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The global hypoellipticity of degenerate elliptic-parabolic operators

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0. Introduction.

As is well-known, there are vast references on (*local*) hypoellipticity of degenerate elliptic-parabolic operators (cf. [**2**], [**3**], [**8**], [**9**], [**13**], [**14**] and their references). However, one can find only few papers concerned with *global* hypoellipticity. Oleinik and Radkevich [**14**] and Fedii [**3**] proved global hypoellipticity of degenerate elliptic-parabolic operators when $S=\{x : \dim \mathrm{Lie}(x)<d\}$ is either a smooth hypersurface or an isolated point. Fedii [**4**] and Fujiwara and Omori [**6**] found an operator which is globally hypoelliptic but not (locally) hypoelliptic; Amano [**1**] generalized their results. In this paper, we shall show sufficient conditions for global hypoellipticity, which are stated in terms of diffusion and drift vector fields. Our results contain Oleinik and Radkevich's, Fedii's and Fujiwara and Omori's theorems as special cases. We can apply our theorems when S is not a smooth hypersurface, and further, since the proof of theorems essentially depends on a certain type of moderate *a priori* estimates (Proposition 2.1), our results are applicable to a wider class of operators.

Recently, Kusuoka and Stroock [**12**] and Omori [**15**] proved similar theorems; their methods are different from ours. Unfortunately, their results are not applicable to the operators which are not of Hörmander type.

Throughout this paper, Ω is an open set of $\boldsymbol{R}^d$, and

$$P=\sum_{i,j=1}^{d} a^{ij}\partial_{x_i}\partial_{x_j}+\sum_{i=1}^{d} b^i\partial_{x_j}+c(x)$$

is a differential operator with real coefficients satisfying

$$a^{ij}(x)=a^{ji}(x)\in C^\infty(\Omega),\quad b^i(x)\in C^\infty(\Omega),\quad c(x)\in C^\infty(\Omega)$$

and

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j\geqq 0 \qquad \text{for any} \quad (x, \xi)\in\Omega\times\boldsymbol{R}^d,$$

i.e., P is a degenerate elliptic-parabolic operator in Ω. We define vector fields $X_0, X_1, \cdots, X_d$ by

$$X_0=\sum_{i=1}^{d}(b^i-\sum_{j=1}^{d}\partial_{x_j}a^{ij})\partial_{x_i}, \qquad X_i=\sum_{j=1}^{d}a^{ij}\partial_{x_j} \quad (1\leqq i\leqq d).$$

X_0 is called *drift vector field* and each X_i $(1\leqq i\leqq d)$ is called *diffusion vector field.* Lie$(\cdot)$ denotes a distribution (in the sense of differential geometry) generated by the Lie algebra Lie$(X_0, X_1, \cdots, X_d)$, i.e., Lie$(x)=\{X(x): X\in \text{Lie}(X_0, X_1, \cdots, X_d)\}$. S stands for the set $\{x\in\Omega: \dim \text{Lie}(x)<d\}$. In this paper, $C^k(\Omega)$ denotes a set of all C^k smooth real-valued functions defined in Ω, and $C_b^k(\Omega)$ denotes a set of functions $f\in C^k(\Omega)$ whose derivatives $\partial_x^\alpha f$ $(|\alpha|\leqq k)$ are bounded in Ω. Unless otherwise specified, we use the same notation of Kumano-go [**10**].

THEOREM 1. *Assume that S is a compact set of Ω. If there is a real-valued function $\Phi\in C^\infty(\Omega)$ such that*

$$S=\{x\in\Omega: \Phi(x)=0\},$$

and if there is a finite sequence of vector fields $\{X^{(k)}\}_{k=1}^N$ such that each $X^{(k)}$ is expressed as $X^{(k)}=\sum_{i=0}^d \lambda_i^{(k)}X_i$, $\lambda_i^{(k)}\in C^\infty(\Omega)$ and

$$S=\bigcup_{k=1}^{N}\{x\in S: X^{(k)}\cdots X^{(2)}X^{(1)}\Phi(x)\neq 0\}, \tag{0.1}$$

then the operator P is globally hypoelliptic in Ω.

Modifying the proof of Theorem 1, we can prove the following fact: *Assume that S is a compact set of Ω, assume that there is a real-valued function $\Phi\in C^\infty(\Omega)$ and there is a constant $\rho>0$ such that*

$$S\subset\{x\in\Omega: |\Phi(x)|<\rho\},$$

and assume that there is a finite sequence of vector fields $\{X^{(k)}\}_{k=1}^N$ satisfying the same conditions of Theorem 1. If $u\in\mathcal{D}'(\Omega)$ and $Pu\in H_s(\Omega)$ for a real number s, and if $\rho>0$ is sufficiently small, then $u\in H_s(\Omega)$. Here ρ depends on s. This shows that *a solution $u\in\mathcal{D}'(\Omega)$ of the equation $Pu=f$ $(f\in C^\infty(\Omega))$ is sufficiently smooth, if the set S is sufficiently thin with respect to the vector fields $X_0, X_1, \cdots, X_d$* (cf. Example 1).

