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A linear prediction problem for symmetric α-stable processes with $1/2<\alpha<1$

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§ 1. Introduction.

Linear prediction problems of weakly stationary processes are well studied when the processes have second moments [2], [4]. For strictly stationary processes with first moments, Urbanik introduces a class which "admits a prediction" and proves parallel results [9]. In this paper we consider a class of processes which have infinite first moments. Process which admits a prediction is the stationary case of the linear processes that P. Lévy introduced as an extension of the class of Gaussian processes by imposing the linear regression property only on regression on the past [5]. Here we study the case where process X_t has a symmetric α-stable, $0<\alpha<1$, distribution. This is also an example of the class of infinitely divisible processes of Maruyama [6]. We prove that when it is completely non-deterministic it has a canonical representation

$$X_t = \int_{-\infty}^{t} f(t-u)M(du)$$

where $M(du)$ is a stochastic measure such that $\{M(du), u\leqq t\}$ has the same information as $\{X_u ; u\leqq t\}$. We call $M(du)$ innovations of the process X_t. Precise meanings are explained in the following sections.

One of difficulties in our case lies on how to take innovations out of the process. Urbanik uses theories of Banach spaces such as theories of Bochner integral and linear functionals. The linear space spanned by an $S\alpha S$ process, $0<\alpha<1$, is only a Fréchet space. We overcome this difficulty under some additional conditions. In § 3, we define a Riemann type integral of functions with values in this Fréchet space and give a sufficient condition for integrability. In § 4 we give several lemmas. Although they are similar to Urbanik's lemmas, technique to prove them is quite different from the Banach space case and more complicated. We use the integral of § 3 to take innovations out of the processes. Theorems are stated in § 5. Depending on our sufficient condition for integrability, we get results only in the case $1/2<\alpha<1$. If we can improve our condition of integrability, we may extend the results to the case $\alpha\leqq 1/2$. This is left for further research.

When $\alpha=1$, our process has a symmetric Cauchy distribution. Still it has no finite first moment, but the linear space spanned by it is a Banach space. Then we can use Urbanik's techniques and get the same results. Of course we can include the case $\alpha=1$ in our scheme.

In this paper the words "linear process" are used as a synonym of "process which admits a prediction". Problem to determine or characterize the class of linear processes is still left for further study. But we note that in the case of $S\alpha S$ processes, $0<\alpha<2$, unlike Gaussian case, the class of harmonizable processes (that is, Fourier transforms of stable random measures) and the class of linear processes are mutually disjoint, [**1**], [**11**]. When processes have second moments, some investigations of linear processes are made by Hida and Ikeda [**3**] including a sufficient condition for N-ple Markov processes to be linear. The process in the example stated in Remarks of § 5 has a simple Markov type kernel.

§ 2. Preliminaries.

1. Let $0<\alpha\leqq 2$. A stochastic process $\{X_t\}$ $(-\infty<t<\infty)$ is called a symmetric α-stable ($S\alpha S$) process if every finite linear combination $Y=\sum_{i=1}^{n}c_iX_{t_i}$ has an $S\alpha S$ distribution with a characteristic function

$$E\exp(iuY)=\exp(-a_Y|u|^{\alpha}), \tag{2.1}$$

where a_Y is a nonnegative constant depending on Y. Hereafter we consider the case $0<\alpha<1$ and define $\|Y\|=a_Y$ whenever Y has a characteristic function (2.1). It is easily seen that for such linear combinations X and Y (i) $\|X\|=0$ if and only if $X=0$ a.s., (ii) $\|X+Y\|\leqq\|X\|+\|Y\|$, (iii) $\|aX\|=|a|^{\alpha}\|X\|$ for real a and moreover (iv) $\|X+Y\|=\|X\|+\|Y\|$ if X and Y are mutually independent (see [8]). So, this induces a metric and the convergence defined by this metric is equivalent to the convergence in probability. The space of all such linear combinations and their limits in probability is denoted by $[X_t]$. Every element $X\in[X_t]$ has an $S\alpha S$ distribution and thus the definition of $\|X\|$ is extended to $[X_t]$. The space $[X_t]$ is a Fréchet space with the quasi-norm $\|X\|$.

2. Let $F(u)$ be a process with independent increments which is continuous in probability such that, for $u_1>u_2$, $F(u_1)-F(u_2)$ has an $S\alpha S$ distribution. We can define stochastic integrals $\int_{-\infty}^{\infty}f(u)dF(u)$ as usual by convergence in probability. The set of all F-integrable functions is denoted by $L(F)$. It is known that $L(F)$ is equal to $L^{\alpha}(d\|F(u)\|)$. $\int_{-\infty}^{\infty}f(u)dF(u)$ is an $S\alpha S$ random variable and

$$\left\| \int_{-\infty}^{\infty} f(u)dF(u) \right\| = \int_{-\infty}^{\infty} |f(u)|^{\alpha} d\|F(u)\| . \tag{2.2}$$

It is known that, for $f(u)$, $g(u) \in L^{\alpha}(d\|F(u)\|)$, the integrals $\int_{-\infty}^{\infty} f(u)dF(u)$ and $\int_{-\infty}^{\infty} g(u)dF(u)$ are independent if and only if $f(u)g(u)=0$ a. e. with respect to $d\|F(u)\|$. Let $[F]$ denote the closed linear space spanned by $\{F(u_1)-F(u_2)\,;\, u_1>u_2\}$. For every element $x \in [F]$, there exists a function $f \in L(F)$ such that $x=\int_{-\infty}^{\infty} f(u)dF(u)$. Refer to [**8**], [**10**].

3. We quote some definitions and properties from Urbanik [**9**]. We use convergence in probability instead of mean convergence.

Let $\{X_t\}$ be a strictly stationary process and be continuous in probability. Let $[X_t\,;\, t \leqq a]$ be the closed linear subspace of $[X_t]$ spanned by $\{X_t\,;\, t \leqq a\}$. We identify two elements in $[X_t]$ which equal a. s.

We say that $\{X_t\}$ is linear or admits a prediction if there is a continuous linear operator A_0 from $[X_t]$ onto $[X_t\,;\, t \leqq 0]$ such that

(i) for every $X \in [X_t\,;\, t \leqq 0]$, $A_0 X = X$,

(ii) if X is independent of Y for every $Y \in [X_t\,;\, t \leqq 0]$, then $A_0 X = 0$,

(iii) for every $X \in [X_t]$, $X - A_0 X$ is independent of every $Y \in [X_t\,;\, t \leqq 0]$.

