$, respectively. Then it is easy to see that

$$(23) \quad (t/\bar{n}) \cdot \bar{a} \cdot \bar{s}_{t \ a_j} = (d/n) \cdot p\bar{a} \cdot s_{dj} \quad (t, d \text{ as above}),$$

where $\bar{a} \cdot \bar{s}_{t \ a_j}$ is defined similarly as $\underline{a} \cdot s_{dj}$.

PROOF OF CLAIM 1. Observing that

$$(24) \quad p\bar{a} \cdot s_{dj} = n/d \quad \text{for } d \mid n, (d, \bar{n}) = \bar{n},$$

we have by Lemma 2.13 and (23) that

$$\langle \chi(\lambda_G)_1, \Delta_{\bar{r}}^{p\bar{a}} \rangle = \langle \chi(\lambda_{\bar{G}})_1, \Delta_{\bar{r}}^{\bar{a}} \rangle \quad (\bar{a} \in \mathbf{Z}).$$

This implies our claim by the Frobenius reciprocity.

PROOF OF CLAIM 2. It follows from Corollary 2.16 and Lemma 2.10 that

$$\begin{aligned} (\text{LHS}) &= \text{the left hand side of (22)} \\ &= \sum_{d|\bar{n}} \#\{j \mid p\bar{a} \cdot s_{dj} \equiv p\bar{q} \pmod{n/d}\} - (2g-2+l) \\ &\quad - \sum_{\substack{t|\bar{n} \\ t \neq \bar{n}}} \sum_{\substack{d|\bar{n} \\ (d, \bar{n})=1}} \#\{j \mid \bar{a} \cdot \bar{s}_{t \ a_j} \equiv \bar{q} \pmod{\bar{n}/t}\} + (2\bar{g}-2+\bar{l}) \end{aligned}$$

where $g=\bar{g}$ and $l=\bar{l}+l_{\bar{n}}$. Hence, using (23) and (24), we obtain that

$$(25) \quad (\text{LHS}) = \sum_{d|n} \#\{j \mid p\bar{a} \cdot s_{aj} \equiv p \cdot \bar{q} \pmod{n/d}\} \\ - \sum_{d|\bar{n}} \#\{j \mid p\bar{a} \cdot s_{aj} \cdot d/p(d, \bar{n}) \equiv \bar{q} \pmod{\bar{n}/(d, \bar{n})}\}.$$

Here we note that for $d|n$,

- (i) if $d \neq (d, \bar{n})$ then $d = p(d, \bar{n})$; and
- (ii) if $d = (d, \bar{n})$ then the two conditions on j in (25) are equivalent.

This implies that (25) yields our relation.

We use an induction to see (21), so we suppose that

$$(26) \quad \chi(\lambda_{\bar{a}})_{\bar{q}} \text{ is "determined",}$$

and shall prove that

$$(27) \quad \chi(\lambda_{\bar{a}})_{\bar{q}+1} \text{ is "determined" if } 1 \leq \bar{q} \leq n'/p.$$

In fact, by the choice of p first we notice that

$$\text{if } d|n \text{ with } d \neq (d, \bar{n}), \text{ then } \mu(d) = 0.$$

Hence it follows from Claim 2, (26) and (17) that

$$\langle \chi(\lambda_{\bar{a}})_{\bar{q}+1}, \Delta_{\bar{q}} \rangle \text{ is "determined" for } \bar{a} \in \mathbf{Z}, 1 \leq \bar{q} \leq n'/p,$$

because we have, in general, that

$$(28) \quad \text{Red } \lambda_G(\tau^d) = d \cdot \sum_{r \in I_{n/a}} \#\{j \mid r_{aj} \equiv r \pmod{n/d}\} \cdot \lambda_{n/a} r.$$

Thus we obtain (27). By Claim 1 this completes our induction and hence the proof of (18).

Finally we shall prove (essentially after [5]) that

$$(29) \quad \lambda_G(\tau) \text{ is "determined",}$$

which, together with (16), completes the proof of Theorem 2.2' by Lemma 2.7 (a).