EXAMPLE 1. Let us consider an operator

$$P_\rho=\partial_{x_1}^2+a_\rho(x_1, x_2)\partial_{x_2}^2$$

in $\boldsymbol{R}^2$, where $\rho\geqq 0$, $a_\rho\in C^\infty(\Omega)$ and

$$a_\rho(x)\begin{cases} >0 & (|\Phi(x)|>\rho) \\ =0 & (|\Phi(x)|\leqq\rho). \end{cases}$$

Here $\Phi(x)=\Phi(x_1, x_2)=(x_1^2+x_2^2-x_1)^2-(x_1^2+x_2^2)$. In case $\rho=0$, Theorem 1 shows the $P_\rho=P_0$ is globally hypoelliptic. It is to be noted that we cannot apply Oleinik and Radkevich's and Fedii's theorems to the operator P_0, since $\Phi(x)=0$ is not a C^1 submanifold of $\boldsymbol{R}^2$ and since $\partial_{x_1}\Phi(0)=\partial_{x_1}^2\Phi(0)=0$. Theorem 1 is not applicable to the operators P_ρ with $\rho>0$. However, modifying the proof, we can prove the following: If $u\in\mathscr{D}'(\boldsymbol{R}^2)$ and $P_\rho u\in H_s(\boldsymbol{R}^2)$ for a real number s, and if $\rho>0$ is sufficiently small with respect to s, then $u\in H_s(\boldsymbol{R}^2)$.

THEOREM 2. *Assume that S is a compact set of Ω with $S=\overline{\mathring{S}}$, and assume that*

$$X_{i_0}(x)\equiv 0,\quad X_0(x)\cdot e_{i_0}\neq 0 \qquad \text{in } S \tag{0.2}$$

holds for some i_0 $(1\leqq i_0\leqq d)$. Then the operator P is globally hypoelliptic in Ω.

Here $e_k=(0,\cdots,0,\overset{k\text{-th}}{\overset{\downarrow}{1}},0,\cdots,0)$, $k=1,2,\cdots,n$. By Lemma 1.3, $S=\overline{\mathring{S}}$, $X_{i_0}(x)=0$ and $X_0(x)\cdot e_{i_0}\neq 0$ imply $a^{i_0i_0}(x)=0$ and $b^{i_0}(x)\neq 0$, i.e., P is (degenerate) parabolic in x_{i_0}. Roughly speaking, Theorem 2 shows that *the operator P is globally hypoelliptic in Ω, if P is (degenerate) parabolic in Ω with respect to a certain variable* (cf. Example 2). Theorem 2 is not contained in Theorem 1. In fact, we cannot apply Theorem 1 to the operators with $\mathring{S}\neq\emptyset$. We can remove the assumption $S=\overline{\mathring{S}}$ of Theorem 2, if we replace (0.2) by $a^{i_0i_0}(x)\equiv 0$, $b^{i_0}(x)\neq 0$ in S.

EXAMPLE 2. Let $a\in C^\infty(\Omega)$ be a nonnegative function such that $\boldsymbol{R}^2\setminus(\operatorname{supp} a)$ is compact. Then the operator

$$\partial_{x_1}+a(x_1,x_2)\partial_{x_2}^2$$

is globally hypoelliptic in $\boldsymbol{R}^2$. It is not necessary that the boundary of $\operatorname{supp} a$ is C^∞ smooth.

If we restrict ourselves to a certain sub-class of formally self-adjoint degenerate elliptic operators, then the global hypoellipticity follows from a moderate assumption which is weaker than (0.2).

THEOREM 3. *Assume that S is a compact set of Ω, and assume that P is a formally self-adjoint operator such that*

$$\partial_x^\alpha a^{ij}(x)=0 \quad \text{in } S\ (1\leqq|\alpha|\leqq 2,\ 1\leqq i,j\leqq d) \quad \text{and} \quad c<0 \text{ in } \Omega. \tag{0.3}$$

Then the operator P is globally hypoelliptic in Ω, if the system

$$\dot{x}=\sum_{i=1}^{d}\lambda_i X_i(x) \qquad (\lambda_i\in\boldsymbol{R}) \tag{0.4}$$

is weakly controllable in Ω, i.e., if for any points $p\in\Omega$ and $q\in\Omega$, and for any neighborhoods $V(p)$ and $V(q)$ of p and q, there is a finite sequence of open sets V_k $(k=0,1,\cdots,N)$ in Ω and there is a finite sequence of C^∞ functions $\phi^k(t,x)$