Let $\{T_t\}$ be a group of shift operators of $\{X_t\}$, that is, $T_t X_s = X_{t+s}$. We define $A_a = T_a A_0 T_{-a}$. The operator A_a is from $[X_t]$ onto $[X_t\,;\, t \leqq a]$ and satisfies the same three properties as above with $[X_t\,;\, t \leqq 0]$ replaced by $[X_t\,;\, t \leqq a]$. For any semi-closed interval $I=(a, b]$, we define $A(I) = A_b - A_a$. The following facts are easily proved:

(2.3) if $J_1 \cap J_2 \neq \emptyset$, then $A(J_1)A(J_2) = A(J_1 \cap J_2)$,

(2.4) if $Y_1, Y_2, \cdots, Y_n \in [X_t]$ and $I_1, I_2, \cdots, I_n$ are disjoint intervals, then $A(I_1)Y_1, A(I_2)Y_2, \cdots, A(I_n)Y_n$ are independent,

(2.5) for every $X \in [X_t]$, there exists a limit $\lim_{t \to -\infty} A_t X$ in probability.

We write this limit in (2.5) as $A_{-\infty}X$. When $A_0 X = X$ holds for every $X \in [X_t]$, we call $\{X_t\}$ deterministic. If $\lim_{t \to -\infty} A_t X = 0$ for every $X \in [X_t]$, then we call $\{X_t\}$ completely non-deterministic. A strictly stationary linear process $\{X_t\}$ is decomposed into two independent stationary linear processes, one of which is deterministic and the other is completely non-deterministic.

§ 3. Riemann type integral.

The author introduced Riemann type integral of functions with values in a Fréchet space F with a quasi-norm such that for every $y \in F$ and real number

c, $\|cy\|=|c|^{\alpha}\|y\|$. We quote definitions and theorems from [7]. Proofs are given in [7].

Let y_t be a function of $t \in I=[a, b]$ with values in F.

DEFINITION 3.1. Let γ, δ_0, K be positive numbers. We say that y_t satisfies Condition $C_\gamma(\delta_0, K)$ if $\|y_t-y_s\| \leqq K|t-s|^{\gamma}$ whenever $t, s \in I$ and $|t-s| \leqq \delta_0$.

Let $\{I_i, 1 \leqq i \leqq n\}$ be a partition of I such that $a=a_0<a_1<\cdots<a_n=b$, $I_i=[a_{i-1}, a_i]$. A pair of $\{I_i\}$ and $\{t_i\}$, $t_i \in I_i$, is denoted by $S=(\{I_i\}, \{t_i\})$. The length of I_i is denoted by $|I_i|$.

DEFINITION 3.2. We say that y_t is Riemann type integrable over I if there is an element $\mathcal{J}$ in F with the following property: For each $\varepsilon>0$, there is $\delta>0$ such that

$$\left\|\sum_{i=1}^{n}|I_i|y_{t_i}-\mathcal{J}\right\|<\varepsilon$$

whenever $S=(\{I_i\}, \{t_i\})$ satisfies $\max_{1 \leqq i \leqq n}|I_i|<\delta$. We call $\mathcal{J}$ the Riemann type integral of y_t over I and write

$$\mathcal{J}=\int_I y_t dt .$$

We have the following theorems.

THEOREM 3.1. *If y_t satisfies Condition $C_\gamma(\delta_0, K)$ for some δ_0, K and γ such that $1 \geqq \gamma > 1-\alpha$, then y_t is Riemann type integrable over I.*

THEOREM 3.2. *Under the same conditions as Theorem 3.1, we have the inequality*

$$\left\|\int_I y_t dt\right\| \leqq M^{1-\alpha}|I|^{\alpha} \sup_{t \in I}\|y_t\|+M^{-\rho}|I|^{\alpha+\gamma}KA_{\alpha\gamma} \tag{3.1}$$

where $\rho=\alpha+\gamma-1$, $A_{\alpha\gamma}=2^{\gamma-\alpha}/(2^{\rho}-1)+2^{\gamma}$ and M is any integer bigger than $2|I|/\delta_0$.

COROLLARY 3.3. *If $|I|$ is smaller than $\delta_0/2$, then*

$$\left\|\int_I y_t dt\right\| \leqq |I|^{\alpha} \sup_{t \in I}\|y_t\|+|I|^{\alpha+\gamma}KA_{\alpha\gamma} . \tag{3.2}$$

§ 4. Lemmas.

Let $\{X_t, -\infty<t<\infty\}$ be a completely non-deterministic linear $S\alpha S$ process, $0<\alpha<1$. We assume that $\{X_t\}$ satisfies Condition $C_\gamma(\delta_0, K)$ for some $\gamma>1-\alpha$, $\delta_0>0$ and $K>0$.

LEMMA 4.1. *Let* $y \in A_a[X_t]$ *for some* a. *If* $y_t = T_t y$ *satisfies Condition* $C_\gamma(\delta_0, C)$, *then*

$$\|T_t y - A_a T_t y\| \leqq C|t|^\gamma \tag{4.1}$$

whenever $t \leqq \delta_0$.

PROOF. Since $T_t y - A_a T_t y$ is independent of $A_a(T_t y - y)$, we have

$$\|T_t y - A_a T_t y\| + \|A_a(T_t y - y)\| = \|T_t y - T_0 y\| \leqq C|t|^\gamma$$

whenever $|t| \leqq \delta_0$. Hence we have (4.1) for $|t| \leqq \delta_0$. The left hand side of (4.1) is zero when $t<0$. Note that (4.1) is the same as

$$\|y - A_{a-t} y\| \leqq C|t|^\gamma . \tag{4.1$'$}$$

LEMMA 4.2. *Let* $y \in A_a[X_t]$ *for some* a. *If* $T_t y$ *satisfies Condition* $C_\gamma(\delta_0, C)$, *then for any* b, c *such that* $|b-c| \leqq \delta_0$,

$$\|(A_b - A_c)y\| \leqq 4C|b-c|^\gamma . \tag{4.2}$$

PROOF. We may assume $b>c$. Let $b-c=t$ and $T_{-c}y=y'$. Since $\|T_t x\| = \|x\|$ for any x, we have $\|(A_b-A_c)y\| = \|(A_t - A_0)y'\|$. Since $\|(T_u - T_v)y'\| = \|(T_u - T_v)y\|$, $T_t y'$ also satisfies Condition $C_\gamma(\delta_0, C)$. So, without loss of generality, we may consider $\|(A_t - A_0)y\|$ instead of $\|(A_b - A_c)y\|$. Assume that there exists δ which satisfies $0<\delta \leqq \delta_0$ and

$$\|(A_\delta - A_0)y\| > 4C\delta^\gamma . \tag{4.3}$$

We will show that this leads to a contradiction.