To show (29) we begin by fixing notation:

$$J = \{a \in \mathbf{Z} \mid (a, n) | n_0, 0 < a \leq n\}.$$

We note that

$$(30) \quad a \mapsto \text{the pair } (r(a), (a, n)) \text{ defines a bijection of } J \\ \text{onto the set } \{\text{pair } (r, d) \mid d | n_0, r \in I_{n/a}\},$$

where $r(a)$ ($a \in \mathbf{Z}$) denotes the integer r ($0 < r \leq n/(a, n)$) such that $r \equiv a \pmod{n/(a, n)}$. Elements of J are ordered as follows:

$$(31) \quad a' \text{ is after } a \text{ if either } (a', n) < (a, n) \text{ or} \\ (a', n) = (a, n) \text{ and } r(a') > r(a).$$

For $a, a' \in J$ we put:

$$(32) \quad c_{aa'} = 1 \quad (\text{resp. } =0) \quad \text{if} \\ (a, n)|(a', n) \text{ and } a \equiv a' \pmod{n/(a', n)} \quad (\text{resp. otherwise}).$$

Then we notice by (31) that

$$(33) \quad \text{the matrix } (c_{aa'}) \text{ is lower triangular with } c_{aa}=1 \quad (a \in J).$$

Moreover we set:

$$m_{ad} = \#\{j \mid r_{dj} \equiv a \pmod{n/d}\} \quad (a \in \mathbf{Z}, d \mid n).$$

Then it is easy to see the followings:

$$(34) \quad m_{ad} = m_{a'd} \quad \text{if } a \equiv a' \pmod{n/d}.$$

$$(35) \quad (a, n/d) = 1 \text{ and } d \neq n \quad \text{if } m_{ad} \neq 0.$$

For $a \in J$ we have by Corollary 2.16 that

$$(36) \quad \sum_{a' \in J} c_{aa'} \cdot m_{a'(a', n)} = \sum_{\substack{d \mid n_0 \\ (a, n) \mid d}} m_{ad} \quad (\text{by (32), (34), (30)}) \\ = \sum_{\substack{d \mid n_0 \\ (a, n/d)=1}} m_{ad} \quad (\text{since } (d, n/d)=1) \\ = \sum_{d \mid n} m_{ad} - \sum_{\substack{d \mid n \\ d \nmid n_0}} m_{ad} \quad (\text{by (35)}) \\ = \langle \chi(\lambda_G)_1 - \chi(\lambda_G)_2, A^a \rangle + 2g - 2 + l - \sum_{\substack{d \mid n \\ d \nmid n_0}} m_{ad}.$$

Since the right hand side of (36) is “determined” by Lemma 2.15 and (18) (cf. also, (28)), it follows from (33) that $m_{a(a, n)}$ are “determined” for $a \in J$, and hence by (30) that

$$\lambda_G(\tau) = \sum_{r \in I_n} m_{r1} \cdot \lambda_{nr} \quad \text{is “determined”,}$$

as asserted. This completes the proof of Theorem 2.2' and hence of Theorem 2.2.

PROOF OF COROLLARY 2.3. This follows from Remark 2.8 and Lemma 2.1'.
Q. E. D.

PROOF OF COROLLARY 2.4. This follows from a theorem of Wiman (cf. [7]).
Q. E. D.

(2.5) Counter example. In his paper [5, p. 219 Corollary], Guerrero asserts substantially that

$$(37) \quad \text{Corollary 2.3 holds even if the condition (c) is replaced} \\ \text{by the condition: (c')} \quad \chi_q = \chi'_q \quad \text{for } q=1, 2.$$

However it seems that the proof given has a gap (cf. [5, p. 222 “the last ... independent”]). Here we make a counter example to (37).

Let g be a non-negative integer. By the existence theorem there exists an automorphism τ of order $n=25$ of some curve X of genus $25g+116$ such that

- (i) the signature of $\langle \tau \rangle$ is $(g, 12; 5, 5, 25, \dots, 25)$;
- (ii) $\phi_{\langle \tau \rangle}(C_j) = \tau^{15}$ (resp. $\tau^{10}, \tau^1, \tau^{21}, \tau^{16}, \tau^{11}, \tau^6, \tau^{19}, \tau^{14}, \tau^9, \tau^4, \tau^{24}$) if $j=1$ (resp. $2, \dots, 12$).