$(0\leqq t\leqq T_k,\ x\in V_k,\ k=1,\cdots,N)$ *such that* $p\in V_0\subset V(p)$, $q\in V_N\subset V(q)$, $V_k=\phi^k(T_k, V_{k-1})$ $(k=1,\cdots,N)$, $\phi^k(t,x)\in\Omega$ $(0\leqq t\leqq T_k,\ x\in V_k,\ k=1,\cdots,N)$ *and*

$$\phi_t^k=\sum_{i=1}^d \lambda_i(\phi^k)X_i(\phi^k), \qquad \phi^k(0,x)=x\,,$$

where $\lambda_i\in C^\infty(\Omega)$, *and such that each* $\phi^k(T_k,\cdot)$ *is a* C^∞ *diffeomorphism from* V_{k-1} *onto* V_k.

Theorem 3 remains valid when Ω is a C^∞ manifold. In fact, we can prove the following fact: *Let P be a degenerate elliptic-parabolic operator defined on a* C^∞ *manifold M. Assume that S is a compact set of M with* $S\subsetneqq M$, *and assume that P is a formally self-adjoint operator such that*

$$P\cdot=\sum_{i,j=1}^d \partial_{x_i}(a^{ij}\partial_{x_j}\cdot)+c$$

in local coordinates, and such that $\partial_x^\alpha a^{ij}=0$ *in S* $(1\leqq|\alpha|\leqq 2,\ 1\leqq i,\ j\leqq d)$ *and* $c\leqq 0$ *in* Ω. *If the system* (0.5) *is weakly controllable on M, then the operator P is globally hypoelliptic on M* (cf. Example 3). In case $S=M$, P is not always globally hypoelliptic (cf. [7]). However, by modifying the proof of Theorem 3, we can show that if the system (0.5) is weakly controllable on M, and if $u\in\mathcal{D}'(M)$, $P\in C^\infty(M)$ and $u\in C^\infty(U)$ for some open set U of M, then $u\in C^\infty(M)$. This shows that C^∞ *regularity of solutions propagates along the diffusion vector fields* $X_1,\cdots,X_d$ (cf. Example 4).

EXAMPLE 3. Let $a(x)=a(x_1,x_2)\in C^\infty(\boldsymbol{T}^2)$ be a nonnegative function such that $a(x)=0$ implies $\partial_x^\alpha a(x)=0$ $(x\in\boldsymbol{T}^2,\ |\alpha|=2)$, where $\boldsymbol{T}^2=\boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$. Then the operator

$$\partial_{x_1}^2\cdot+\partial_{x_2}(a(x_1,x_2)\partial_{x_2}\cdot)$$

is globally hypoelliptic on $\boldsymbol{T}^2$ if and only if the system

$$\dot{x}=\sum_{i=1}^2 \lambda_i X_i \qquad (\lambda_i\in\boldsymbol{R})$$

is weakly controllable on $\boldsymbol{T}^2$, where $X_1=\partial_{x_1}$ and $X_2=a(x)\partial_{x_2}$.

EXAMPLE 4. Let

$$P=\sum_{i,j=1}^2 a^{ij}\partial_{x_i}\partial_{x_j}$$

be a degenerate elliptic operator with real constant coefficients defined on a torus $\boldsymbol{T}^2=\boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$. Assume that the system

$$\dot{x}=\sum_{i=1}^2 \lambda_i X_i \qquad (\lambda_i\in\boldsymbol{R})$$

is weakly controllable on $\boldsymbol{T}^2$, where $X_i=\sum_{j=1}^2 a^{ij}\partial_{x_j}$ $(i=1, 2)$. If $u\in\mathcal{D}'(\boldsymbol{T}^2)$, $Pu\in C^\infty(\boldsymbol{T}^2)$ and if $u\in C^\infty(U)$ for some open set U of $\boldsymbol{T}^2$, then $u\in C^\infty(\boldsymbol{T}^2)$.

1. Preliminaries.

In this section, we shall prove elementary lemmas on nonnegative functions, positive semi-definite quadratic forms and degenerate elliptic-parabolic operators.