Let $[a/\delta]=k$, where $[a/\delta]$ denotes the largest integer that does not exceed a/δ. Let

$$u_i = (A_{(i+1)\delta} - A_{i\delta})T_\delta y , \qquad i=0, \cdots, k-1 ,$$

$$u'_i = (A_{(i+1)\delta} - A_{i\delta})y , \qquad i=0, \cdots, k-1 ,$$

$$v_i = u_i - u'_i .$$

From Condition $C_\gamma(\delta_0, C)$ we have

$$\|(A_{k\delta} - A_\delta)(T_\delta y - y)\| \leqq \|T_\delta y - y\| \leqq C\delta^\gamma .$$

On the other hand

$$\begin{aligned}\|(A_{k\delta} - A_\delta)(T_\delta y - y)\| &= \|(A_{k\delta} - A_{(k-1)\delta})(T_\delta y - y)\| \\ &\quad + \cdots + \|(A_{2\delta} - A_\delta)(T_\delta y - y)\| \\ &= \|v_{k-1}\| + \|v_{k-2}\| + \cdots + \|v_1\| .\end{aligned}$$

Let $C_i=\|v_i\|/\delta^{\gamma}$, then we have

$$C_1+C_2+\cdots+C_{k-1}\leqq C. \tag{4.4}$$

Since $|\|u_i'\|-\|u_i\||\leqq\|v_i\|=C_i\delta^{\gamma}$,

$$\|u_i\|-C_i\delta^{\gamma}\leqq\|u_i'\|\leqq\|u_i\|+C_i\delta^{\gamma}. \tag{4.5}$$

From stationariness we have $\|u_i'\|=\|u_{i+1}\|$. Thus,

$$\|u_1'\|\geqq\|u_1\|-C_1\delta^{\gamma}=\|u_0'\|-C_1\delta^{\gamma},$$

$$\|u_2'\|\geqq\|u_2\|-C_2\delta^{\gamma}=\|u_1'\|-C_2\delta^{\gamma}$$

$$\geqq\|u_0'\|-(C_1+C_2)\delta^{\gamma},$$

and finally

$$\|u_{k-1}'\|\geqq\|u_0'\|-(C_1+\cdots+C_{k-1})\delta^{\gamma}. \tag{4.6}$$

Since $u_0'=(A_\delta-A_0)y$ by the definition, we have, by (4.3) and (4.4),

$$\|u_{k-1}'\|>4C\delta^{\gamma}-(C_1+\cdots+C_{k-1})\delta^{\gamma}\geqq 3C\delta^{\gamma}. \tag{4.7}$$

Now $\|(A_a-A_{k\delta})T_\delta y\|+\|(A_{(k+1)\delta}-A_a)T_\delta y\|=\|u_k\|=\|u_{k-1}'\|$ and, from Lemma 4.1, we get

$$\|T_\delta y-A_{(k+1)\delta}T_\delta y\|+\|A_{(k+1)\delta}T_\delta y-A_aT_\delta y\|=\|T_\delta y-A_aT_\delta y\|\leqq C\delta^{\gamma}. \tag{4.8}$$

Hence

$$\|(A_a-A_{k\delta})T_\delta y\|>3C\delta^{\gamma}-\|(A_{(k+1)\delta}-A_a)T_\delta y\|\geqq 2C\delta^{\gamma}. \tag{4.9}$$

We know

$$\|(A_a-A_{k\delta})y-(A_a-A_{k\delta})T_\delta y\|=\|(A_a-A_{k\delta})(T_\delta y-y)\|\leqq C\delta^{\gamma}. \tag{4.10}$$

On the other hand, we have, by (4.1) and (4.9),

$$\|(A_a-A_{k\delta})y-(A_a-A_{k\delta})T_\delta y\|\geqq\|(A_a-A_{k\delta})T_\delta y\|-\|y-A_{k\delta}y\| \tag{4.11}$$

$$>2C\delta^{\gamma}-C\delta^{\gamma}=C\delta^{\gamma}.$$

This is a contradiction. Thus Lemma 4.2 is proved.

COROLLARY 4.3. *Let $y\in A_a[X_t]$ for some a. If T_ty satisfies Condition $C_\gamma(\delta_0, C)$, then for $J=(b, c]$ such that $b<c\leqq a$, $T_tA(J)y$ satisfies Condition $C_\gamma(\delta_0, 9C)$.*

PROOF. Let $0<t<\delta_0$. If $t<c-b$, then

$$\|T_tA(J)y-A(J)y\|=\|(A_{c+t}-A_{b+t})T_ty-(A_c-A_b)y\|$$

$$=\|(A_{c+t}-A_c)T_ty\|+\|(A_c-A_{b+t})(T_ty-y)\|+\|(A_{b+t}-A_b)y\|.$$

By Lemma 4.2, the first and the third terms are less than $4C|t|^{\gamma}$. The second term is bounded by $\|T_t y - y\|$, which is less than $C|t|^{\gamma}$ from the assumption. Hence

$$\|T_t A(J)y - A(J)y\| \leqq 9Ct^{\gamma}.$$

In case $t \geqq |c-b|$, we have

$$\|T_t A(J)y - A(J)y\| = \|T_t A(J)y\| + \|A(J)y\| \leqq 8C|t|^{\gamma}$$

applying Lemma 4.2 for both terms.

