J. Math. Soc. Japan
Vol. 40, No. 1, 1988

The linearity question for Abelian groups on translation planes

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(Received July 1, 1986)

1. Introduction.

Let Π denote a translation plane of order q^k with kernel $GF(q)$ and let $\mathcal{G}$ be a collineation group of Π in the translation complement. That is, $\mathcal{G}$ is a subgroup of $\Gamma L(2k, q)$. Normally, $\mathcal{G}$ is *taken* to belong to the linear translation complement while simultaneous disclaimers are made as to the differences between the situations linear and nonlinear.

If $\mathcal{G}$ is nonsolvable then there is a nonsolvable subgroup in the linear translation complement. This usually suffices for the study in question. However, when $\mathcal{G}$ is solvable, the fact that $\mathcal{G}$ may not be linear creates many problems.

In several recent articles, translation planes of order q^2 with kernel $GF(q)$ which admit collineation groups of order q^2 have been studied. In order to apply various analyses of functions on finite fields, the group $\mathcal{G}$ is *required* to be in the linear translation complement.

For a general study, we must therefore consider the following:

LINEARITY QUESTION. *If Π is a translation plane of order $q^s=p^{sr}$ with kernel $GF(q)$ admitting a group $\mathcal{G}$ of order q^s in the translation complement, is $\mathcal{G}$ a subgroup of the linear translation complement?*

If Π is a semifield plane of even order q^2 (for example Desarguesian) which admits a Baer involution then there is a group $\mathcal{G}$ of order q^2 such that $|\mathcal{G}\cap GL(\Pi)|=q^2/2$ or q^2 depending on the kernel.

Hence, in order to study the linearity question in dimension 2, we must make an additional assumption.

In the odd order case, a linear group of order q^2 which acts on translation plane of order q^2 and kernel $GF(q)$ turns out to be Abelian (see e.g. [3]). So,

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This article was conceived while the authors were visiting the University of Palermo in the Spring of 1985. The authors gratefully acknowledge the support of the C.N.R. of Italy. Also, the authors would like to thank Professors C. Bartolone and F. Bartolozzi for making the arrangements for the visits.

it makes sense to consider the linearity question for Abelian groups in dimension 2.

LINEARITY QUESTION FOR ABELIAN GROUPS IN DIMENSION 2. *If Π is a translation plane of order q^2 and kernel $GF(q)$ which admits an Abelian collineation group $\mathcal{G}$ of order q^2 in the translation complement, is $\mathcal{G}$ linear?*

To see how strongly dependent the question is on the translation plane, we illustrate an Abelian group $\mathcal{G}$ of order $p^{(r^2+1)}$ in $\Gamma L(2r, p^p)$ which is not in $GL(2r, p^p)$. Let $\mathcal{E}=\left\{\begin{pmatrix} I & C \\ O & I \end{pmatrix}\right.$, O, I $r\times r$ zero and identity matrices respectively and C any $r\times r$ matrix whose entries are in $\left.GF(p)\right\}$. Then $\mathcal{E}$ is a group of order p^{r^2}.

Let $\hat{\sigma}:(x, y)\to(x^\sigma, y^\sigma)$, $\sigma\in \mathrm{Aut}\, GF(p^p)$ so $|\hat{\sigma}|=p$. Then $\hat{\sigma}$ centralizes $\mathcal{E}$ so that $\langle\mathcal{E}, \hat{\sigma}\rangle$ is an Abelian group of order p^{r^2+1} which is not in $GL(2r, p^p)$. For dimension 4 groups, and $p=2$,

$$|\langle\mathcal{E}, \hat{\sigma}\rangle| = 2^5.$$

So there are Abelian groups in $\Gamma L(4, q)$ which *are not* in $GL(4, q)$ of order $\geqq q^2$.

Nevertheless, we prove:

(2.9) THEOREM. *Let Π be a translation plane of order $p^{2r}=q^2$, p a prime, r an integer with kernel $GF(q)$. If Π admits an Abelian collineation group of order q^2 in the translation complement then $\mathcal{G}$ is in the linear translation complement.*

(2.10) THEOREM. *Let Π be a translation plane of order $q^2=p^{2r}$, p a prime, r an integer with kernel $GF(q)$. Assume Π admits an Abelian collineation group of order q^2 in the translation complement.*

(1) If q is even then Π is a semifield plane or a Betten plane.