LEMMA 1.1. *Let $f(t)\in C_b^2(\boldsymbol{R})$ be a nonnegative function. Then we have*

$$(f'(t))^2 \leqq 2(\sup_{s\in\boldsymbol{R}}|f''(s)|)f(t) \qquad (t\in\boldsymbol{R}). \tag{1.1}$$

PROOF. For any $t\in\boldsymbol{R}$, Taylor's formula gives

$$0\leqq f(t+h)\leqq f(t)+f'(t)h+\frac{C}{2}h^2 \qquad (h\in\boldsymbol{R}),$$

where $C=\sup_{s\in\boldsymbol{R}}|f''(s)|$. Hence, we obtain

$$(f'(t))^2-2Cf(t)\leqq 0 \qquad (t\in\boldsymbol{R}). \quad \blacksquare$$

LEMMA 1.2 ([**14**]). *Let (a^{ij}) be a symmetric positive semi-definite $d\times d$ matrix. Then we have*

$$\Big(\sum_{j=1}^d a^{ij}\xi_j\Big)^2 \leqq a^{ii}\Big(\sum_{k,l=1}^d a^{kl}\xi_k\xi_l\Big) \qquad (\xi\in\boldsymbol{R}^d,\ 1\leqq i\leqq d). \tag{1.2}$$

PROOF. By applying Lemma 1.1 to the nonnegative functions

$$\xi_i \longrightarrow \frac{1}{2}\sum_{k,l=1}^d a^{kl}\xi_k\xi_l \qquad (\xi=(\xi_1,\cdots,\xi_d)\in\boldsymbol{R}^d,\ 1\leqq i\leqq d),$$

we obtain (1.2). $\blacksquare$

LEMMA 1.3 ([**14**]). *Let $a^{ij}(x)\in C_b^2(\boldsymbol{R}^d)$ $(1\leqq i, j\leqq d)$ be real-valued functions such that $a^{ij}(x)=a^{ji}(x)$ $(x\in\boldsymbol{R}^d)$ and*

$$\sum_{i,j=1}^d a^{ij}(x)\xi_i\xi_j\geqq 0 \qquad (x\in\boldsymbol{R}^d,\ \xi\in\boldsymbol{R}^d).$$

Then there is a constant $C\geqq 0$ depending only on $\sup|\partial_x^\alpha a^{ij}|$ $(1\leqq i, j\leqq d,\ 1\leqq|\alpha|\leqq 2)$ *such that*

$$\Big(\sum_{i,j=1}^d a_{x_k}^{ij}(x)u_{x_ix_j}\Big)^2 \leqq C\sum_{l=1}^d\Big(\sum_{i,j=1}^d a^{ij}(x)u_{x_lx_i}u_{x_lx_j}\Big) \tag{1.3}$$

$$(u\in C_b^2(\boldsymbol{R}^d),\ x\in\boldsymbol{R}^d,\ 1\leqq k\leqq d).$$

PROOF. Since the matrix $(a^{ij}(x))$ is positive semi-definite, $a^{ii}(x)\geqq 0$, $a^{ii}(x)+2a^{ij}(x)+a^{jj}(x)\geqq 0$ and $(a^{ij}(x))^2\leqq a^{ii}(x)a^{jj}(x)$. By Lemma 1.1, we have

$$(a^{ii}_{x_k}(x))^2 \leqq 2C_1 a^{ii}(x),$$

$$(a^{ii}_{x_k}(x)+2a^{ij}_{x_k}(x)+a^{jj}_{x_k}(x))^2 \leqq 4C_2(a^{ii}(x)+a^{jj}(x))$$

for $1 \leqq i,\ j,\ k \leqq d$, where

$$C_1 = \sup\{|a^{ii}_{x_k}(x)| : x \in \boldsymbol{R}^d,\ 1 \leqq i,\ k \leqq d\},$$

$$C_2 = \sup\{|a^{ii}_{x_k x_k}(x)+2a^{ij}_{x_k x_k}(x)+a^{jj}_{x_k x_k}(x)| : x \in \boldsymbol{R}^d,\ 1 \leqq i,\ j,\ k \leqq d\}.$$

Hence, we obtain

$$(a^{ij}_{x_k}(x))^2 \leqq \left(\frac{3}{2}C_1+3C_2\right)(a^{ii}(x)+a^{jj}(x));$$

this gives

$$\left(\sum_{i,j=1}^{d} a^{ij}_{x_k}(x)u_{x_i x_j}\right)^2 \leqq 2d^2\left(\frac{3}{2}C_1+3C_2\right)\sum_{l=1}^{d}\left(\sum_{i=1}^{d} a^{ii}(x)u_{x_l x_i}u_{x_l x_i}\right).$$