LEMMA 4.4. *Let* $\delta_0 > 0$, $K_1 > 0$, $1 \geqq \gamma > 1-\alpha$ *and* $g(t)$ *be a continuous function such that* $g(0) \neq 0$ *and* $|g(t)-g(s)|^{\alpha} \leqq K_1 |t-s|^{\gamma}$ *whenever* $|t-s| \leqq \delta_0$. *For positive* a *and* K_2, *let* y *be an element of* $A(0, a][X_t]$, *such that* $\{T_t y\}$ *satisfies Condition* $C_{\gamma}(\delta_0, K_2)$. *Let* $\mathcal{L}$ *be a closed linear subspace of* $[X_t]$. *If*

$$A(I)\int_{-a}^{a} g(t) T_t A(J) y dt \in \mathcal{L} \tag{4.12}$$

for every pair of intervals I, J *in* $(0, a]$, *then* y *itself is an element of* $\mathcal{L}$.

If we take $\mathcal{L} = \{0\}$, we have

COROLLARY 4.5. *Let* $g(t)$ *and* y *be as above. If*

$$A(I)\int_{-a}^{a} g(t) T_t A(J) y dt = 0$$

for every pair of intervals I, $J \subset (0, a]$, *then* $y = 0$.

(We note that in case $\alpha \leqq 1/2$, the function $g(t)$ in the lemma is automatically constant.)

PROOF OF LEMMA 4.4. 1) First we have to check that the integral in the lemma is well-defined. Let $Y(t) = T_t A(J) y$. By Corollary 4.3 $Y(t)$ satisfies Condition $C_{\gamma}(\delta_0, 9K_2)$. Hence

$$\begin{aligned} \|g(t)Y(t) - g(s)Y(s)\| &= \|(g(t)-g(s))Y(t) + g(s)(Y(t)-Y(s))\| \\ &\leqq |g(t)-g(s)|^{\alpha}\|Y(t)\| + |g(s)|^{\alpha}\|Y(t)-Y(s)\| \\ &\leqq K_1|t-s|^{\gamma}\|A(J)y\| + |g(s)|^{\alpha} 9K_2 |t-s|^{\gamma} \leqq K_3|t-s|^{\gamma}, \end{aligned}$$

where $K_3 = K_1\|A(J)y\| + 9K_2 \sup_{s \in [-a, a]} |g(s)|^{\alpha}$. Now we obtain that $g(t)Y(t)$ satisfies Condition $C_{\gamma}(\delta_0, K_3)$. So, it is Riemann type integrable over any finite interval by virtue of Theorem 3.1.

2) We will show that

$$A(0, a]\int_{-h}^{h} g(t) T_t A(0, a] y dt \in \mathcal{L} \tag{4.13}$$

for every h satisfying $0<h<a$. Partition the interval $(0, a]$ into n subintervals $I_1^n, I_2^n, \cdots, I_n^n$ of equal length a/n, where n is taken to be so large that a/n is smaller than t_0 and $a-h$. For brevity we write $I_j=I_j^n$. Write $I_j=(a_j, b_j]$ and define $J_j=(a_j-h, b_j+h]\cap(0, a]$. From the assumption we have

$$A(0, a]\int_{-a}^{a} g(t)T_t A(0, a]y dt \in \mathcal{L}.$$

We write it as the sum of n independent terms:

$$A(0, a]\int_{-a}^{a} g(t)T_t A(0, a]y dt = \sum_{j=1}^{n} A(I_j)\int_{-a}^{a} g(t)T_t A(0, a]y dt. \tag{4.14}$$

Each term is further written as the sum of two terms as follows:

$$A(I_j)\int_{-a}^{a} g(t)T_t A(0, a]y dt = A(I_j)\int_{-a}^{a} g(t)T_t A(J_j)y dt + A(I_j)\int_{-a}^{a} g(t)T_t A((0, a]\setminus J_j)y dt. \tag{4.15}$$

Again by the assumption these two terms are elements of $\mathcal{L}$. We decompose each term as follows:

$$\begin{aligned} A(I_j)\int_{-a}^{a} g(t)T_t A(J_j)y dt &= A(I_j)\int_{-h}^{h} g(t)T_t A(J_j)y dt \\ &\quad + A(I_j)\int_{h<|t|\leq(a/n)+h} g(t)T_t A(J_j)y dt \\ &\quad + A(I_j)\int_{(a/n)+h<|t|\leq a} g(t)T_t A(J_j)y dt. \end{aligned} \tag{4.16}$$

We denote the terms in the right hand side by (i), (ii) and (iii) in this order.

$$\begin{aligned} &A(I_j)\int_{-a}^{a} g(t)T_t A((0, a]\setminus J_j)y dt \\ &= A(I_j)\int_{-h}^{h} g(t)T_t A((0, a]\setminus J_j)y dt \\ &\quad + A(I_j)\int_{h\leq|t|\leq a} g(t)T_t A((0, a]\setminus J_j)y dt. \end{aligned} \tag{4.17}$$

Denote the first and the second terms in the right hand side by (iv) and (v), respectively. We have

$$\begin{aligned} \sum_{j=1}^{n}((\mathrm{i})+(\mathrm{iv})) &= \sum_{j=1}^{n} A(I_j)\int_{-h}^{h} g(t)T_t A(0, a]y dt \\ &= A(0, a]\int_{-h}^{h} g(t)T_t A(0, a]y dt. \end{aligned}$$

From the definition of I_j and J_j, it is easily seen that $A(I_j)T_t A(J_j)y=0$ for $|t|>(a/n)+h$. So, (iii)=0. Also $A(I_j)T_t A((0, a]\setminus J_j)=0$ for $|t|\leqq h$. Thus (iv)=0. Then we see (v)$\in\mathcal{L}$. These considerations show that

$$A(0,\ a]\int_{-h}^{h} g(t)T_t A(0,\ a]y dt = Y_n - \sum_{j=1}^{n} A(I_j)\int_{h<|t|\leq|I_j|+h} g(t)T_t A(J_j)y dt$$

where Y_n is some element of $\mathcal{L}$. We will show that

$$\sum_{j=1}^{n} A(I_j)\int_{h<|t|\leq|I_j|+h} g(t)T_t A(J_j)y dt \longrightarrow 0 \quad \text{as } n\to\infty, \tag{4.18}$$

from which (4.13) follows. We have

$$\Big\|\sum_{j=1}^{n} A(I_j)\int_{h<|t|\leq(a/n)+h} g(t)T_t A(J_j)y dt\Big\| = \Big\|\int_{h<|t|\leq(a/n)+h} \sum_{j=1}^{n} A(I_j)g(t)T_t A(J_j)y dt\Big\|.$$

