J. Math. Soc. Japan
Vol. 40, No. 2, 1988

An asymptotic estimation of dimension of harmonic spinors

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(Received Nov. 25, 1986)

§ 1. Introduction.

The purpose of the present paper is to give an asymptotic bounds of the dimension of twisted harmonic spinors.

Let (X, g) be an oriented $2n$-dimensional compact spinnable Riemannian manifold, and $\{L, h\}$ a C^∞ complex line bundle over X with a Hermitian fibre metric h. We consider the twisted Dirac's operator $D_k : \Gamma(S\otimes L^k)\to\Gamma(S\otimes L^k)$ which is naturally induced from the Levi-Civita connection of (X, g) and the Hermitian connection of $\{L, h\}$. Here S denotes the spinor bundle of X. Let Δ_k be the Laplace-Beltrami operator of D_k. Then, by Bochner-Weizenböck formula, we obtain that, for $u\in\Gamma(S\otimes L^k)$,

$$\int_X \langle u, \Delta_k u\rangle dV_g = \int_X \left\{|\nabla_k u|^2+\frac{\kappa}{4}|u|^2+k\langle\hat{\Theta}_h u, u\rangle\right\} dV_g,$$

where $\langle\,,\rangle$ represents the inner product on $S\otimes L^k$ with respect to the metric g and h, κ is the scalar curvature of (X, g), and $\hat{\Theta}_h$ is an element of $\mathrm{End}_C(S\otimes L^k)$ which is defined as

$$\hat{\Theta}_h := \frac{1}{2}\sum_{i,j}(e_i e_j\otimes\Theta_h(e_i, e_j)).$$

Here $\{e_1, \cdots, e_{2n}\}$ is an oriented orthonormal base of $T_x X$, and Θ_h is the curvature form of $\{L, h\}$. Now, following Demailly's observation ([3]), we consider the operator $\kappa/4+k\hat{\Theta}_h$ as a potential of the Dirac's operator D_k, and we shall show that the dimension of harmonic spinors of D_k can be asymptotically estimated in terms of the operator $\hat{\Theta}_h$ as k goes to infinity. In fact, using Theorem 2.3 of [3], we shall show the following asymptotic estimation which is a Dirac's operator-version of Demailly's result on $\bar{\partial}$-operator.

THEOREM. *For the curvature form Θ_h of $\{L, h\}$, we define a subset X_+ (resp. X_-) of X as*

$$X_+ \ (resp.\ X_-) := \{x\in X|\ ((i\Theta_h)^n/dV_g)(x)>0\ (resp.\ <0)\},$$

where dV_g is the volume form of (X, g), and we define $H_k^+(0)$ (resp. $H_k^-(0)$) as

$$H_k^+(0)\ (resp.\ H_k^-(0)) := \{harmonic\ sections\ of\ S_+\otimes L^k\ (resp.\ S_-\otimes L^k)\}.$$

Then we have the following estimation.

(1) *If* $X_-=\emptyset$, *we have*

$$\lim_{k\to\infty} k^{-n}\dim H_k^+(0) = \frac{1}{n!}\int_X c_1(L)^n, \qquad \lim_{k\to\infty} k^{-n}\dim H_k^-(0) = 0.$$

(2) *If* $X_+=\emptyset$, *we have*

$$\lim_{k\to\infty} k^{-n}\dim H_k^-(0) = -\frac{1}{n!}\int_X c_1(L)^n, \qquad \lim_{k\to\infty} k^{-n}\dim H_k^+(0) = 0.$$

Here S_+ (respectively S_-) denotes the positive (respectively negative) spinor bundle of X, and we call $H_k^+(0)$ (respectively $H_k^-(0)$) *twisted positive k-harmonic spinors* (respectively *twisted negative k-harmonic spinors*).

Intuitively, the theorem above can be explained as follows. Twisted positive (resp. negative) k-harmonic spinors tend to concentrate on the set where the determinant of the curvature form Θ_h is positive (resp. negative).

The author would like to thank M. Furuta for his kind advice.

§ 2. Morse inequality for Dirac's complex.

Let (X, g) be an oriented $2n$-dimensional compact spinnable Riemannian manifold, let $\{L, h\}$ be a smooth Hermitian line bundle with the Hermitian connection D, and let ∇ be Levi-Civita's connection of (X, g). Since (X, g) is spinnable, there exists the spin($2n$) bundle Spin(X) over X. We lift the Levi-Civita's connection ∇ to Spin(X) and the connection is represented by ∇ again. In the following, for any complex vector bundle E over X, $\Gamma(E)$ represents smooth sections of E over X.