We now fix a point p of $\boldsymbol{R}^d$ arbitrarily. Since a symmetric matrix is diagonalized by a suitable orthogonal matrix, and since for any orthonormal transformation $(x_1, x_2, \cdots, x_d) \rightarrow (y_1, y_2, \cdots, y_d)$ direct computation gives

$$a^{ij}_{x_k}u_{x_i x_j} = \sum_{n=1}^{d}\frac{\partial y_n}{\partial x_k}\left(\sum_{l,m=1}^{d} a^{ij}_{y_n}\frac{\partial y_l}{\partial x_i}\frac{\partial y_m}{\partial x_j}u_{y_l y_m}\right)$$

and

$$a^{ij}u_{x_k x_i}u_{x_k x_j} = \sum_{n=1}^{d}\left(\sum_{l,m=1}^{d} a^{ij}\frac{\partial y_l}{\partial x_i}\frac{\partial y_m}{\partial x_j}u_{y_n y_l}u_{y_n y_m}\right),$$

we may assume that the matrix $(a^{ij}(p))$ is diagonal. Therefore, it follows that

$$\left(\sum_{i,j=1}^{d} a^{ij}_{x_k}(p)u_{x_i x_j}\right)^2 \leqq C_3\sum_{l=1}^{d}\left(\sum_{i,j=1}^{d} a^{ij}(p)u_{x_l x_i}u_{x_l x_j}\right),$$

where $C_3 \geqq 0$ is a constant independent of the choice of p. ■

REMARK. By applying (1.3) to the function

$$u(x) = \frac{1}{2}\sum_{i,j=1}^{d}\xi_i\eta_j x_i x_j \qquad (x \in \boldsymbol{R}^d,\ \xi,\ \eta \in \boldsymbol{R}^d),$$

we obtain

$$\left(\sum_{i,j=1}^{d} a^{ij}_{x_k}(x)\xi_i\eta_j\right)^2 \leqq C\left(|\xi|\sum_{i,j=1}^{d} a^{ij}(x)\eta_i\eta_j+|\eta|^2\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j\right) \tag{1.4}$$

$$(x \in \boldsymbol{R}^d,\ \xi,\ \eta \in \boldsymbol{R}^d,\ 1 \leqq k \leqq d)$$

(cf. [**16**]). (1.4) shows that $a^{ij}_{x_k}(x)=0$ if $a^{ii}(x)=a^{jj}(x)=0$.

LEMMA 1.4. *For any compact set $K \subset \Omega$ and for any $\sigma \geqq 0$, any multi-index $\beta \neq 0$ and any $\delta > 0$, there is a constant $C \geqq 0$ such that*

$$\|P_{(\beta)}u\|_{-|\beta|} \leqq \delta\|Pu\|_{-\sigma}+C\|u\|_{\sigma} \qquad (u\in C_K^\infty(\Omega)). \tag{1.5}$$

PROOF. Since K is a compact set contained in Ω, we may assume that P is a degenerate elliptic-parabolic operator with real coefficients belonging to the class $C_0^\infty(\boldsymbol{R}^d)$. (1.3) gives

$$\sum_{|\beta|=1}\|P_{(\beta)}u\|_0^2 \leqq C_1\Big\{\sum_{k=1}^d\Big(\sum_{i,j=1}^d\int a^{ij}u_{x_kx_i}u_{x_kx_j}dx\Big)+\|u\|_1^2\Big\} \quad (u\in C_K^\infty(\Omega)).$$

Integrating by parts, we obtain

$$\begin{aligned}&\int a^{ij}u_{x_kx_i}u_{x_kx_j}dx\\ &=\int a^{ij}u_{x_ix_j}u_{x_kx_k}dx+\int a_{x_k}^{ij}u_{x_ix_j}u_{x_k}dx+\frac{1}{2}\int a_{x_ix_j}^{ij}u_{x_k}^2dx\\ &=\int(a^{ij}u_{x_ix_j}+b^iu_{x_i}+cu)u_{x_kx_k}dx+\int(a_{x_k}^{ij}u_{x_ix_j}+b_{x_k}^iu_{x_i}+c_{x_k}u)u_{x_k}dx\\ &\quad+\int\Big(\frac{1}{2}a_{x_ix_j}^{ij}-\frac{1}{2}b_{x_i}^i+c\Big)u_{x_k}^2dx \qquad (u\in C_K^\infty(\Omega)).\end{aligned}$$

Hence, we have

$$\sum_{|\beta|=1}\|P_{(\beta)}u\|_0^2 \leqq C_2\Big\{\|Pu\|_{1-\sigma}\|u\|_{1+\sigma}+\Big(\sum_{|\beta|=1}\|P_{(\beta)}u\|_{-\sigma}\Big)\|u\|_{1+\sigma}+\|u\|_1^2\Big\}$$
$$(u\in C_K^\infty(\Omega));$$

this implies that

$$\sum_{|\beta|=1}\|P_{(\beta)}u\|_0^2 \leqq C_3(\|Pu\|_{1-\sigma}\|u\|_{1+\sigma}+\|u\|_{1+\sigma}^2) \qquad (u\in C_K^\infty(\Omega)). \tag{1.6}$$