Put $G = \mathbf{Z}/n\mathbf{Z}$ and let λ, χ_q (resp. λ', χ'_q) denote $\lambda_G, \text{Tr}(G|H^0(X, \Omega_X^{\otimes q}))$, respectively, defined for G considered as a subgroup of $\text{Aut}(X)$ via the injection: $(u \bmod n \cdot \mathbf{Z}) \rightarrow \tau^u$ (resp. τ^{2u}). For $a \in \mathbf{Z}$, we denote by Δ^a the homomorphism: $G \rightarrow k^\times$ defined by $(1 \bmod n \cdot \mathbf{Z}) \rightarrow \zeta_n^a$. Then it is easy to see the following:

$$\begin{aligned} \langle \chi_1, \Delta^a \rangle &= \langle \chi'_1, \Delta^a \rangle = g \quad (\text{resp. } g+5, g+4) \\ &\text{if } a \equiv 0 \pmod{25} \quad (\text{resp. if } 5 \nmid a, \text{ otherwise}); \\ \langle \chi_2, \Delta^a \rangle &= \langle \chi'_2, \Delta^a \rangle = 3g+9 \quad (\text{resp. } 3g+14) \\ &\text{if } a \equiv 0 \pmod{25} \quad (\text{resp. otherwise}). \end{aligned}$$

Hence $\chi_q = \chi'_q$ for $q=1, 2$. On the other hand, since now $\lambda \neq \lambda'$ we have by Remark 2.8 and Corollary 2.3 that

$$\chi_q \neq \chi'_q \quad \text{for some } q.$$

In fact this is so for $q=5+1$.

3. Topological equivalence.

The purpose of this section is to give a proof to Theorem in Introduction. So throughout this section we suppose that $k = \mathbf{C}$.

(3.1) A group-theoretic lemma. Before giving the proof, we prepare a lemma which is a modification of Theorem 14 in [8].

LEMMA 3.1. *Let $(g, l; m_1, \dots, m_l), \Gamma, A_i, B_i$ and C_j be as in Notation 1.2. Assume that τ is a generator of our cyclic group G , and $\phi: \Gamma \rightarrow G$ is a surjective homomorphism. Then for any permutation k of $\{1, \dots, l\}$ with $m_{k(j)} = m_j$ ($j=1, \dots, l$), there exists an automorphism θ of Γ such that*

- (i) $\phi \circ \theta(A_i) = \phi \circ \theta(B_i) = \tau \quad (i=1, \dots, g)$;
- (ii) $\theta(C_j) = D_j \cdot C_{k(j)} \cdot D_j^{-1} \quad \text{for some } D_j \in \Gamma \quad (j=1, \dots, l)$.

PROOF. We list the automorphisms of Γ : $\theta_{(1)}, \theta_{(2)}, \theta_{(3)}, \theta_u, \theta'_v$ and θ''_w ($u=1, \dots, g-1$; $v=1, \dots, l$; $w=1, \dots, l-1$ with $m_w = m_{w+1}$), giving their non-trivial part of actions on the generators:

$$\begin{aligned}
\theta_{(1)} &: A_1 \mapsto A_1 B_1; \\
\theta_{(2)} &: A_1 \mapsto A_1 B_1 A_1^{-1}, \quad B_1 \mapsto A_1^{-1}; \\
\theta_{(3)} &: C_j \mapsto A_2 C_j A_2^{-1}, \quad A_1 \mapsto A_2 A_1, \quad A_2 \mapsto B_1 A_2 B_1^{-1}, \quad B_2 \mapsto A_2 B_2 A_2^{-1} B_1^{-1}, \\
&\quad A_i \mapsto A_2 A_i A_2^{-1}, \quad B_i \mapsto A_2 B_i A_2^{-1} \quad (i=3, \dots, g); \\
\theta_u &: A_u \mapsto A_{u+1}, \quad B_u \mapsto B_{u+1}, \quad A_{u+1} \mapsto E_{u+1}^{-1} A_u E_{u+1}, \quad B_{u+1} \mapsto E_{u+1}^{-1} B_u E_{u+1}, \\
&\quad \text{where } E_{u+1} = [A_{u+1}, B_{u+1}]; \\
\theta'_v &: C_j \mapsto A_1^{-1} C_j A_1 \quad (j=v, \dots, l), \\
&\quad A_1 \mapsto [A_1^{-1}, (C_v \cdots C_l)^{-1}] \cdot A_1, \quad B_1 \mapsto B_1 A_1^{-1} (C_v \cdots C_l) A_1; \\
\theta''_w &: C_w \mapsto C_{w+1}, \quad C_{w+1} \mapsto C_{w+1}^{-1} C_w C_{w+1} \quad (\text{assuming } m_w = m_{w+1}).
\end{aligned}$$

We begin with a reduction. In fact, replacing ϕ by $\phi \circ \theta$ for some $\theta \in \langle \theta''_w \mid w=1, \dots, l-1 \text{ with } m_w = m_{w+1} \rangle$, we may assume that $k=1$.