(2.1) DEFINITION. For any $k\geqq 1$, we define a differential operator

$$D_k : \Gamma(S\otimes L^k) \longrightarrow \Gamma(S\otimes L^k)$$

as

$$D_k(f\otimes g)(x) := \left\{\sum_{i=1}^{2n} e_i(\nabla_i f)\otimes g + e_i f\otimes D_i^k g\right\}(x)$$

where $\{e_1, \cdots, e_{2n}\}$ is an orthonormal basis of T_xX, D^k is the connection of L^k induced from D, f (resp. g) is a local section of S (resp. L^k) near $x\in X$, and $\nabla_i := \nabla_{e_i}$, $D_i^k := D_{e_i}^k$.

We define differential operators D_k^+, D_k^-, Δ_k^+, Δ_k^- as

$$D_k^+ := D_k\,|\,\Gamma(S_+\otimes L^k) : \Gamma(S_+\otimes L^k) \longrightarrow \Gamma(S_-\otimes L^k),$$

$$D_k^- := D_k\,|\,\Gamma(S_-\otimes L^k) : \Gamma(S_-\otimes L^k) \longrightarrow \Gamma(S_+\otimes L^k),$$

$$\Delta_k^+ := D_k^- D_k^+ : \Gamma(S_+\otimes L^k) \longrightarrow \Gamma(S_+\otimes L^k), \quad \text{and}$$

$$\Delta_k^- := D_k^+ D_k^- : \Gamma(S_-\otimes L^k) \longrightarrow \Gamma(S_-\otimes L^k).$$

It is well-known that D_k^- is the formal adjoint of D_k^+ with respect to the inner product on $\Gamma(S_\pm\otimes L^k)$ which is canonically defined by g and h, and that D_k^+, D_k^- are first order elliptic operators.

(2.2) DEFINITION. We define Dirac's complex as

$$0 \longrightarrow \Gamma(S_+\otimes L^k) \longrightarrow \Gamma(S_-\otimes L^k) \longrightarrow 0.$$

Obviously, Dirac's complex is an elliptic complex.

We shall construct a subcomplex of Dirac's complex.

(2.3) DEFINITION. For any $\lambda\in\boldsymbol{R}$ and $k\geqq 1$, we define $H_k^+(\lambda)$ and $H_k^-(\lambda)$ as

$$H_k^\pm(\lambda) := \{u\in\Gamma(S_\pm\otimes L^k) \mid u=\sum_\alpha u_\alpha \text{ where } \Delta_k^\pm u_\alpha=\lambda_\alpha u_\alpha,\ \lambda_\alpha\leqq k\lambda\}$$

respectively. Since $\Delta_k^\pm$ are positive operator, $H_k^+(\lambda)=H_k^-(\lambda)=0$ for $\lambda<0$, and it is obvious that

$$H_k^-(0) = \mathrm{Ker} D_k^- \quad\text{and}\quad H_k^+(0) = \mathrm{Ker} D_k^+.$$

(2.4) LEMMA. *We have*

(1) $\quad \Delta_k^+ D_k^- = D_k^- \Delta_k^-,$

(2) $\quad \Delta_k^- D_k^+ = D_k^+ \Delta_k^+.$

PROOF. $\Delta_k^+ D_k^- = (D_k^- D_k^+) D_k^- = D_k^- (D_k^+ D_k^-) = D_k^- \Delta_k^-$. Hence we have obtain (1). Similarly we will have (2). Q.E.D.

(2.5) LEMMA. *For any $\lambda\in\boldsymbol{R}$, we have*

(1) $\quad D_k^- : H_k^-(\lambda) \longrightarrow H_k^+(\lambda),$

(2) $\quad D_k^+ : H_k^+(\lambda) \longrightarrow H_k^-(\lambda).$

PROOF. We will only prove (1). (2) can be proved in the same way. Let $u\in H_k^-(\lambda)$. Then

$$u = \sum_\alpha u_\alpha$$

where $\Delta_k^- u_\alpha = \lambda_\alpha u_\alpha$ and $\lambda_\alpha \leqq k\lambda$. From (2.4) (1), we have

$$\Delta_k^+(D_k^- u_\alpha) = D_k^-(\Delta_k^- u_\alpha) = \lambda_\alpha D_k^- u_\alpha.$$

Since $D_k^- u = \sum_\alpha D_k^- u_\alpha$, we have obtained (1). Q.E.D.

(2.6) DEFINITION. We define $L^2(S_+\otimes L^k)$ (resp. $L^2(S_-\otimes L^k)$) as

$$L^2(S_\pm\otimes L^k) := \{L^2\text{-sections of } S_\pm\otimes L^k \text{ with respect to the metric on } S_\pm\otimes L^k \text{ induced from } g \text{ and } h \text{ in the canonical way}\},$$

respectively. Let $\hat{P}_\lambda^+$ (resp. $\hat{P}_\lambda^-$): $L^2(S_+\otimes L^k)\to H_k^+(\lambda)$ (resp. $L^2(S_-\otimes L^k)\to H_k^-(\lambda)$) be the orthogonal projections. Then we define

$$P_\lambda^\pm := \hat{P}_\lambda^\pm \circ i^\pm : \Gamma(S_\pm\otimes L^k) \longrightarrow H_k^\pm(\lambda),$$

respectively. Here $i^+ : \Gamma(S_+\otimes L^k)\to L^2(S_+\otimes L^k)$ and $i^- : \Gamma(S_-\otimes L^k)\to L^2(S_-\otimes L^k)$ are the natural inclusions.