Since $[P, \Lambda^{-1}]$, $[P_{(\beta)}, \Lambda^{-1}]\in S^0$, we have

$$\|P_{(\beta)}u\|_{-1}^2 \leqq \|P_{(\beta)}\Lambda^{-1}u\|_0+C_4\|u\|_0 \qquad (u\in C_K^\infty(\Omega)) \tag{1.7}$$

and

$$\|P\Lambda^{-1}u\|_{1-\sigma} \leqq \|Pu\|_{-\sigma}+C_5\|u\|_0 \qquad (u\in C_K^\infty(\Omega)). \tag{1.8}$$

Therefore, it follows from (1.6), (1.7) and (1.8) that

$$\sum_{|\beta|=1}\|P_{(\beta)}u\|_{-|\beta|}^2 \leqq C_6(\|Pu\|_{-\sigma}\|u\|_{\sigma}+\|u\|_{\sigma}^2) \qquad (u\in C_K^\infty(\Omega)). \tag{1.9}$$

It is easy to show that

$$\|P_{(\beta)}u\|_{-|\beta|}^2 \leqq C_7\|u\|_0^2 \qquad (u\in C_K^\infty(\Omega))$$

for $|\beta|\geqq 2$. Thus, (1.5) is proved. ∎

REMARK. Since we may assume $P\in S^2$, and since

$$[P_{(\beta)}, \Lambda^s] \in S^{s+1}, \qquad [P, \Lambda^s] - \sum_{|\beta|=1} C_\beta \Lambda^{s-1} P_{(\beta)} \in S^s$$

for some constants C_β, we can prove, by (1.5), the following fact: For any compact set $K \subset \Omega$ and for any $s \in \boldsymbol{R}$, any $\sigma \geqq 0$, any multi-index $\beta \neq 0$ and any $\delta > 0$, there is a constant $C \geqq 0$ such that

$$\|P_{(\beta)} u\|_{s-|\beta|} \leqq \delta \|Pu\|_{s-\sigma} + C\|u\|_{s+\sigma} \qquad (u \in C_K^\infty(\Omega)). \tag{1.10}$$

LEMMA 1.5. *For any compact set $K \subset \Omega$ and for any $\sigma \geqq 0$, any multi-index $\alpha \neq 0$ and any $\delta > 0$, there is a constant $C \geqq 0$ such that*

$$\|P^{(\alpha)} u\|_0 \leqq \delta \|Pu\|_{-\sigma} + C\|u\|_\sigma \qquad (u \in C_K^\infty(\Omega)). \tag{1.11}$$

PROOF. (1.2) gives

$$\sum_{|\alpha|=1} \|P^{(\alpha)} u\|_0^2 \leqq C_1 \Big(\sum_{i,j=1}^d \int a^{ij} u_{x_i} u_{x_j} dx + \|u\|_0^2 \Big) \qquad (u \in C_K^\infty(\Omega)).$$

Integrating by parts, we obtain

$$\int a^{ij} u_{x_i} u_{x_j} dx = -\int a^{ij} u_{x_i x_j} u dx + \frac{1}{2} \int a^{ij}_{x_i x_j} u^2 dx$$

$$= -\int (a^{ij} u_{x_i x_j} + b^i u_{x_i} + cu) u dx + \int \Big(\frac{1}{2} a^{ij}_{x_i x_j} - \frac{1}{2} b^i_{x_i} + c \Big) u^2 dx$$

$$(u \in C_K^\infty(\Omega)).$$

Hence, we have

$$\sum_{|\alpha|=1} \|P^{(\alpha)} u\|_0^2 \leqq C_2(|(Pu, u)| + \|u\|_0^2) \qquad (u \in C_K^\infty(\Omega)). \tag{1.12}$$

It is easy to show that

$$\|P^{(\alpha)} u\|_0^2 \leqq C_3 \|u\|_0^2 \qquad (u \in C_K^\infty(\Omega))$$

for $|\alpha| \geqq 2$. Therefore, (1.11) is proved. ■

REMARK. Since we may assume $P \in S^2$, and since

$$[P^{(\alpha)}, \Lambda^s] \in S^s \quad (\alpha \neq 0), \qquad [P, \Lambda^s] - \sum_{|\beta|=1} C_\beta \Lambda^{s-1} P_{(\beta)} \in S^s$$

for some constants C_β, we can prove, by (1.10) and (1.11), the following fact: For any compact set $K \subset \Omega$ and for any $s \in \boldsymbol{R}$, any $\sigma \geqq 0$, any multi-index $\alpha \neq 0$ and any $\delta > 0$, there is a constant $C \geqq 0$ such that

$$\|P^{(\alpha)} u\|_s \leqq \delta \|Pu\|_{s-\sigma} + C\|u\|_{s+\sigma} \qquad (u \in C_K^\infty(\Omega)). \tag{1.13}$$