(2) If q is odd then Π is a semifield plane or a "desirable" plane.

2. Abelian collineation groups.

In this section, we consider translation planes Π of order $p^{2r}=q^2$, p a prime, r an integer, with kernel $F\cong GF(q)$. We assume Π admits an Abelian collineation group $\mathcal{G}$ of order q^2 in the translation complement.

(2.1) LEMMA.

(i) *There exists an elation τ in $\mathcal{G}$.*

(ii) *We may choose coordinates so that $\mathcal{T}=\begin{pmatrix} I & I \\ O & I \end{pmatrix}$ where I, O are 2×2 matrices over $F\cong GF(q)$.*

(iii) *The elements of* $\mathcal{G}$ *may be represented in the form* $(x, y)\to(x^\sigma, y^\sigma)\left(\begin{array}{c|c} A & B \\ \hline O & A \end{array}\right)$ *where* x, y *are* 2-*vectors*, A, B *are* 2×2 *matrices and* $\sigma\in \mathrm{Aut}\,F$.

PROOF. $|\mathcal{G}\cap GL(4, q)|\geqq q^2/r$ where $q=p^r$. Also, $\mathcal{G}$ acting as a linear group over the prime field P fixes some 1-space over P pointwise. Hence, there is a component $\mathcal{L}$ of Π which is left invariant by $\mathcal{G}$. (Note, by Jha-Johnson [**2**], $\mathcal{G}$ is transitive on $l_\infty - l_\infty\cap\mathcal{L}$.) Choose $\mathcal{L}$ as $(x=O)$ where $x=(x_1, x_2)$, $x_i\in F$. $\mathcal{G}|\mathcal{L}\leqq \Gamma L(2, q)$. So, $|\mathcal{G}/\mathcal{E}|\leqq rq$ where $\mathcal{E}$ denotes the elation group of $\mathcal{G}$ (with axis $\mathcal{L}$). Hence, $\mathcal{E}\neq\langle 1\rangle$ (i.e., $|\mathcal{E}|\geqq[q/r]$).

Let $\tau\in\mathcal{E}-\{1\}$ and assume the image of $y=0$ is $y=x$. Then $\tau=\begin{pmatrix} I & I \\ O & I\end{pmatrix}$. Since $g\in\mathcal{G}\subseteqq\Gamma L(4, q)$, clearly, g has the form $(x, y)\xrightarrow{g}(x^\sigma, y^\sigma)\left(\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right)$. But, τ commutes with g (*and* with $(x, y)\to(x^\sigma, y^\sigma)$) so that $A=C$.

This proves (2.1).

(2.2) LEMMA. *Assume* $\mathcal{G}\cap GL(4, q)\neq\mathcal{E}$ (*where* $\mathcal{E}$ *is the elation subgroup*). *Then* $|\mathcal{E}|\leqq q$.

PROOF. By (2.1), elements of $\mathcal{G}\cap GL(4, q)$ have the form $\left(\begin{array}{c|c} A & B \\ \hline O & A \end{array}\right)$. Since $\mathcal{G}\cap GL(4, q)$ is a p-group and $A\in GL(2, q)$, we may choose coordinates so that $A=\begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix}$ for some $a\in F$.

If $\mathcal{G}\cap GL(4, q)\neq\mathcal{E}$ then some corresponding $a\neq 0$. That is $h=\left(\begin{array}{c|c} \begin{matrix}1 & a\\ 0 & 1\end{matrix} & B \\ \hline O & \begin{matrix}1 & a\\ 0 & 1\end{matrix} \end{array}\right)$ $\in\mathcal{G}\cap GL(4, q)$ for some $a\neq 0$.

Let g be an arbitrary elation of $\mathcal{G}$. g must be of the form $\begin{pmatrix} I & \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4\end{pmatrix} \\ O & I\end{pmatrix}$. Form the commutator

$$ghg^{-1}h^{-1}=\begin{pmatrix} I & \begin{pmatrix} ac_3 & a(c_1-c_4)+a^2c_3 \\ 0 & -ac_3\end{pmatrix} \\ O & I\end{pmatrix}.$$

Since this commutator is $\begin{pmatrix} I & O \\ O & I\end{pmatrix}$, it must be that $ac_3=0$ or $c_3=0$ and $a(c_1-c_4)+a^2c_3=0$ or $c_1=c_4$. Hence,

$$\mathcal{E}\subseteqq\left\{\begin{pmatrix} I & \begin{pmatrix} u & m(u) \\ 0 & u\end{pmatrix} \\ O & I\end{pmatrix}\ \middle|\ u\in F \text{ and } m \text{ is some function (additive) of } F\right\}.$$

Thus, $|\mathcal{E}|\leqq q$.