(2.7) LEMMA. *We have*

(1) $P_\lambda^- D_k^+ = D_k^+ P_\lambda^+$.

(2) $P_\lambda^+ D_k^- = D_k^- P_\lambda^-$.

PROOF. We will only prove (1). (2) can be proved in the same way. Let $u=\sum_\alpha u_\alpha$ be the orthogonal decomposition of $u\in\Gamma(S_+\otimes L^k)$ with respect to the eigenfuntions $\{u_\alpha\}$ of Δ_k^+ with eigenvalues $\{\lambda_\alpha\}$. Then

$$(D_k^+ P_\lambda^+)(u) = \sum_{\lambda_\alpha \leq \lambda} D_k^+ u_\alpha .$$

On the other hand, from (2.4) (2), we have

$$\Delta_k^-(D_k^+ u_\alpha) = D_k^+(\Delta_k^+ u_\alpha) = \lambda_\alpha D_k^+ u_\alpha \qquad \text{and} \qquad P_\lambda^- D_k^+ u = \sum_{\lambda_\alpha \leq \lambda} D_k^+ u_\alpha .$$

Therefore we have obtained (1). Q.E.D.

From (2.5) and (2.7), we have obtained a chain homomorphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(S_+\otimes L^k) & \xrightarrow{D_k^+} & \Gamma(S_-\otimes L^k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle P_\lambda^+} & & \downarrow{\scriptstyle P_\lambda^-} & & \\ 0 & \longrightarrow & H_k^+(\lambda) & \xrightarrow[D_k^+]{} & H_k^-(\lambda) & \longrightarrow & 0. \end{array}$$

Then we have the following theorem.

(2.8) THEOREM. *For any $\lambda \geqq 0$, there exists a linear operator H_λ such that*

$$\mathrm{Id} - P_\lambda^+ = H_\lambda D_k^+ \qquad \textit{and} \qquad \mathrm{Id} - P_\lambda^- = D_k^+ H_\lambda .$$

PROOF. Let

$$E_+ \text{ (resp. } E_-) := \{\text{eigenvalues of } \Delta_k^+ \text{ (resp. } \Delta_k^-)\},$$

and for any $\nu\in E_+$ (resp. $\mu\in E_-$), we define

$$\rho_\nu^+ : \Gamma(S_+\otimes L^k) \longrightarrow \oplus \boldsymbol{C}\{u\in\Gamma(S_+\otimes L^k) \mid \Delta_k^+ u=\nu u\}$$
$$(\text{resp. } \rho_\mu^- : \Gamma(S_-\otimes L^k) \longrightarrow \oplus \boldsymbol{C}\{v\in\Gamma(S_-\otimes L^k) \mid \Delta_k^- v=\mu v\})$$

as the orthogonal projection with respect to the L^2-inner product respectively. From (2.4), we have

$$\rho_\nu^- D_k^+ = D_k^+ \rho_\nu^+ \quad \text{and} \quad \rho_\nu^+ D_k^- = D_k^- \rho_\nu^- .$$

Let

$$G_+ \text{ (resp. } G_-) := \sum_{\nu \in E_+} \nu^{-1} \rho_\nu^+ \quad \text{(resp. } \sum_{\mu \in E_-} \mu^{-1} \rho_\mu^-)$$

be the Green's operator of Δ_+^k (resp. Δ_-^k). Then we have

$$G_- D_k^+ = D_k^+ G_+ , \qquad G_+ D_k^- = D_k^- G_- ,$$
$$P_\lambda^+ G_+ = G_+ P_\lambda^+ , \qquad P_\lambda^- G_- = G_- P_\lambda^- ,$$
$$\mathrm{Id} - P_0^+ = \Delta_k^+ G_+ , \qquad \mathrm{Id} - P_0^- = \Delta_k^+ G_- .$$

Therefore for any $\lambda \geqq 0$, we obtain

$$\mathrm{Id} - P_\lambda^+ = (P_0^+ + \Delta_k^+ G_+)(\mathrm{Id} - P_\lambda^+) = D_k^- D_k^+ G_+ (\mathrm{Id} - P_\lambda^+) = \{D_k^- G_- (\mathrm{Id} - P_\lambda^-)\} D_k^+ ,$$

and

$$\mathrm{Id} - P_\lambda^- = (\mathrm{Id} - P_\lambda^-)(P_0^- + \Delta_k^- G_-) = (\mathrm{Id} - P_\lambda^-) D_k^+ D_k^- G_- = D_k^+ (\mathrm{Id} - P_\lambda^+) D_k^- G_-$$
$$= D_k^+ \{D_k^- G_- (\mathrm{Id} - P_\lambda^-)\} .$$

So if we set

$$H_\lambda = D_k^- G_- (\mathrm{Id} - P_\lambda^-) ,$$

we have (2.8). Q.E.D.