Next we list the non-trivial part of the effect on the generators, letting:

$$\begin{aligned}
(1) \quad \phi &\text{ be defined by } C_j \mapsto x_j, \quad A_i \mapsto a_i, \quad B_i \mapsto b_i, \\
\phi \circ \theta_{(1)} &: A_1 \mapsto a_1 b_1; \\
\phi \circ \theta_{(2)} &: A_1 \mapsto b_1, \quad B_1 \mapsto a_1^{-1}; \\
\phi \circ \theta_{(3)} &: A_1 \mapsto a_1 a_2, \quad B_2 \mapsto b_2 b_1^{-1}; \\
\phi \circ \theta_u &: A_u \mapsto a_{u+1}, \quad A_{u+1} \mapsto a_u, \quad B_u \mapsto b_{u+1}, \quad B_{u+1} \mapsto b_u; \\
\phi \circ \theta'_v &: B_1 \mapsto b_1 \cdot x_v \cdots x_l.
\end{aligned}$$

Using the list we reduce ϕ systematically. The case where $g=0$ is trivial, so we assume that $g \geq 1$. We denote by $A\Gamma$ the subgroup of $\text{Aut}(\Gamma)$:

$$\langle \theta_{(1)}, \theta_{(2)}, \theta_{(3)}, \theta_u, \theta'_v \mid u=1, \dots, g-1; v=1, \dots, l \rangle.$$

Let ϕ be as in (1). We describe the effect in $\phi \circ \theta$ as the above manner.

Step 1. There exists θ in $A\Gamma$ such that

$$\phi \circ \theta : A_1 \mapsto 1, \quad B_1 \mapsto (\text{an element of } G).$$

In fact, this follows from the "Euclidean algorithm" by using $\theta_{(1)}$ and $\theta_{(2)}$.

Step 2. There exists θ in $A\Gamma$ such that

$$\phi \circ \theta : A_i \mapsto 1, \quad B_i \mapsto (\text{an element of } G).$$

In fact, this follows from Step 1 by using θ_u .

Step 3. There exists θ in $A\Gamma$ such that

$$\phi \circ \theta : B_1 \mapsto (\text{an element of } G), \quad \text{the other } A_i, B_i \mapsto 1.$$

In fact, by Step 2 we may assume that $g \geq 2$ and $a_1 = \cdots = a_g = 1$. Then there exists θ' in $A\Gamma$ such that

$$\phi \circ \theta' : B_1 \mapsto 1, \quad B_2 \mapsto (\text{an element of } G),$$

which follows from the “Euclidean algorithm” by using $\theta_{(3)}$ and θ_1 . Using θ_u and repeating as above we obtain Step 3.

Step 4. There exists θ in $A\Gamma$ such that

$$\phi \circ \theta : B_1 \mapsto \tau, \quad \text{the other } A_i, B_i \mapsto 1.$$

In fact, this follows from the surjectivity of ϕ by using $\theta_{(1)}$, $\theta_{(2)}$, θ'_v and Step 3.

Finally our lemma follows by using $\theta_{(1)}$, $\theta_{(3)}$, θ_u and Step 4. Q. E. D.

(3.2) Proof of Theorem. The implication: (a) \Rightarrow (b) follows from the trace formula (Lemma 2.1) by using the following:

REMARK 3.2. Let $\sigma : \mathcal{C} \rightarrow \mathcal{C}$ be the mapping defined by $z \mapsto \exp(2\pi\sqrt{-1}/m) \cdot z$ ($m \in \mathbf{Z}, m \geq 1$). Let $w : D \rightarrow D'$ be a homeomorphism such that $w \circ \sigma \circ w^{-1} = \sigma \circ \dots \circ \sigma$ (r -times), where D and D' denote open neighbourhoods of the origin of the plane \mathcal{C} . If w is orientation-preserving (resp. -reversing) then $r \equiv 1$ (resp. $\equiv -1$) mod m .