(2.3) LEMMA. *Let* $q=p^r>4$ *where* $r=p^t\cdot s$ *for* $(p, s)=1$. *Then*

$$|\mathcal{E}| \geqq \frac{q}{p^t} = p^{p^t \cdot s - t} > \sqrt{q}\,.$$

PROOF. $|\mathcal{E}| \geqq q/[r]_p$ where $[r]_p$ is the largest power of p dividing r. $p^{p^t \cdot s-t} > p^{p^t \cdot s/2}$ if and only if $p^t \cdot s - t > p^t \cdot s/2$ if and only if $p^t \cdot s/2 > t$. Since $p^t \cdot s \geqq 2^t \cdot s$ and $2^t \cdot s > 2t$ unless $(t, s) \in \{(1, 1), (2, 1)\}$ we may assume $q=2$ or 4. (However, the translation planes of order 4 and 16 are known [**1**].)

(2.4) LEMMA. *If $\mathcal{G} \cap GL(4, q) \neq \mathcal{E}$ then $\mathcal{G} \subseteqq GL(4, q)$.*

PROOF. By lemmas (2.2) and (2.3), $\sqrt{q} < q/p^t \leqq |\mathcal{E}| \leqq q$ and

$$\mathcal{E} = \left\{ \begin{pmatrix} I & \begin{pmatrix} u & m(u) \\ 0 & u \end{pmatrix} \\ O & I \end{pmatrix} \,\middle|\, u \in \Sigma \subseteqq F \right\}$$

where $|\Sigma| \geqq q/p^t$ (see the proof to (2.2)). Then let $g \in \mathcal{G}$ and have the form $(x, y) \rightarrow (x^\sigma, y^\sigma)\left(\begin{array}{c|c} A & B \\ \hline O & A \end{array}\right)$ for $\sigma \in \mathrm{Aut}\, F$. g commutes with $\mathcal{E}$ so that for $h \in \mathcal{E}$, gh is

$$(x, y) \longrightarrow (x^\sigma, y^\sigma)\begin{pmatrix} A & D \\ O & A \end{pmatrix}\begin{pmatrix} I & \begin{pmatrix} u & m(u) \\ 0 & u \end{pmatrix} \\ O & I \end{pmatrix}$$

and hg is

$$(x, y) \longrightarrow (x^\sigma, y^\sigma)\begin{pmatrix} I & \begin{pmatrix} u^\sigma & (m(u))^\sigma \\ 0 & u^\sigma \end{pmatrix} \\ O & I \end{pmatrix}\begin{pmatrix} A & D \\ O & A \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} u^\sigma & (m(u))^\sigma \\ 0 & u^\sigma \end{pmatrix} A = A \begin{pmatrix} u & m(u) \\ 0 & u \end{pmatrix}.$$

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ so that

(1) $u^\sigma a_1 + (m(u))^\sigma a_3 = a_1 u$,

(2) $u^\sigma a_3 = a_3 u$.

If $a_3 \neq 0$ then $u^\sigma = u$ for all $u \in \Sigma$. Since $|\Sigma| > \sqrt{q}$ it must be that $\sigma = 1$. If $a_3 = 0$ then $u^\sigma a_1 = a_1 u$. However, $\mathcal{G}$ acts transitively on $l_\infty - (\infty)$ so that A is nonsingular. Therefore, if $a_3 = 0$ then $a_1 \neq 0$. Hence $u^\sigma = u$ for all $u \in \Sigma$. Hence, $\sigma = 1$. Since this argument is valid for all elements $g \in \mathcal{G}$, it follows that $\mathcal{G} \subseteqq GL(4, q)$.

(2.5) LEMMA. *If $\mathcal{G} \cap GL(4, q)$ is the elation subgroup of $\mathcal{G}$, assume $\mathcal{G} \not\subseteqq GL(4, q)$.*

(1) *Then $\mathcal{G} \cap GL(4, q) = \left\{ \left(\begin{array}{c|c} I & C \\ \hline O & I \end{array}\right) \right\}$ for some additive set of nonsingular matrices in $GL(2, q)$.*

(2) *Let the subgroup of* $GL(2, q)$ *which is generated by the set of matrices* C *of* (1) *be denoted by* $\mathfrak{R}$. *Then* $|\mathfrak{R}\cap(\text{scalar group})|\leqq\sqrt{q}$.