As a consequence of (2.8), we obtain

(2.9) COROLLARY (Morse inequality for Dirac's complex). *For any* $\lambda \geqq 0$, *we have*

(1) $\dim \mathrm{Ker}\{D_k^+ : \Gamma(S_+ \otimes L^k) \to \Gamma(S_- \otimes L^k)\} \leqq \dim H_k^+(\lambda)$,

(2) $\dim \mathrm{Ker}\{D_k^- : \Gamma(S_- \otimes L^k) \to \Gamma(S_+ \otimes L^k)\} \leqq \dim H_k^-(\lambda)$,

(3) $\dim H_k^+(\lambda) - \dim H_k^-(\lambda) = \mathrm{ind}(D_k^+)$,

where

$$\mathrm{ind}(D_k^+) := \dim H_k^+(0) - \dim H_k^-(0) .$$

§ 3. Spinor representation.

In this section, we recall some well-known facts about Clifford algebra. First of all, we shall give some notations.

(3.1) NOTATIONS.

E: an oriented $2n$-dimensional vector space over $\boldsymbol{R}$ with an inner product,
$\{\gamma_1, \cdots, \gamma_{2n}\}$: an oriented orthonormal basis of E over $\boldsymbol{R}$,
$c(E)$: the Clifford algebra of E.

It is well-known that there exists a 2^n-dimensional complex vector space Δ such that

$$c(E)\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C} \cong \mathrm{End}_{\boldsymbol{C}}(\Delta)$$

as a $\boldsymbol{C}$-algebra. Set

$$\tau := i^n\gamma_1 \cdots \gamma_{2n},$$

then it is easy to see that $\tau^2=1$. Now we define Δ_+ and Δ_- as

$$\Delta_+ := \{\delta\in\Delta \mid \tau\delta=\delta\}, \qquad \Delta_- := \{\delta\in\Delta \mid \tau\delta=-\delta\}.$$

From the above argument, we have

$$\Delta = \Delta_+\oplus\Delta_-.$$

Since any element of spin$(2n)$ commutes with τ, spin$(2n)$ preserves Δ_+ and Δ_- respectively. The following fact is well-known.

(3.2) FACT (cf. [6]). *Let S^0 be $\{-1, 1\}$. Then there exists a base of Δ over $\boldsymbol{C}$*

$$\{u_{\varepsilon_1,\dots,\varepsilon_{n-1}}, v_{\varepsilon_1,\dots,\varepsilon_{n-1}}\}_{(\varepsilon_1,\dots,\varepsilon_{n-1})\in(S^0)^{n-1}}$$

such that

(1) $\gamma_1\gamma_2(u_{\varepsilon_1,\dots,\varepsilon_{n-1}}) = (i\varepsilon_1)u_{\varepsilon_1,\dots,\varepsilon_{n-1}}$,

$\gamma_{2k-1}\gamma_{2k}(u_{\varepsilon_1,\dots,\varepsilon_{n-1}}) = (-i\varepsilon_{k-1}\varepsilon_k)u_{\varepsilon_1,\dots,\varepsilon_{n-1}}$ *for* $2\leqq k\leqq n-1$,

$\gamma_{2n-1}\gamma_{2n}(u_{\varepsilon_1,\dots,\varepsilon_{n-1}}) = (i\varepsilon_{n-1})u_{\varepsilon_1,\dots,\varepsilon_{n-1}}$,

and

(2) $\gamma_1\gamma_2(v_{\varepsilon_1,\dots,\varepsilon_{n-1}}) = (-i\varepsilon_1)v_{\varepsilon_1,\dots,\varepsilon_{n-1}}$,

$\gamma_{2k-1}\gamma_{2k}(v_{\varepsilon_1,\dots,\varepsilon_{n-1}}) = (-i\varepsilon_{k-1}\varepsilon_k)v_{\varepsilon_1,\dots,\varepsilon_{n-1}}$ *for* $2\leqq k\leqq n-1$,

$\gamma_{2n-1}\gamma_{2n}(v_{\varepsilon_1,\dots,\varepsilon_{n-1}}) = (i\varepsilon_{n-1})v_{\varepsilon_1,\dots,\varepsilon_{n-1}}$.

(3.3) COROLLARY.