We shall prove the converse by using the following:

LEMMA 3.3 ([12], [15]). *Let Γ and Γ' be Fuchsian groups (acting on the unit disk U) with compact orbit spaces. Then any isomorphism $\theta : \Gamma \rightarrow \Gamma'$ can be geometrically realized, i. e., there is a homeomorphism $w : U \rightarrow U$ such that for all $A \in \Gamma$ we have $w \circ A = \theta(A) \circ w$.*

We begin by fixing some notation. In fact, for our G and X let $K, \Gamma, (g, l; m_1, \dots, m_l), \phi : \Gamma \rightarrow G, R, z_i, e_j, A_i, B_i, C_j$ and the permutation k be as in Remark 1.1. For G' and X' we put ' to represent the corresponding object. Applying Corollary 2.3 to our assumption, we have that

$$(2) \quad \lambda_G(\sigma) = \lambda_{G'}(\sigma') \quad \text{for } \sigma \in G$$

(cf. also, Remark 2.8). This implies in particular that

$$(g, l; m_1, \dots, m_l) = (g', l'; m'_1, \dots, m'_l).$$

Here we may assume that k' is the identity and by (2) (cf. also, (4) of § 1) moreover that

$$(3) \quad \phi(C_j)' = \phi'(C'_{k(j)}) \quad j=1, \dots, l.$$

Next, applying Lemma 3.1 to ϕ and ϕ' , we obtain by virtue of (3) that

$$(4) \quad \text{there exists an isomorphism } \theta : \Gamma \rightarrow \Gamma' \text{ such that} \\ \text{(i) } \phi' \circ \theta = ' \circ \phi; \quad \text{(ii) } \theta(C_j) = D'_j \cdot C'_{k(j)} \cdot D_j'^{-1} \quad (D'_j \in \Gamma').$$

Let $w:U \rightarrow U$ be a homeomorphism which induces θ . If w is orientation-preserving then we put: $w_1=1_U$ and $\theta_1=1_\Gamma$. Now assume that w is orientation-reversing. Then, by (ii) of (4) and Remark 3.2 we have that

$$(5) \quad m_1 = \cdots = m_l = 2.$$

Here we claim that

$$(6) \quad \text{there exists an automorphism } \theta_1 \text{ of } \Gamma \text{ such that (i) } \phi \circ \theta_1 = \phi; \text{ (ii) } \theta_1 \text{ is induced by an orientation-reversing homeomorphism } w_1: U \rightarrow U.$$

To see (6), by using Lemma 3.1 it suffices to show that

$$(7) \quad \text{there exists an orientation-reversing homeomorphism } w_1: U \rightarrow U \text{ which induces a permutation of } \{A_i, B_i, C_j\}.$$

By Bers' version of the "continuity method" via the theory of quasiconformal mappings (cf. [9, p. 7]), to show (7) we may assume moreover that

$$\begin{aligned} \Gamma &\text{ is a standard group (cf. [1, p. 224]),} \\ R &\text{ is equal to } K \text{ in [1, p. 223].} \end{aligned}$$

In such a case, by (5) R is a regular non-Euclidean polygon with $4g+l$ sides. Put: $w_1(z) = \exp(-2\pi\sqrt{-1}l/(4g+l)) \cdot \bar{z}$ ($z \in R$). It is obvious that w_1 extends to an orientation-reversing homeomorphism of U by the action of Γ . Since $w_1 \circ A_i \circ w_1^{-1}$ is a Möbius transformation which sends the arc $\widehat{z_{4i'-3} z_{4i'-2}}$ onto the arc $\widehat{z_{4i'} z_{4i'-1}}$ (with $i' = g+1-i$), we have that $w_1 \circ A_i = B_{g+1-i} \circ w_1$. Similarly we have that $w_1 \circ B_i = A_{g+1-i} \circ w_1$ and $w_1 \circ C_j = C_{l+1-j} \circ w_1$, as desired.

Finally it is obvious by the definition of ϕ, ϕ' that

$$(8) \quad \text{the mapping } h: U/K \rightarrow U/K' \text{ induced from } w \circ w_1 \text{ has the desired property.}$$

This completes the proof of our Theorem.