PROOF. (1) The form is clear by the previous lemma (2.2). Since $y=0 \rightarrow y=xC$ under this group and we have a translation plane, the matrices C are nonsingular. Since $\mathcal{G}\cap GL(4, q)$ is elementary abelian, this set is also additive.

(2) Assume $|\mathfrak{R}\cap(\text{scalar group})|>\sqrt{q}$. Then if $(x, y)\xrightarrow{g}(x^{\sigma}, y^{\sigma})\left(\begin{array}{c|c} A & B \\ \hline O & A \end{array}\right)$ is an arbitrary element of $\mathcal{G}$ we must have $C^{\sigma}A=AC$ since $\mathcal{G}$ is abelian. Since $\langle C\rangle=\mathfrak{R}$, we can extend this equation to $D^{\sigma}A=AD$ for all $D\in\mathfrak{R}$. If $|\mathfrak{R}\cap(\text{scalar group})|>\sqrt{q}$ then there exist at least $[\sqrt{q}]+1$ elements $\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$ such that $\begin{pmatrix}\alpha^{\sigma} & 0\\ 0 & \alpha^{\sigma}\end{pmatrix}A=A\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$. That is, $\alpha^{\sigma}=\alpha$ or $\sigma=1$ since A is nonsingular (see (2.4)). So, $\mathcal{G}\subseteqq GL(4, q)$. Hence, we have the proof to (2).

We now consider the group $\mathfrak{R}$.

(2.6) LEMMA. $\mathfrak{R}$ *contains a subset of order* $\geqq q^2/[r]_p$ *which acts semiregularly on a 2-dimension vector space* V.

PROOF. The set of C-matrices of (2.5) have cardinality $q^2/[r]_p$. But $y=xC$ is a component of the associated translation plane. Hence, $xC=x$ for some $x\in V$ implies $C=I$ and $xC=x\bar{C}$ implies $C=\bar{C}$.

$\mathfrak{R}/\mathfrak{R}\cap\mathfrak{Z}\leqq PGL(2, q)$ where $\mathfrak{Z}$ denotes the center of $GL(2, q)$.

(2.7) LEMMA. *Assume* $q\neq 4$. *Under the assumptions of* (2.5), *if* $\mathfrak{R}/\mathfrak{R}\cap\mathfrak{Z}$ *has a subgroup* $PSL(2, p^s)$ *of index dividing* 4 *then* $SL(2, q)\subseteqq\mathfrak{R}$.

PROOF. Let $q=p^r$ so s/r. Let $r=p^t\cdot k$ where $(p, k)=1$ and $t\geqq 1$. Clearly $SL(2, p^s)\subseteqq\mathfrak{R}\mathfrak{Z}$ so $SL(2, p^s)\subseteqq\mathfrak{R}$ as the p-elements must belong to $\mathfrak{R}$.

Hence, $|\mathfrak{R}|\leqq 4(\sqrt{q}-1)\cdot|PSL(2, p^s)|$. If $p^s\neq q$ then $p^s\leqq\sqrt{q}$. Let $d=(2, q-1)$. So $|\mathfrak{R}|\leqq(4(\sqrt{q}-1)/d)p^s(p^{2s}-1)$. However, $SL(2, p^s)\subseteqq\mathfrak{R}$ and the orbit length of any vector $v\neq O$ under $\mathfrak{R}\geqq q^2/[r]_p-1$ but is $\leqq 4(\sqrt{q}-1)/d\cdot$(orbit length under $SL(2, p^s)$). However, the minimum orbit length of $SL(2, p^s)$ is $p^{2s}-1\leqq q-1$ as each Sylow p-subgroup must fix some nonzero vector and $p^s\leqq\sqrt{q}$.