(1) $\Delta_+ = \bigoplus_{\varepsilon\in(S^0)^{n-1}} \boldsymbol{C}u_\varepsilon$, *and*

(2) $\Delta_- = \bigoplus_{\varepsilon\in(S^0)^{n-1}} \boldsymbol{C}v_\varepsilon$.

For

$$a = \frac{1}{i}\sum_{k=1}^{n} a_k\gamma_{2k-1}\gamma_{2k}, \qquad a_k\in\boldsymbol{R},$$

we define a_+ and a_- as

$$a_+ := a\,|\,\Delta_+ : \Delta_+ \longrightarrow \Delta_+ \quad \text{and} \quad a_- := a\,|\,\Delta_- : \Delta_- \longrightarrow \Delta_-.$$

Using (3.3), it is easy to see that the following proposition holds.

(3.4) PROPOSITION. (1) *The eigenvalues of a_+ are*

$$\{-a_{\sigma(1)}-\cdots-a_{\sigma(2k)}+a_{\sigma(2k+1)}+\cdots+a_{\sigma(n)}\}_{0\leq k\leq n/2,\,\sigma\in\mathfrak{S}_n}$$

when n is even, and

$$\{-a_{\sigma(1)}-\cdots-a_{\sigma(2k+1)}+a_{\sigma(2k+2)}+\cdots+a_{\sigma(n)}\}_{0\leq k\leq (n-1)/2,\,\sigma\in\mathfrak{S}_n}$$

when n is odd.

(2) *The eigenvalues of a_- are*

$$\{-a_{\sigma(1)}-\cdots-a_{\sigma(2k+1)}+a_{\sigma(2k+2)}+\cdots+a_{\sigma(n)}\}_{0\leq k\leq (n-2)/2,\,\sigma\in\mathfrak{S}_n}$$

when n is even, and

$$\{-a_{\sigma(1)}-\cdots-a_{\sigma(2k)}+a_{\sigma(2k+2)}+\cdots+a_{\sigma(n)}\}_{0\leq k\leq (n-1)/2,\,\sigma\in\mathfrak{S}_n}$$

when n is odd. Here $\mathfrak{S}_n$ represents the n-th permutation group.

§ 4. Bochner's identity.

In this section, we shall recall Bochner's identity for Dirac's operator. Let (X, g) be an oriented $2n$-dimensional compact spinnable Riemannian manifold and let $\{L, h\}$ be a smooth complex line bundle over X with a Hermitian fibre metric h. Let D be the Hermitian connection of $\{L, h\}$. Then we define a connection ∇_k of $S\otimes L^k$ as

$$\nabla_k := \nabla\otimes \mathrm{Id}_{L^k}+\mathrm{Id}_S\otimes D_k \tag{4.1}$$

where D_k is the connection of L^k induced from D.

(4.2) FACT (Bochner's identity, cf. [2]).

(1) *For $u\in\Gamma(S_+\otimes L^k)$, we have*

$$\int_X \langle u, \Delta_k^+ u\rangle dV_g=\int_X\left\{|\nabla_k u|^2+\frac{\kappa}{4}|u|^2+k\langle\hat{\Theta}_h u, u\rangle\right\}dV_g.$$

(2) *For $v\in\Gamma(S_-\otimes L^k)$, we have*

$$\int_X \langle v, \Delta_k^- v\rangle dV_g=\int_X\left\{|\nabla_k v|^2+\frac{\kappa}{4}|v|^2+k\langle\hat{\Theta}_h v, v\rangle\right\}dV_g.$$

Where $\langle\,,\rangle$ represents the inner product on $S\otimes L^k$ with respect to the metric g and h, κ is the scalar curvature of (X, g), and $\hat{\Theta}_h$ is an element of $\mathrm{End}_C(S\otimes L^k)$ which is defined as

$$\hat{\Theta}_h := \frac{1}{2}\sum_{i,j}(e_ie_j\otimes\Theta_h(e_i, e_j)).$$

Here $\{e_1, \cdots, e_{2n}\}$ is an oriented orthonormal base of T_xX, and Θ_h is the curvature form of $\{L, h\}$.

§ 5. An asymptotic estimation of $\dim H_k^{\pm}(\lambda)$.

First of all, we shall give some notations and definitions.

(5.1) NOTATIONS.

$E \to X$: a smooth complex vector bundle over X $\mathrm{rank}_C E = r$ with a Hermitian metric ρ and with a connection $\hat{\nabla}$,

$L \to X$: a smooth complex line bundle over X with a Hermitian metric h and with the Hermitian connection D,

and $\hat{\nabla}_k := \hat{\nabla} \otimes \mathrm{Id}_{L^k} + \mathrm{Id}_E \otimes D_k$ where D_k is the connection on L^k induced from D.