REMARK. Theorem 1 in [13] implies that we may choose as our w an orientation-preserving one (cf. Lemma 3.1).

(3.3) Interpretation in terms of the Teichmüller theory. First we recall some definitions after [2]. Let X_0 be a fixed compact Riemann surface. Two quasiconformal homeomorphisms, $f: X_0 \rightarrow X$ and $f': X_0 \rightarrow X'$, are called equivalent if there is a conformal mapping $h: X \rightarrow X'$ such that $f'^{-1} \circ h \circ f$ is homotopic to the identity. Let $\{f\}$ denote the equivalence class of f . The set of all $\{f\}$ is the Teichmüller space $T(X_0)$ of X_0 . The modular group $\text{Mod}(X_0)$ is defined as the factor group of all quasiconformal selfmappings g of X_0 over the normal

subgroup of those homotopic to the identity. The element of $\text{Mod}(X_0)$ defined by a selfmapping g will be denoted by $[g]$. The group $\text{Mod}(X_0)$ acts on $T(X_0)$ by: $\{f\} \mapsto \{f \circ g^{-1}\}$ ($[g] \in \text{Mod}(X_0)$). It is well-known that

for $\{f: X_0 \rightarrow X\} \in T(X_0)$, if $\text{genus}(X_0) \geq 2$, the homomorphism η_f defined by $\sigma \mapsto [f^{-1} \circ \sigma \circ f]$ is in fact an isomorphism of $\text{Aut}(X)$ onto the isotropy subgroup: $\{[g] \in \text{Mod}(X_0) \mid \{f \circ g^{-1}\} = \{f\}\}$.

By Proposition 2 of [2] (cf. also, [5, p. 222]) we have:

THEOREM^{bis}. *In the situation of Theorem in Introduction, let X_0 be a compact Riemann surface of genus \tilde{g} and take $\{f: X_0 \rightarrow X\}, \{f': X_0 \rightarrow X'\} \in T(X_0)$. Then the conditions (a), (b) are equivalent to the condition:*

(c) *There exists an element $[g] \in \text{Mod}(X_0)$ such that*

$$[g] \cdot \eta_f(\sigma) = \eta_{f'}(\sigma') \cdot [g] \quad (\sigma \in G).$$

In [4], Gilman formulated an invariant (the same as our “rotation datum”) of $[g] \in \text{Mod}(X_0)$ of prime order with respect to the relation of $\text{Mod}(X_0)$ -conjugacy. We remark here that such a special case of our Theorem^{bis} is proved in [5].

4. Realizability and representability.

Throughout this section we suppose that

(1) our k is of characteristic zero, and H denotes a group of order n .

(4.1) **Statement.** Our motivation to introduce the rotation datum λ (defined below) is the fact in the following:

COROLLARY 4.1 (to Theorem 2.2). *Let X be a curve of genus $\tilde{g} \geq 2$. Let $\lambda_{\text{Aut}(X)}: \text{Aut}(X) \rightarrow \text{Map}(\mathbf{Z}, \mathbf{Q})$ be defined as in Definition 2.6. Then $\lambda_{\text{Aut}(X)}$ is “determined” by the sequence $(\text{Tr}(\text{Aut}(X) | H^0(X, \mathbf{Q}_X^{\otimes q})))_{q \geq 1}$ and vice versa.*

DEFINITION 4.2. (a) A mapping $\lambda: H \rightarrow \text{Map}(\mathbf{Z}, \mathbf{Q})$ (resp. $H \setminus \{1\} \rightarrow \text{Map}(\mathbf{Z}, \mathbf{Q})$) such that

$$\lambda(\sigma) \in \text{Map}((\mathbf{Z} / \# \sigma \cdot \mathbf{Z})^\times, \mathbf{Q}) \quad \text{for } \sigma \in H \text{ (resp. } H \setminus \{1\})$$

is called a rotation datum (resp. semi-rotation datum) of H if the following two conditions are satisfied:

- (i) If $\sigma, \sigma' \in H \setminus \{1\}$ are H -conjugate then $\lambda(\sigma) = \lambda(\sigma')$;
 - (ii) For $\sigma \in H \setminus \{1\}$ and $(u, n) = 1$, the \mathbf{Q} -automorphism of $\text{Map}((\mathbf{Z} / \# \sigma \cdot \mathbf{Z})^\times, \mathbf{Q})$ defined by $\lambda_{\# \sigma r} \mapsto \lambda_{\# \sigma r u}$ sends $\lambda(\sigma)$ to $\lambda(\sigma^u)$.
- (b) We say that a rotation datum λ is of genus $\tilde{g}(\lambda) = \tilde{g}$ ($\in \mathbf{Q}$) if $\text{deg}(\lambda(1)) =$

$2-2\tilde{g}$.