As $q^2/[r]_p=p^{2p^t\cdot k-t}$, hence we must have $q^2/[r]_p-1\leqq(4(\sqrt{q}-1)/d)(q-1)$ or

$$(1)\qquad p^{(2p^t\cdot k-t)}-1\leqq\frac{4(p^{p^t\cdot k/2}-1)(p^{p^t\cdot k}-1)}{d}.$$

We shall show that

$$(2)\qquad p^{2p^t\cdot k-t}>\frac{4p^{p^t\cdot k/2}p^{p^t\cdot k}}{d}.$$

However, (1) implies

$$p^{2p^{t}\cdot k-t}-1<\frac{4p^{p^{t}\cdot k/2}p^{p^{t}\cdot k}}{d}$$

or

$$p^{2p^{t}\cdot k-t}\leqq\frac{4p^{p^{t}\cdot k/2}p^{p^{t}\cdot k}}{d}.$$

To prove (2), we first assume $p\geqq 5$. Then $5^{2\cdot 5^{t}\cdot k-t}\geqq 5\cdot 5^{5^{t}\cdot k/2}\cdot 5^{5^{t}\cdot k}$ if and only if $5^{t}\cdot k/2-t\geqq 1$ if and only if $5^{t}\cdot k\geqq 2t+2$. Hence, (2) is valid for $p=5$.

If $p=3$ then

$$3^{2\cdot 3^{t}\cdot k-t}\geqq 3\cdot 3^{3^{t}\cdot k/2}\cdot 3^{3^{t}\cdot k}$$

if and only if

$$\frac{3^{t}\cdot k}{2}-t\geqq 1$$

or $3^{t}\cdot k\geqq 2t+2$ which is valid unless $t=1$ and $k=1$. That is, $p^{p^{t}\cdot k}=3^{3}=q$. However, $3^{3/2}>2$.

If $p=2$ then $2^{(2^{t}\cdot k-2t)/2}>2$ if and only if $2^{t-1}k-t>1$ unless $t=2$ and $k=1$ or $t=1$ and $k=1$, 3 contrary to (1). Hence, (2) is valid unless $q=2^{2}$ or $q=2^{2^{2}}$. However, for $q=2^{4}$, the index contributes to the order of the Sylow 2-subgroups and the Sylow 2-subgroup must fix a vector. Hence, the equation should be actually

$$\frac{q^{2}}{[r]_{p}}-1\leqq(\sqrt{q}-1)(q-1)$$

if *q is even.*

Hence, (2.7) is proved.

Now $\mathcal{G}\cap GL(4, q)=\left\{\begin{pmatrix} I & C \\ O & I\end{pmatrix}\right\}$ and $\langle C\rangle=\mathcal{R}\supseteqq SL(2, q)$. If $g\in\mathcal{G}$ has the form

$$(x, y)\longrightarrow(x^{\sigma}, y^{\sigma})\left(\begin{array}{c|c} A & B \\ \hline O & A\end{array}\right)$$

for A, B are 2×2 matrices and $\sigma\in\operatorname{Aut}F$ then $C^{\sigma}A=AC$. However, this extends to $\mathcal{R}$ so, for example, $\begin{pmatrix} 1 & d \\ 0 & 1\end{pmatrix}^{\sigma}A=A\begin{pmatrix} 1 & d \\ 0 & 1\end{pmatrix}$ for *all* $d\in F$. If $A=\begin{pmatrix} a_{1} & a_{2} \\ a_{3} & a_{4}\end{pmatrix}$ then $\begin{pmatrix} a_{1}+d^{\sigma}a_{3} & a_{2}+d^{\sigma}a_{4} \\ a_{3} & a_{4}\end{pmatrix}=\begin{pmatrix} a_{1} & a_{1}d+a_{2} \\ a_{3} & a_{3}d+a_{4}\end{pmatrix}$ so that $a_{3}=0$ and $a_{1}d=a_{4}d^{\sigma}$ for all d and fixed a_{1}, a_{4} nonzero (as A is nonsingular).

Hence, letting $d=1$, we have $a_{1}=a_{4}$, so that $d=d^{\sigma}$ for all $d\in F$. Hence, $\sigma=1$.

Thus we have

(2.8) LEMMA. *If* $\mathcal{R}/\mathcal{R}\cap\mathcal{Z}$ *has a subgroup isomorphic to* $PSL(2, p^{s})$ *then* $\mathcal{G}\subseteqq GL(4, q)$.

By (2.8), we may assume that $\mathfrak{R}/\mathfrak{R}\cap\mathfrak{Z}$

1) is a subgroup of a group of order dividing $2(q\pm 1)/d$, where $d=(2, q-1)$,

or

2) a subgroup of a group of order dividing $2q(q-1)/d$,

3) is an index 1 or 2 group isomorphic to A_4, S_4 or A_5.