From 1), the integrand $A(I_j)g(t)T_tA(J_j)y$ satisfies Condition $C_\gamma(\delta_0,\ K_3')$ where

$$K_3' = K_1\|A(J_j)y\| + \sup_{s\in(-a,\ a]}|g(s)|^{\alpha}9K_2 \leqq K_3''$$

where

$$K_3'' = K_1\|A(0,\ a]y\| + \sup_{s\in(-a,\ a]}|g(s)|^{\alpha}9K_2. \tag{4.19}$$

Therefore $\sum_{j=1}^{n}A(I_j)g(t)T_tA(J_j)y$ satisfies Condition $C_\gamma(\delta_0,\ nK_3'')$. Using the estimation formula (3.2) of an integral over a small interval, we have

$$\Big\|\int_{h<|t|\leq a/n+h} \sum_{j=1}^{n} A(I_j)g(t)T_t A(J_j)y dt\Big\|$$
$$\leqq 2\{(a/n)^{\alpha}\sup_{t\in I}\|g(t)A(0,\ a]T_t y\| + a^{\alpha+\gamma}n^{1-(\alpha+\gamma)}K_3''A_{\alpha\gamma}\}.$$

This tends to 0 as $n\to\infty$. This shows (4.18).

3) Let

$$\mathcal{J}(h) = \frac{1}{2h}\int_{-h}^{h}\{A(0,\ a]g(t)T_tA(0,\ a]y - g(0)A(0,\ a]y\}dt.$$

Since $y\in A(0,\ a][X_t]$, $A(0,\ a]y=y$. Hence

$$\mathcal{J}(h) = \frac{1}{2h}\int_{-h}^{h}\{A(0,\ a]g(t)T_t y - g(0)y\}dt.$$

If we show $\mathcal{J}(h)\to 0$ as $h\to 0$, we obtain that $g(0)y$ is an element of $\mathcal{L}$ as the limit of $\frac{1}{2h}\int_{-h}^{h}A(0,\ a]g(t)T_t y dt$ in $\mathcal{L}$. Let $0<h\leqq\delta_0/2$. Going back to the definition of the integral, partition $[-h,\ h]$ into $2n$ subintervals of equal length and let

$$Y_n^h = \frac{1}{2h}\frac{2h}{2n}\sum_{i=1}^{2n}[A(0,\ a]g(t_i)T_{t_i}y - g(0)y] \tag{4.20}$$

where t_i is taken from the i-th subinterval. Our $\mathcal{J}(h)$ is the limit of Y_n^h as $n\to\infty$. Let M be such that $2h/M<\delta_0/2$. If n and m are bigger than M, then, from [7]

$$\|Y_n^h - Y_m^h\| \leqq \left(\frac{1}{2h}\right)^\alpha K_3''|2h|^{\alpha+\gamma} M^{-\rho}\{2^{1-\alpha+\gamma}/(2^{\alpha+\gamma-1}-1)+3\cdot 2^\gamma\} \tag{4.21}$$
$$\leqq (2h)^\gamma M^{-(\alpha+\gamma-1)} \times \text{const}$$
$$\leqq |\delta_0|^\gamma M^{-(\alpha+\gamma-1)} \times \text{const}$$

where K_3'' is given by (4.19). This shows that the convergence of Y_n^h to $\mathcal{J}(h)$ is uniform in h. We have

$$\|Y_n^h\| \leqq (2n)^{-\alpha}\left\| \sum_{i=1}^{2n} \{(g(t_i)-g(0))A(0, a]T_{t_i}y - g(0)(A(0, a]T_{t_i}y-y)\}\right\| \tag{4.22}$$
$$\leqq (2n)^{-\alpha} \sum_{i=1}^{2n} \{|g(t_i)-g(0)|^\alpha \|y\| + |g(0)|^\alpha \|A(0, a]T_{t_i}y-y\|\}$$
$$\leqq (2n)^{-\alpha} \sum_{i=1}^{2n} [K_1|t_i|^\gamma \|y\| + |g(0)|^\alpha K_2|t_i|^\gamma]$$
$$= (2n)^{-\alpha}[K_1\|y\| + |g(0)|^\alpha K_2] \sum_{i=1}^{2n} |t_i|^\gamma .$$

Since each t_i is a point in I_i, we have

$$\sum_{i=1}^{2n} |t_i|^\gamma \leqq 2[(h/n)^\gamma + (2h/n)^\gamma + \cdots + (nh/n)^\gamma] . \tag{4.23}$$

Hence

$$\|Y_n^h\| \leqq (2n)^{-\alpha}(K_1\|y\| + |g(0)|^\alpha K_2) 2(h/n)^\gamma [1+2+\cdots+n] \tag{4.24}$$
$$= 2^{-\alpha}(K_1\|y\| + |g(0)|^\alpha K_2) h^\gamma (n+n^2)/n^{\alpha+\gamma} .$$

Therefore, $\|Y_n^h\|$ tends to zero as $h \to 0$ for fixed n. Given $\varepsilon > 0$, choose n_0 such that $\|\mathcal{J}(h) - Y_{n_0}^h\| < \varepsilon/2$ for all small $h > 0$. Then choose h_0 such that $\|Y_{n_0}^h\| < \varepsilon/2$ whenever $h \leqq h_0$. Now, for any $h < h_0$,

$$\|\mathcal{J}(h)\| \leqq \|\mathcal{J}(h) - Y_{n_0}^h\| + \|Y_{n_0}^h\| < \varepsilon .$$

This completes the proof of the lemma.

Let $x \in [X_t]$. If $\{T_t x\}$ satisfies Condition $C_\gamma(\delta_0, K)$ for some δ_0 and K, we say that x belongs to (C_γ).

LEMMA 4.6. *Assume $\alpha > 1/2$. For any $x \in (C_\gamma)$,*

$$\int_a^\infty e^{-t} T_t x dt$$

is well-defined for any number a.