We need a fundamental result for later use:

LEMMA 4.3. *Let λ be a rotation datum of G . Let $\text{Red}\lambda$ denote the semi-rotation datum defined by the relations in Lemma 2.7. Put $l(\sigma)=[N_G(\langle\sigma\rangle):\langle\sigma\rangle]^{-1}\cdot\text{deg}(\text{Red}\lambda(\sigma))$ for $\sigma\in G\setminus\{1\}$ with $l(1)=0$, and $g(\lambda)=(1/\#G)\cdot\sum_{\sigma\in G}\{1-(1/2)\cdot\text{deg}(\lambda(\sigma))\}$. Then we have the following relation:*

$$(RH) \quad 2\tilde{g}(\lambda)-2 = n(2g(\lambda)-2) + n \cdot \sum_{\substack{m|n \\ m\neq 1}} l(\sigma_m)\{1-1/m\}$$

where σ_m denotes an element of G of order m .

PROOF. Let τ denote a generator of G . From the definition of $g(\lambda)$, we have that

$$(2) \quad 2\tilde{g}(\lambda)-2 = n(2g(\lambda)-2) + \sum_{u=1}^{n-1} \text{deg}(\lambda(\tau^u)).$$

On the other hand, since $l(\sigma_m)$ is independent of the choice of σ_m , we have by Lemma 2.7 that

$$(3) \quad \sum_{u=1}^{n-1} \text{deg}(\lambda(\tau^u)) = n \cdot \sum_{\substack{m|n \\ m\neq 1}} l(\tau^{n/m})\{1-1/m\}.$$

Combining (2) and (3), we obtain our lemma.

Q. E. D.

The normality (defined below) is a basic necessary condition for $\lambda=\lambda_G$.

DEFINITION 4.4. Let λ be a rotation datum of G .

- (a) We say that λ is normal if $[N_G(\langle\sigma\rangle):\langle\sigma\rangle]^{-1}\cdot\text{Red}\lambda(\sigma)$ is of the form: $\sum_{j=1}^{l(\sigma)}\lambda_{\#\sigma u_j}$ with $u_j\in I_{\#\sigma}$ and $l(\sigma)\in\mathbf{Z}$ ($l(\sigma)\geq 0$) for each $\sigma\in G\setminus\{1\}$.
- (b) For a normal rotation datum λ the datum:

$$(g(\lambda), \sum_t l(\sigma_{m_t}); \underbrace{m_1, \dots, m_1}_{l(\sigma_{m_1})\text{-times}}, \dots, \underbrace{n, \dots, n}_{l(\sigma_n)\text{-times}})$$

is called the signature of λ , where σ_m denotes an element of order m and $m_0=1, m_1, m_2, \dots, n$ are the divisors of n such that $m_0 < m_1 < \dots < n$.

- (c) *Remark.* In the case where $\lambda=\lambda_G$ with G being an automorphism group of a curve, we have by Lemma 2.7 that λ is normal and the signature of λ is equal to the signature of G .

Finally, for a rotation datum λ of H we define the class functions $\chi(\lambda)_q: H\rightarrow k$ ($q\in\mathbf{Z}$) as in Notation 2.11.

The purpose of this section is to prove the following:

PROPOSITION 4.5. *Let $(\chi_q: G\rightarrow k)_{q\geq 1}$ be a sequence of class functions such that $\chi_1(1)\in\mathbf{Z}$ with $\chi_1(1)\geq 2$. Then the following two conditions are equivalent:*

- (a) $(\chi_q)_{q \geq 1}$ is realizable (over k), i. e., $\chi_q = \text{Tr}(G|H^0(X, \Omega_X^{\otimes q}))$ ($q \geq 1$) with $G \cong \text{Aut}(X)$ for some curve X .
- (b) (i) χ_1 is a character of some representation of G ;
(ii) there exists a normal rotation datum λ of G such that $\chi_1 = 1 + \chi(\lambda)_1$ and $\chi_q = \chi(\lambda)_q$ ($q \geq 2$).