We consider each case in turn.

Case 1. $|\mathfrak{R}|\leqq(\sqrt{q}-1)2(q\pm 1)/d$. Since $|\mathfrak{R}|\geqq q^2/[r]_p-1$, and since $(\sqrt{q}-1)(q+1)<\sqrt{q}\cdot q$, we may apply the argument of (2.7) to obtain a contradiction.

Case 2. $|\mathfrak{R}|\,\Big|\,|\mathfrak{R}\cap\mathfrak{Z}|(2/d)q(q-1)$. Let $\mathcal{S}_p$ be a Sylow p-subgroup of $\mathfrak{R}$. Then $\mathcal{S}_p$ fixes a nonzero vector v so that the $\mathfrak{R}$ orbit of v has length $\leqq\dfrac{|\mathfrak{R}|}{|\mathcal{S}_p|}\,\Big|\,|\mathfrak{R}\cap\mathfrak{Z}|(2/d)(q-1)$. Again, by the argument of (2.7), we obtain a contradiction.

Case 3. $\mathfrak{R}/\mathfrak{R}\cap\mathfrak{Z}$ has A_4, A_5, S_5 as an index 1 or 2 subgroup.

Since $|\mathfrak{R}|\geqq q^2/[r]_p-1$, let $q=p^r$, $r=(p^t\cdot s)$ for $(p, s)=1$.
For $G=A_4$, A_5, S_5, we have

$$(2|G|\sqrt{q})-1\geqq 2|G|(\sqrt{q}-1)$$

so we consider the inequality

$$(2|G|\sqrt{q})-1\geqq\frac{q^2}{[r]_p}-1$$

or rather

$$(*)\quad 2|G|\geqq p^{(3/2)p^t\cdot s-t}.$$

If $G=A_4$, then $24\geqq p^{(3/2)p^t\cdot s-t}$ implies $p=2$, $(3^{(3/2)3^t\cdot s-t}>27)$. If $p=2$ then $24\geqq 2^{3\cdot 2^{t-1}\cdot s-t}$ only if $(t, s)\in\{(1, 1), (2, 1)\}$ so $q=2^2$ or 2^{2^2}. However, when q is even, the Sylow 2-groups fix a nonzero vector so the smallest orbit argument implies that $3(\sqrt{q}-1)\geqq q^2/[r]_2-1$ for $q=2^{2^2}$ which cannot be the case.

If $G=A_5$ then $(120\sqrt{q})\geqq q^2/[r]_p$ if and only if $120\geqq p^{(3/2)p^t\cdot s-t}$. For $p\geqq 5$ then $p^{(3/2)5\cdot s-1}>120$. For $p=3$, $3^{(3/2)3^t\cdot s-t}\geqq 3^{(3/2)3-1}=3^{7/2}$. So a possible problem occurs when $(t, s)=(1, 1)$ or $q=3^3$.

However, the Sylow 3-group fixes a vector $\neq O$ so that we must have $40(\sqrt{3^3}-1)\geqq 3^{2\cdot 3}/3-1$. Also, $|\mathfrak{R}\cap\mathfrak{Z}|\,\Big|\,3^3-1$ and $\leqq\sqrt{3^3}-1$. So $|\mathfrak{R}\cap\mathfrak{Z}|=1, 2$. Hence, we must have $40\cdot 2\geqq 3^5-1$, a contradiction.

If $G=S_5$, $240\geqq p^{(3/2)p^t\cdot s-t}$ and, as above, we must have $p\leqq 3$. If $p=3$ and $t=2$ then $3^{(3/2)9\cdot s-2}>240$ so again $(t, s)=(1, 1)$. In this case, we reduce to $120\geqq 3^5-1$. If $p=2$, since the Sylow 2-subgroups fix a nonzero vector, the equation reduces to $15 \geqq 2^{(3/2)2^{t} \cdot s - t}$ which implies $(t, s)=(1, 1)$ or $q=2^2$. Since the translation planes of order 16 are determined we do not need to consider this case.

So by the preceding lemmas, we have the proof to (2.9). The proof of (2.10) now follows directly from Johnson, Wilke [**3**]. Also, the reader is referred to [**3**] for the definition of Betten planes and desirable planes.

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