We define a vector bundle $\mathrm{Herm}_\rho(E)$ as

$$\mathrm{Herm}_\rho(E) := \bigcup_{x \in X} \mathrm{Herm}_\rho(E_x)$$

where E_x is the fibre of $E \to X$ at $x \in X$, and

$$\mathrm{Herm}_\rho(E_x) := \{\phi \in \mathrm{End}(E_x) \mid \langle u, \phi v\rangle_\rho = \langle \phi u, v\rangle_\rho \text{ for any } u, v \in E_x\}.$$

Let $V \in \Gamma(\mathrm{Herm}_\rho(E))$. Then the homomorphism

$$V \otimes \mathrm{Id}_{L^k} : \Gamma(E \otimes L^k) \longrightarrow \Gamma(E \otimes L^k)$$

will be represented by V again.

(5.2) DEFINITION. Let $W^{1,2}(L^k \otimes E)$ be the completion of $\Gamma(L^k \otimes E)$ with respect to the norm

$$\|u\|_{1,2} := \left[\int_X \{|u|^2 + |\hat{\nabla}_k u|^2\} dV_g\right]^{1/2},$$

where $|\ |$ is the norm induced by g, h and ρ. We define a quadratic form Q_k on $W^{1,2}(L^k \otimes E)$ as

$$Q_k(u) := \int_X \left\{\frac{1}{k} |\hat{\nabla}_k u|^2 - \langle Vu, u\rangle\right\} dV_g,$$

and for any $\lambda \in \boldsymbol{R}$, we define $N_k(\lambda)$ as

$$N_k(\lambda) := \#\{\text{eigenvalues of } Q_k \leq \lambda\}.$$

(5.3) DEFINITION. For $B \in \Omega^2(X) := \Gamma(\Lambda^2 T^* X)$, we define a real valued function $\nu_B(X)$ on X for any $\lambda \in \boldsymbol{R}$ as follows. For $x \in X$, we represent $B(x)$ as

$$B(x) = \sum_{i=1}^n B_i(x) \gamma_{2i-1} \wedge \gamma_{2i}$$

with respect to an oriented orthonormal base $\{\gamma_1, \cdots, \gamma_{2n}\}$ of $T_x^* X$, where $\{B_i(x)\}$ are real numbers satisfying

$$|B_1(x)| \geqq \cdots \geqq |B_s(x)| > 0 = B_{s+1}(x) = \cdots = B_n(x).$$

Then we define $\nu_B(\lambda)(x)$ as

$$\nu_B(\lambda)(x):=\frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)}|B_1(x)\cdots B_s(x)|\sum_{(p_1,\cdots,p_s)\in N^s}\left(\left[\lambda-\sum_{j=1}^{s}(2p+1)|B_j(x)|\right]_+\right)^{n-s}$$

where

$$N:=\{l\in Z\,|\,l\geqq 0\},$$

$$\lambda_+:=\begin{cases}\lambda & \text{if } \lambda\geqq 0\\ 0 & \text{if } \lambda<0\end{cases} \quad\text{and}\quad \lambda_+^0:=\begin{cases}1 & \text{if } \lambda>0\\ 0 & \text{if } \lambda\leqq 0.\end{cases}$$

Since the function $\nu_B(\lambda)(x)$ is monotone increasing with respect to λ, we define $\bar{\nu}_B(\lambda)(x)$ as

$$\bar{\nu}_B(\lambda)(x):=\lim_{0<\varepsilon\to 0}\nu_B(\lambda+\varepsilon)(x).$$

Now we state Demailly's result.

(5.4) FACT (cf. [3]). *Let $V_1(x),\cdots,V_r(x)$ be the eigenvalues of $V(x)\in \mathrm{Herm}_\rho(E_x)$. Then there exists a countable set $\mathfrak{D}$ of $\boldsymbol{R}$ such that for any $\lambda\in\boldsymbol{R}\backslash\mathfrak{D}$, there exists $\lim_{k\to\infty}k^{-n}N_k(\lambda)$ which satisfies*

$$\lim_{k\to\infty}k^{-n}N_k(\lambda)=\sum_{j=1}^{r}\int_X\nu_{i\Theta_h}(V_j+\lambda)dV_g.$$

We shall apply this result to the case of $E=S_+$ or S_-, and of $\langle Vu, u\rangle=-(\kappa/4k)|u|^2-\langle\hat{\Theta}_h u, u\rangle$.