If these conditions are satisfied, then λ in (b) is unique and in fact $\lambda = \lambda_G$ for G in (a).

(4.2) Proof of Proposition 4.5. The implication: (a) \Rightarrow (b) is already seen. To prove the converse, we suppose (b). First we note by (i) that

$$(4) \quad g(\lambda|_{G'}) \text{ is a non-negative integer for any subgroup } G' \subseteq G,$$

because we have $g(\lambda|_{G'}) = \langle \chi_1|_{G'}, 1 \rangle$ (cf. Lemma 2.15).

Next we set some notation: assume $G = \langle \tau \rangle$ and let $(g(\lambda), l; m_1, \dots, m_l)$ denote the signature of λ . Assume that for $d|n$, $d \neq n$

$$(5) \quad \text{Red } \lambda(\tau^d) = d \sum_{n/d = m_j} \lambda_{m_j} r_j \quad (r_j \in I_{m_j}),$$

and that $s_j \in I_{m_j}$ such that $s_j r_j \equiv 1 \pmod{m_j}$. Then from the proof of Lemma 2.13 we have that

$$\langle \chi(\lambda)_1, A_\tau^1 \rangle \equiv g(\lambda) - \sum_j s_j / m_j \pmod{1}.$$

Hence it follows from (4) and (i) that

$$\sum_j s_j n / m_j \equiv 0 \pmod{n}.$$

This implies that we have a well-defined group homomorphism $\phi: \Gamma \rightarrow G$ such that

$$(6) \quad A_i, B_i \mapsto \tau \quad \text{and} \quad C_j \mapsto (\tau^{n/m_j})^{s_j},$$

where Γ , A_i , B_i and C_j are as in Notation 1.2. We shall show that

$$(7) \quad \phi \text{ is surjective and } \text{Ker}(\phi) \text{ is torsion-free.}$$

To see (7) it suffices to show that

$$(8) \quad \text{l. c. m. } \{m_1, \dots, m_l\} = n \quad \text{if } g(\lambda) = 0,$$

by virtue of Lemma 1.7. To verify (8), we put

$$\text{l. c. m. } \{m_j\} = m, \quad \bar{n} = n/m, \quad G' = \langle \tau^{\bar{n}} \rangle \quad \text{and} \quad \tau' = \tau^{\bar{n}}.$$

Then by the proof of Lemma 2.9 we have that

$$l'(\tau'^s) = \sum_{d=(\bar{n}, \bar{n})s}^{d|n} (d, \bar{n}) \cdot l(\tau^d) \quad (s|m, s \neq m),$$

where $l'(\tau'^s)$ is defined for $\lambda|_{G'}$ similarly as $l(\tau^d)$ for λ . Using this and the relations (RH) for $\lambda, \lambda|_{G'}$, we obtain that

$$\begin{aligned} (9) \quad & 2g(\lambda|_{G'}) - 2 - \bar{n}(2g(\lambda) - 2) \\ &= \bar{n} \cdot \sum_{d|n} l(\tau^d) \{1 - (d/n)\} - \sum_{s|m} l'(\tau'^s) \{1 - (s/m)\} \\ &= \sum_{d|n} l(\tau^d) \{\bar{n} - (d, \bar{n})\}. \end{aligned}$$

The right hand side of (9) is equal to 0 by the choice of \bar{n} . This yields that $g(\lambda|_{G'}) + \bar{n} = 1$ and hence that $\bar{n} = 1$, i. e., $n = m$ by (4), as desired.

Applying Theorem 1.6 to our ϕ , we conclude that G is an automorphism group of some curve X and

$$(10) \quad \phi = \phi_G \text{ up to the equivalence.}$$

This means $\chi_q = \text{Tr}(G|H^0(X, \Omega_X^q))$, because by comparing (3) in § 2 with (5), (6), it follows from (10) that $\text{Red } \lambda = \text{Red } \lambda_G$ and hence that $\lambda = \lambda_G$, $\chi(\lambda)_q = \chi(\lambda_G)_q$.

The rest part follows from Corollary 2.3 (cf. also, Remark 2.8). Q. E. D.

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