PROOF. Suppose $\{T_t x\}$ satisfies Condition $C_\gamma(\delta_0, K)$. We may assume $\delta_0 < 1$. Fix t_1. For t and s such that $t, s \geqq t_1$ and $|t-s| \leqq \delta_0$, we have

$$|e^{-t} - e^{-s}|^\alpha \leqq e^{-t_1\alpha}(1-\delta_0)^{-\alpha}|t-s|^\alpha$$

by an easy calculation. Since $T_t x$ satisfies Condition $C_\gamma(\delta_0, K)$, $y_t = e^{-t}T_t x$, $t \geqq t_1$, satisfies Condition $C_{\gamma'}(\delta_0, K_{t_1})$, where $K_{t_1} = e^{-t_1\alpha}(1-\delta_0)^{-\alpha}\|x\| + e^{-t_1\alpha}$ and $\gamma' = \min(\gamma, \alpha)$. Hence we see that $\int_a^N e^{-t}T_t x dt$ is well-defined for finite a and N. Now we show that we can define $\int_a^\infty e^{-t}T_t x dt$. Using (3.1), for $M > 2/\delta_0$, $0 < s \leqq 1$ and any n, we see that

$$\left\|\int_n^{n+s} y_t dt\right\| \leqq M^{1-\alpha} \sup_{t \in [n, n+s]} \|y_t\| + M^{-\rho} K_n A_{\alpha\gamma} = e^{-n\alpha} A \tag{4.25}$$

where $A = M^{1-\alpha}\|x\| + M^{-\rho} A_{\alpha\gamma}((1-\delta_0)^{-\alpha}\|x\| + 1)$. Note that A does not depend on n nor s. For any N_1 and $N_2 > N_1$, we write $N_2 = N_1 + k + s$ (k is an integer and $0 \leqq s < 1$). Then

$$\begin{aligned} \left\|\int_{N_1}^{N_2} y_t dt\right\| &\leqq \left\|\int_{N_1}^{N_1+1} y_t dt\right\| + \cdots + \left\|\int_{N_1+k-1}^{N_1+k} y_t dt\right\| + \left\|\int_{N_1+k}^{N_2} y_t dt\right\| \\ &\leqq (e^{-N_1\alpha} + e^{-(N_1+1)\alpha} + \cdots + e^{-(N_1+k)\alpha})A \\ &< e^{-N_1\alpha} A (1-e^{-\alpha})^{-1}. \end{aligned} \tag{4.26}$$

Therefore, for any $\varepsilon > 0$, there exists N_0 such that for $N_2 > N_1 > N_0$, $\left\|\int_{N_1}^{N_2} y_t dt\right\| < \varepsilon$. This shows that $\int_a^N y_t dt$ has a limit as $N \to \infty$. We define $\int_a^\infty e^{-t}T_t x dt$ as this limit.

With all above preparations, now we get for our processes the results which are parallel to Urbanik's ([**9**], Lemma 4.2, Theorems 4.1 and 4.2).

From now on let $\mathfrak{R}_0$ be the family of all bounded intervals of the form $(a, b]$, $\mathfrak{R}_*$ be the ring of all finite unions of elements of $\mathfrak{R}_0$, and $\mathfrak{R}$ be the family of all bounded Borel subsets of the real line.

LEMMA 4.7. *Assume $\alpha > 1/2$. Suppose that the process $\{X_t\}$ is nontrivial.* (i) *The stochastic interval function M_0, defined on $\mathfrak{R}_0$ by the formula*

$$M_0((a, b]) = A(a, b] \int_a^\infty e^{-t}T_t X_0 dt \tag{4.27}$$

can be extended to an $[X_t]$-valued measure on $\mathfrak{R}$. (ii) *The class of M_0-null sets coincides with the class of Lebesgue null sets.* (iii) *Moreover, for any interval $I \in \mathfrak{R}_0$, the equation*

$$[M_0(J)\ ;\ J \in \mathfrak{R}_0, J \subset I] = A(I)[X_t] \tag{4.28}$$

holds and for $E \in \mathfrak{R}$, $f \in L(M_0)$

$$A(I)\int_E f(u) M_0(du) = \int_{E \cap I} f(u) M_0(du)\,. \tag{4.29}$$

PROOF. The stochastic interval function M_0 is well-defined by Lemma 4.6. Then, as we see in the following, (i), (ii) and (4.29) are proved in the same way as in [**9**].

Since

$$A(a, b]\int_c^a e^{-t}T_t X_0 dt = 0 \tag{4.30}$$

for any $c \leqq a$, we can also write

$$M_0(a, b] = A(a, b]\int_c^\infty e^{-t}T_t X_0 dt \tag{4.31}$$

for any $c \leqq a$. This shows that, for disjoint intervals $J_1, J_2 \in \mathfrak{R}_0$ with $J_1 \cup J_2 \in \mathfrak{R}_0$, we have

$$M_0(J_1 \cup J_2) = M_0(J_1) + M_0(J_2). \tag{4.32}$$

Moreover, if $I_1, I_2, \cdots, I_n$ are disjoint intervals in $\mathfrak{R}_0$, then $M_0(I_1), \cdots, M_0(I_n)$ are independent. Since A_a is a continuous mapping onto $[X_\tau; \tau \leqq a]$ and $A(a, b] = A_b - A_a$, $\lim_{c \to b+} M_0(a, c] = M_0(a, b]$. Then stochastic interval function M_0 on $\mathfrak{R}_0$ is extended to the ring $\mathfrak{R}_*$ if we define $M_0(\bigcup_{j=1}^n I_j) = \sum_{j=1}^n M_0(I_j)$ for disjoint intervals $I_1, \cdots, I_n$ in $\mathfrak{R}_0$. When $I_1, I_2, \cdots \in \mathfrak{R}_*$ are disjoint and $\bigcup_{i=1}^\infty I_i \in \mathfrak{R}_*$,

$$\sum_{i=1}^n \|M_0(I_i)\| = \left\| \sum_{i=1}^n M_0(I_i) \right\| = \left\| M_0\Big(\bigcup_{i=1}^n I_i\Big) \right\| \leqq \left\| M_0\Big(\bigcup_{i=1}^\infty I_i\Big) \right\| < \infty.$$

This shows that $\sum_{i=1}^n M_0(I_i)$ converges in probability and $\sum_{i=1}^\infty M_0(I_i) = M_0(\bigcup_{i=1}^\infty I_i)$. Hence M_0 is countably additive on $\mathfrak{R}_*$. When $I_1, I_2, \cdots \in \mathfrak{R}_*$ are disjoint and $\bigcup_{i=1}^\infty I_i \in \mathfrak{R}$, there is an interval $I_0 \in \mathfrak{R}_0$ such that $\bigcup_{i=1}^\infty I_i \subset I_0$. Then $\|\sum_{i=1}^n M_0(I_i)\| \leqq \|M_0(I_0)\|$. So, we know that $\sum_{i=1}^\infty M_0(I_i)$ converges and M_0 can be extended uniquely to a stochastic measure on $\mathfrak{R}$.