(5.5) THEOREM. *There exists a countable set $\mathfrak{D}$ of $\boldsymbol{R}$ such that there exists $\lim_{k\to\infty}k^{-n}\dim H_k^{\pm}(\lambda)$ for any $\lambda\in\boldsymbol{R}\backslash\mathfrak{D}$ and* (1) *when n is even,*

$$\lim_{k\to 8}k^{-n}\dim H_k^+(\lambda)$$
$$=\sum_{0\leq k\leq n/2,\,\sigma\in\mathfrak{S}_n}\int_X\nu_{i\Theta_h}(\lambda+(a_{\sigma(1)}+\cdots+a_{\sigma(2k)}-a_{\sigma(2k+1)}-\cdots-a_{\sigma(n)}))dV_g,$$

$$\lim_{k\to\infty}k^{-n}\dim H_k^-(\lambda)$$
$$=\sum_{0\leq k\leq(n/2)-1,\,\sigma\in\mathfrak{S}_n}\int_X\nu_{i\Theta_h}(\lambda+(a_{\sigma(1)}+\cdots+a_{\sigma(2k+1)}-a_{\sigma(2k+2)}-\cdots-a_{\sigma(n)}))dV_g,$$

(2) *when n is odd,*

$$\lim_{k\to\infty}k^{-n}\dim H_k^+(\lambda)$$
$$=\sum_{0\leq k\leq(n-1)/2,\,\sigma\in\mathfrak{S}_n}\int_X\nu_{i\Theta_h}(\lambda+(a_{\sigma(1)}+\cdots+a_{\sigma(2k+1)}-a_{\sigma(2k+2)}-\cdots-a_{\sigma(n)}))dV_g,$$

$$\lim_{k\to\infty}k^{-n}\dim H_k^-(\lambda)$$
$$=\sum_{0\leq k\leq(n-1)/2,\,\sigma\in\mathfrak{S}_n}\int_X\nu_{i\Theta_h}(\lambda+(a_{\sigma(1)}+\cdots+a_{\sigma(2k)}-a_{\sigma(2k+1)}-\cdots-a_{\sigma(n)}))dV_g,$$

where

$$\Theta_h(x)=\frac{1}{i}\sum_{i=1}^{n}a_i(x)\gamma_{2i-1}\wedge\gamma_{2i}, \qquad a_i(x)\in \boldsymbol{R}$$

with respect to an oriented orthonormal base $\{\gamma_1, \cdots, \gamma_{2n}\}$ *of* T_x^*X.

PROOF. Since

$$S_+ = \Delta_+ \underset{\mathrm{spin}(2n)}{\times} \mathrm{Spin}(X) \quad \text{and} \quad S_- = \Delta_- \underset{\mathrm{spin}(2n)}{\times} \mathrm{Spin}(X),$$

we have the identity above from (3.4), (4.1) and (5.4). Q. E. D.

As a consequence of (5.5), we obtain the following theorem.

(5.6) THEOREM.

(1) $$\lim_{0<\lambda\to 0}\lim_{k\to\infty}k^{-n}\dim H_k^+(\lambda)=\frac{1}{n!}\int_{X_+}\left(\frac{i}{2\pi}\Theta_h\right)^n,$$

(2) $$\lim_{0<\lambda\to 0}\lim_{k\to\infty}k^{-n}\dim H_k^-(\lambda)=-\frac{1}{n!}\int_{X_-}\left(\frac{i}{2\pi}\Theta_h\right)^n,$$

where

$$X_+ := \{x\in X \mid (i\Theta_h(x))^n/dV_g(x)>0\} \quad \text{and} \quad X_- := \{x\in X \mid (i\Theta_h(x))^n/dV_g(x)<0\}.$$

PROOF. Fix $x\in X$. We represent $\Theta_h(x)$ as

$$\Theta_h(x)=\frac{1}{i}\sum_{i=1}^{n}a_i\gamma_{2i-1}\wedge\gamma_{2i}$$

with respect to an oriented orthonormal base $\{\gamma_1, \cdots, \gamma_{2n}\}$ of T_x^*X, where $a_i\in\boldsymbol{R}$ satisfying

$$|a_1|\geqq|a_2|\geqq\cdots\geqq|a_s|>0=|a_{s+1}|=\cdots=|a_n|.$$

Then it is easy to see

$$\lim_{0<\lambda\to 0}\sum_{(p_1,\cdots,p_s)\in N^s}\left(\left[\lambda+a_1+\cdots+a_p-a_{p+1}-\cdots-a_n-\sum_{j=1}^{s}(2p_j+1)|a_j|\right]_+\right)^{n-s}$$
$$=\begin{cases}1 & \text{if } n=s,\ a_1,\cdots,a_p>0, \text{ and } a_{p+1},\cdots,a_n<0\\ 0 & \text{otherwise.}\end{cases}$$