For a real-valued function $\Phi \in C^\infty(\Omega)$, we put

$$U(\Phi, t)=\left\{x\in\Omega:|\Phi(x)|<\frac{1}{t}\right\}\qquad(t>0),$$

$$U(\Phi, \infty)=\{x\in\Omega:|\Phi(x)|=0\}$$

and

$$V(\Phi, t)=\left\{x\in\Omega:\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)^{1/2}(x)+|\Phi(x)|<\frac{1}{t}\right\}\qquad(t>0),$$

$$V(\Phi, \infty)=\left\{x\in\Omega:\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)^{1/2}(x)+|\Phi(x)|=0\right\}.$$

LEMMA 1.6. *Let $\Phi\in C^\infty(\Omega)$ be a real-valued function such that $U(\Phi, \infty)\neq\emptyset$ and $U(\Phi, t_0)\Subset\Omega$ for some $t_0>0$. Then there are constants $C_0>0$ and $C\geqq 0$ independent of $t\geqq t_0$ such that for any $t\geqq t_0$*

$$\begin{aligned}&C_0t^2\inf_{U(\Phi,t)}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)\|u\|_0^2+t\inf_{U(\Phi,t)}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_ix_j}+\sum_{i=1}^{d}b^i\Phi_{x_i}\Big)\|u\|_0^2\\&\leqq C(\|Pu\|_0\|u\|_0+\|u\|_0^2)\qquad\qquad(u\in C_0^\infty(U(\Phi, t))).\end{aligned}\tag{1.14}$$

PROOF. For a real-valued function $u\in C_0^\infty(U(\Phi, t))$, we put

$$v=(T-e^{t\Phi})^{-1}u,\qquad T=\text{const}>e^2.$$

Direct computation gives

$$\begin{aligned}Pu=(T-e^{t\Phi})\Big(\sum_{i,j=1}^{d}a^{ij}v_{x_ix_j}+\sum_{i=1}^{d}b^iv_{x_i}+cv\Big)-e^{t\Phi}\Big\{t^2\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)v\\+t\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_ix_j}+\sum_{i=1}^{d}b^i\Phi_{x_i}\Big)v+2t\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}v_{x_j}\Big\}.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\int(T-e^{t\Phi})^{-1}Pu\cdot v\,dx=&-\int\sum_{i,j=1}^{d}a^{ij}v_{x_i}v_{x_j}dx+\int\Big(\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}_{x_ix_j}-\frac{1}{2}\sum_{i=1}^{d}b^i_{x_i}+c\Big)v^2dx\\&-t^2\int e^{t\Phi}(T-e^{t\Phi})^{-1}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)v^2dx\\&-t\int e^{t\Phi}(T-e^{t\Phi})^{-1}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_ix_j}+\sum_{i=1}^{d}b^i\Phi_{x_i}\Big)v^2dx\\&-2\int\sum_{i,j=1}^{d}a^{ij}\{te^{t\Phi}(T-e^{t\Phi})^{-1}\Phi_{x_i}v\}v_{x_j}dx.\end{aligned}$$

Since

$$\begin{aligned}&\Big|2\int\sum_{i,j=1}^{d}a^{ij}\{te^{t\Phi}(T-e^{t\Phi})^{-1}\Phi_{x_i}v\}v_{x_j}dx\Big|\\&\leqq t^2\int e^{2t\Phi}(T-e^{t\Phi})^{-2}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)v^2dx+\int\sum_{i,j=1}^{d}a^{ij}v_{x_i}v_{x_j}dx\end{aligned}$$

and

$$e^{t\Phi}(T-e^{t\Phi})^{-1}-e^{2t\Phi}(T-e^{t\Phi})^{-2}=e^{t\Phi}(T-2e^{t\Phi})(T-e^{t\Phi})^{-2},$$

we have

$$\begin{aligned}\int(T-e^{t\Phi})^{-2}Pu\cdot udx \leqq & -t^2\int e^{t\Phi}(T-e^{2t\Phi})(T-e^{t\Phi})^{-4}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)u^2dx \\ & -t\int e^{t\Phi}(T-e^{t\Phi})^{-3}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_ix_j}+\sum_{i=1}^{d}b^i\Phi_{x_i}\Big)u^2dx \qquad (1.15) \\ & +\int(T-e^{t\Phi})^{-2}\Big(\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}_{x_ix_j}-\frac{1}{2}\sum_{i=1}^{d}b^i_{x_i}+c\Big)u^2dx.\end{aligned}$$