Since, for intervals I and J in $\mathfrak{R}_0$,

$$A(I)M_0(J) = M_0(I \cap J), \qquad T_t M_0(I) = e^t M_0(I+t), \tag{4.33}$$

we have from the uniqueness of the extension

$$A(I)M_0(E) = M_0(I \cap E), \qquad T_t M_0(E) = e^t M_0(E+t) \tag{4.34}$$

for $I \in \mathfrak{R}_0$, $E \in \mathfrak{R}$. From (3.31)

$$\|M_0(E)\| = \|T_t M_0(E)\| = e^{t\alpha}\|M_0(E+t)\|.$$

Hence $M_0(E)=0$ implies $M_0(E+t)=0$. That means that the class of all $\|M_0\|$-null sets is translation invariant. So, it coincides with the class of Lebesgue null sets. When $f(u)$ is a simple function, the first equation of (4.33) is equal to

(4.35) $$A(I)\int_E f(u)M_0(du) = \int_{E\cap I} f(u)M_0(du)\,.$$

For general $f(u)$, we can find a sequence of simple functions $f_n(u)$ that converges to $f(u)$ in $L^{\alpha}(d\|M_0\|)$, where $d\|M_0(u)\|$ is the measure such that $\int_a^b d\|M_0(u)\| = \|M_0(a, b]\|$. Thus (4.29) is proved.

Now we prove (4.28). Let

(4.36) $$\mathcal{L}_0 = \left\{x \,:\, x\in(C_\gamma),\ A(a, b]\int_a^\infty e^{-t}T_t x dt \in [M_0]\right\}.$$

Here $[M_0]$ denotes the closed linear space spanned by $M_0(I)$, $I\in\mathfrak{R}_0$. Then $\mathcal{L}_0$ is a subspace of $[X_t]$. For $h\leqq 0$, we have

(4.37) $$\begin{aligned} A(a, b]\int_a^\infty e^{-t}T_t T_h X_0 dt &= e^h A(a, b]\int_{a+h}^\infty e^{-t}T_t X_0 dt \\ &= e^h M_0(a, b] \in [M_0]\,. \end{aligned}$$

So, for every $h\leqq 0$, $T_h X_0 \in \mathcal{L}_0$. Then every $x\in[X_t\,;\,t\leqq 0]\cap(C_\gamma)$ belongs to $\mathcal{L}_0$. It is clear that for any $I\in\mathfrak{R}_0$,

$$[M_0(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset I\in\mathfrak{R}_0] \subset A(I)[X_t]\,.$$

We show the converse. Without loss of generality we prove in the case $I=(0, a]$. For any $y\in(C_\gamma)\cap A(0, a][X_t]$, with I, $J\in\mathfrak{R}_0$ and contained in $(0, a]$, let

(4.38) $$Z(I, J) = e^a A(I)\int_0^\infty e^{-t}T_t T_{-a}A(J)y dt\,.$$

Because of $T_{-a}A(J)y\in[X_t\,;\,t\leqq 0]\cap(C_\gamma)$, $T_{-a}A(J)y$ belongs to $\mathcal{L}_0$. That means $Z(I, J)\in[M_0]$. Since $A(0, a]T_t A(J)=0$ for $t\geqq a$,

$$\begin{aligned} Z(I, J) &= A(I)\int_0^\infty e^{a-t}T_{t-a}A(J)y dt \\ &= A(I)\int_{-a}^{a} e^{-t}T_t A(J)y dt \\ &\in [M_0(U)\,:\,U\in\mathfrak{R}_0 \text{ and } U\subset(0, a]]\,. \end{aligned}$$

By Lemma 4.4, we know $y\in[M_0(U): U\in\mathfrak{R}_0,\ U\subset(0, a]]$, that is,

(4.39) $$[M_0(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset(0, a]] \supset A(0, a][X_t]\cap(C_\gamma)\,.$$

By Lemma 4.1 and Lemma 4.2, for any number b and an interval $J\subset(0, a]$, $A(J)X_b\in(C_\gamma)$. This means that, $A(0, a][X_t]\cap(C_\gamma)$ is dense in $A(0, a][X_t]$. Thus we have

(4.40) $$[M_0(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset(0, a]] \supset A(0, a][X_t]\,.$$

This completes the proof of Lemma 4.7.

§ 5. Theorems.

THEOREM 5.1. *Let* X_t *be a stationary linear completely nondeterministic* $S\alpha S(1/2<\alpha<1)$ *process such that Condition* $C_\gamma(\delta_0, K)$ *is satisfied for some positive* δ_0, K *and* $\gamma>1-\alpha$. *There exists an* $[X_t]$*-valued* $\{T_t\}$*-homogeneous stochastic measure* M *such that for any interval* $I\in\mathfrak{R}_0$

$$[M(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset I]=A(I)[X_t]\,. \tag{5.1}$$