Therefore when n is even, from (5.5), we obtain

$$\lim_{0<\lambda\to 0}\lim_{k\to\infty}k^{-n}\dim H_k^+(\lambda)$$
$$=\sum_{0\leq k\leq n/2,\,\sigma\in\mathfrak{S}_n}\int_X \bar{\nu}_{i\Theta_h}(a_{\sigma(1)}+\cdots+a_{\sigma(2k)}-a_{\sigma(2k+1)}-\cdots-a_{\sigma(n)})dV_g$$
$$=\left(\frac{1}{2\pi}\right)^n\int_{X_+}|a_1|\cdots|a_n|dV_g=\frac{1}{n!}\int_{X_+}\left(\frac{i}{2\pi}\Theta_h\right)^n,$$

and when n is odd, we obtain

$$\lim_{0<\lambda\to 0}\lim_{k\to\infty}k^{-n}\dim H_k^+(\lambda)$$

$$= \sum_{0 \leq k \leq (n-1)/2,\, \sigma \in \mathfrak{S}_n} \int_X \nu_{i\Theta_h}(a_{\sigma(1)} + \cdots + a_{\sigma(2k+1)} - a_{\sigma(2k+2)} - \cdots - a_{\sigma(n)}) dV_g$$

$$= \left(\frac{1}{2\pi}\right)^n \int_{X_+} |a_1| \cdots |a_n| dV_g = \frac{1}{n!}\int_{X_+}\left(\frac{i}{2\pi}\Theta_h\right)^n.$$

Similarly we will obtain the identity in the case of negative spinors. Q. E. D.

§ 6. An asymptotic estimation of dimension of harmonic spinors.

In this section, we shall apply (5.6) to an asymptotic estimation of the dimension of harmonic spinors. The following theorem follows from (2.8) and (5.6).

(6.1) THEOREM.

(1) $\limsup_{k\to\infty} k^{-n} \dim H_k^+(0) \leqq \frac{1}{n!}\int_{X_+}\left(\frac{i}{2\pi}\Theta_h\right)^n$, *and*

(2) $\limsup_{k\to\infty} k^{-n} \dim H_k^-(0) \leqq \frac{-1}{n!}\int_{X_-}\left(\frac{i}{2\pi}\Theta_h\right)^n$.

As a consequence, we obtain the following theorem.

(6.2) THEOREM. (1) *If* $[(i\Theta_h)^n/dV_g](x) \geqq 0$ *for any* $x \in X$, *there exists*

$$\lim_{k\to\infty} k^{-n} \dim H_k^{\pm}(0)$$

and

$$\lim_{k\to\infty} k^{-n} \dim H_k^+(0) = \frac{1}{n!}\int_X c_1(L)^n, \qquad \lim_{k\to\infty} k^{-n} \dim H_k^-(0) = 0.$$

(2) *If* $[(i\Theta_h)^n/dV_g](x) \leqq 0$ *for any* $x \in X$, *there exists*

$$\lim_{k\to\infty} k^{-n} \dim H_k^{\pm}(0)$$

and

$$\lim_{k\to\infty} k^{-n} \dim H_k^-(0) = \frac{-1}{n!}\int_X c_1(L)^n, \qquad \lim_{k\to\infty} k^{-n} \dim H_k^+(0) = 0.$$

PROOF. We shall only prove (1). (2) will be proved in the same way. From (2.9), we obtain

$$k^{-n} \dim H_k^+(0) - k^{-n} \dim H_k^-(0) = k^{-n} \operatorname{ind}(D_k^+)$$

and from the Atiyah-Singer's index theorem (cf. [**1**]),

$$= k^{-n}(\operatorname{ch}(kL)\hat{\mathfrak{A}}(X))[X].$$

Therefore we have obtained

$$\liminf_{k\to\infty} k^{-n}\dim H_k^-(0)+\frac{1}{n!}\int_X c_1(L)^n = \liminf_{k\to\infty} k^{-n}\dim H_k^+(0)$$

$$\leqq \limsup_{k\to\infty} k^{-n}\dim H_k^+(0) = \limsup_{k\to\infty} k^{-n}\dim H_k^-(0)+\frac{1}{n!}\int_X c_1(L)^n .$$

From (6.1), we obtain

$$\lim_{k\to\infty} k^{-n}\dim H_k^+(0) = \frac{1}{n!}\int_X c_1(L)^n, \quad \text{and}$$

$$\lim_{k\to\infty} k^{-n}\dim H_k^-(0) = 0. \qquad \text{Q.E.D.}$$

Addendum (added in June 1987). Let (X, g) be an oriented $2n$-dimensional compact Riemannian manifold and let $\{L, h\}$ be a C^∞ complex line bundle with a Hermitian fibre metric h. Assume that X possesses a spin^c structure. Then, associated to the exact sequence

$$1 \longrightarrow \mathrm{Spin} \longrightarrow \mathrm{Spin}^c \longrightarrow S^1 \longrightarrow 1,$$

we obtain $\pm 1/2$ spinor bundle over X; $S^\pm$ respectively. As in Section 4, we obtain a connection ∇_k^c of $S \otimes L^k$ and its Laplace-Beltrami operator Δ_k^c. Now, since we have Bochner-Weizenböck formula for Δ_k^c which is same as (4.2) (see for instance [4], Theorem 1.1), we shall obtain the same results as in Section 6 in the case that X is spin^c.

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