PROOF. Suppose $\{X_t\}$ be nontrivial. First we show that there is y_0 in $A(0, 1][X_t]\cap(C_\gamma)$ such that

$$A(0, 1]\int_{-1}^{1}T_t y_0 dt\neq 0\,. \tag{5.2}$$

It is clear that $A(0, 1][X_t]\cap(C_\gamma)$ is not empty. For example, let $x=X_1-A_0X_1$. Since X_t is nontrivial and completely nondeterministic, $x\neq 0$ and $x\in A(0, 1][X_t]$. We know $x\in(C_\gamma)$ by Corollary 4.3. Choose a nonzero element y of $A(0, 1][X_t]\cap(C_\gamma)$. If

$$A(0, 1]\int_{-1}^{1}T_t A(J)y dt=0$$

for every $J\in\mathfrak{R}_0$ with $J\subset(0, 1]$, then,

$$A(I)\int_{-1}^{1}T_t A(J)y dt=0 \qquad \text{for every } I\in\mathfrak{R}_0 \text{ with } I\subset(0, 1]\,, \tag{5.3}$$

and by Lemma 4.4 (5.3) implies $y=0$, a contradiction. Thus, there exists $J_0\in\mathfrak{R}_0$ with $J_0\subset(0, 1]$, which satisfies

$$A(0, 1]\int_{-1}^{1}T_t A(J_0)y dt\neq 0\,.$$

Let $y_0=A(J_0)y$. This satisfies (5.2) and belongs to (C_γ) by Corollary 4.3. Let us define

$$M(a, b]=A(a, b]\int_{a-1}^{b}T_t y_0 dt\,. \tag{5.4}$$

This is a stochastic interval function on $\mathfrak{R}_0$ taking values in $[X_t]$. Using this M, we can prove the theorem in the same way as in [**9**]. Like in the case of Lemma 4.7, M is extended to a stochastic measure M on $\mathfrak{R}$. Moreover, from the definition, $T_tM(I)=M(I+t)$ for $I\in\mathfrak{R}_0$. It holds also for $I\in\mathfrak{R}$. That means M is T_t-homogeneous. Since

$$A(I)M(J)=M(I\cap J) \qquad \text{for all } I, J\in\mathfrak{R}_0\,, \tag{5.5}$$

we have

$$[M(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset I]\subset A(I)[X_t]\,. \tag{5.6}$$

To prove the converse inclusion, we show

$$[M(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset I]\supset[M_0(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset I]\,, \tag{5.7}$$

where M_0 is the stochastic measure defined by (4.27). By (4.28), (5.7) will complete the proof of (5.1). It follows from (4.28), (4.29) and the facts in §2, 2) that there exists $g\in L(M_0)$ such that g is M_0-integrable over any finite interval and

$$M(E)=\int_E g(u)M_0(du) \tag{5.8}$$

for all $E\in\mathfrak{R}$. Since M is T_t-homogeneous, the class of M-null sets is translation invariant. It is the class of Lebesgue null sets. Hence the class of M-null sets equals the class of M_0-null sets. From (5.8) we get

$$\int_E f(u)M(du)=\int_E f(u)g(u)M_0(du) \tag{5.9}$$

for all sets $E\in\mathfrak{R}$ and simple functions f. If we take a sequence of simple functions $\{f_n\}$ such that $|f_n(u)|\leqq|g(u)|^{-1}$ and $\lim_{n\to\infty}f_n(u)=g(u)^{-1}$ M_0-a. e., then we have

$$M_0(I)=\lim_{n\to\infty}\int_I f_n(u)g(u)M_0(du)\in[M(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset I]\,.$$

This shows (5.7). Thus the theorem is proved.

Now we get a canonical representation theorem for linear completely nondeterministic $S\alpha S$ $(1/2<\alpha<1)$ processes.

THEOREM 5.2. *Let X_t be a nontrivial linear completely nondeterministic $S\alpha S$ $(1/2<\alpha<1)$ process satisfying Condition $C_\gamma(\delta_0, K)$ for some positive δ_0, K and $\gamma>1-\alpha$. Then there exist an $[X_t]$-valued nontrivial T_t-homogeneous stochastic measure M and a function $f\in L(M)$ such that*

$$[M(J)\ ;\ J\in\mathfrak{R}_0,\ J\subset(-\infty,\ t]]=[X_u\,;\,u\leqq t] \tag{5.10}$$

and

$$X_t=\int_{-\infty}^t f(t-u)M(du)\,. \tag{5.11}$$

PROOF. By Theorem 5.1 there exists an $[X_t]$-valued T_t-homogeneous stochastic measure M satisfying condition (5.10). Thus there is a function $f\in L(M)$ such that $X_0=\int_{-\infty}^0 f(-u)M(du)$. Using translation operator $\{T_t\}$, we have

$$X_t=T_tX_0=\int_{-\infty}^t f(t-u)M(du)\,. \tag{5.12}$$

REMARKS. The converse of Theorem 5.2 is as follows: Suppose that M is a nontrivial T_t-homogeneous $S\alpha S$ stochastic measure. Let $f\in L(M)$ and $f(u)=0$ for $u<0$ and define a process X_t by (5.11). Assume that it satisfies (5.10). We have $[X_t]=[M]$. So, each element $x\in[X_t]$ has a representation $x=\int_{-\infty}^{\infty}g(u)M(du)$ where $g\in L(M)$. Let

$$A_0x = \int_{-\infty}^{0} g(u)M(du)\,. \tag{5.13}$$

This linear operator A_0 transforms $[X_t]$ onto $[X_t\,;t\leqq 0]$ and satisfies conditions (i), (ii), (iii) of § 2, 3). And we get

$$A_tx = \int_{-\infty}^{t} g(u)M(du)\,. \tag{5.14}$$

It follows that $\lim_{t\to-\infty}A_tx=0$. Thus X_t is a completely nondeterministic stationary $S\alpha S$ linear process.

For example, let M be a homogeneous $S\alpha S$ $(0<\alpha<1)$ motion. Let

$$X_t = \int_{-\infty}^{t} e^{-(t-u)}dM(u)\,. \tag{5.15}$$

X_t is a stationary $S\alpha S$ process. $[X_t]\subset[M]$ holds. Define the operator A_0 in the same way as in (5.13). Then

$$\begin{aligned} A_0X_t &= \int_{-\infty}^{0} e^{-(t-u)}dM(u) \qquad (t\geqq 0) \\ &= e^{-t}\int_{-\infty}^{0} e^{u}dM(u) = e^{-t}X_0 \in [X_t\,;t\leqq 0]\,. \end{aligned}$$

Thus A_0 is a linear operator from $[X_t]$ onto $[X_t\,;t\leqq 0]$ satisfying (i), (ii), (iii) of § 2, 3). By easy calculation we see that X_t satisfies Condition $C_\alpha(\delta_0, K)$ for some positive δ_0 and K. So $\gamma=\alpha$ in this example. Hence, if $\gamma=\alpha>1-\alpha$ (i.e. $\alpha>1/2$) we can prove (5.1) for this process. Therefore we know the representation (5.15) is canonical for $\alpha>1/2$.

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