Combining (1.15) and

$$e^{-1}\leqq e^{t\Phi}\leqq e, \qquad (T-e^{-1})^{-1}\leqq(T-e^{t\Phi})^{-1}\leqq(T-e)^{-1} \quad (x\in U(\Phi, t)),$$

we obtain (1.14). ■

LEMMA 1.7. *Let $\Phi\in C^{\infty}(\Omega)$ be a real-valued function such that $V(\Phi, \infty)\neq\emptyset$ and $V(\Phi, t_0)\Subset\Omega$ for some $t_0>0$. Then there is a constant $C\geqq 0$ independent of $t\geqq t_0$ such that for any $t\geqq t_0$*

$$t\inf_{V(\Phi,t)}\sum_{i=1}^{d}\Big(b^i-\sum_{j=1}^{d}a^{ij}_{x_j}\Big)\Phi_{x_i}\|u\|_0^2\leqq C(\|Pu\|_0\|u\|_0+\|u\|_0^2) \qquad (1.16)$$

$$(u\in C_0^{\infty}(V(\Phi, t)))$$

PROOF. For a real-valued function $u\in C_0^{\infty}(V(\Phi, t))$, we put

$$v=(T-e^{t\Phi})^{-1}u, \qquad T=\text{const}>e.$$

As in the proof of Lemma 1.6, we have

$$\begin{aligned}\int(T-e^{t\Phi})^{-1}Pu\cdot vdx = & -\int\sum_{i,j=1}^{d}a^{ij}v_{x_i}v_{x_j}dx+\int\Big(\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}_{x_ix_j}-\frac{1}{2}\sum_{i=1}^{d}b^i_{x_i}+c\Big)v^2dx \\ & -t^2\int e^{t\Phi}(T-e^{t\Phi})^{-1}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)v^2dx \\ & -t\int e^{t\Phi}(T-e^{t\Phi})^{-1}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_ix_j}+\sum_{i=1}^{d}b^i\Phi_{x_i}\Big)v^2dx \\ & -2t\int e^{t\Phi}(T-e^{t\Phi})^{-1}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}v_{x_j}\Big)vdx.\end{aligned}$$

Since

$$\begin{aligned}& -2\int e^{t\Phi}(T-e^{t\Phi})^{-1}a^{ij}\Phi_{x_i}v_{x_j}vdx \\ & =t\int Te^{t\Phi}(T-e^{t\Phi})^{-2}a^{ij}\Phi_{x_i}\Phi_{x_j}v^2dx+\int e^{t\Phi}(T-e^{t\Phi})^{-2}(a^{ij}\Phi_{x_i})_{x_j}v^2dx\end{aligned}$$

and

$$Te^{t\Phi}(T-e^{t\Phi})^{-2}-e^{t\Phi}(T-e^{t\Phi})^{-1}=e^{2t\Phi}(T-e^{t\Phi})^{-2},$$

we obtain

$$\int(T-e^{t\Phi})^{-2}Pu\cdot udx \leqq t^2\int e^{2t\Phi}(T-e^{t\Phi})^{-4}\Big(\sum_{i,j=1}^{d} a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)u^2dx$$

$$-t\int e^{t\Phi}(T-e^{t\Phi})^{-3}\Big\{\frac{1}{2}\sum_{i=1}^{d}\Big(b^i-\sum_{j=1}^{d}a^{ij}_{x_j}\Big)\Phi_{x_i}\Big\}u^2dx \quad (1.17)$$

$$+\int(T-e^{t\Phi})^{-2}\Big(\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}_{x_ix_j}-\frac{1}{2}\sum_{i=1}^{d}b^i_{x_i}+c\Big)v^2dx.$$

Hence, combining (1.17) and

$$e^{-1}\leqq e^{t\Phi}\leqq e,\quad (T-e^{-1})^{-1}\leqq(T-e^{t\Phi})^{-1}\leqq(T-e)^{-1}\qquad (x\in V(\Phi,t)),$$

$$t^2\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)\leqq 1\qquad (x\in V(\Phi,t)),$$

we have (1.16). ■

LEMMA 1.8 ([**14**]). *Assume that* $\dim \mathrm{Lie}(x)\equiv d$ *in* Ω. *Then for any* $\phi\in C_0^\infty(\Omega)$ *and any* $N>0$, *there is a function* $\psi\in C_0^\infty(\Omega)$ *with* $\phi\Subset\psi$ *and there are constants* $C\geqq 0$ *and* $\kappa>0$ *such that*

$$\|\phi u\|_0\leqq C(\|\psi Pu\|_{-\kappa}+\|\psi u\|_{-N})\qquad (u\in C^\infty(\Omega)). \quad (1.18)$$

REMARK. By applying (1.14), (1.16) and (1.18) to $\chi\Lambda^s u=\chi(x)\langle D_x\rangle^s u$ instead of u, where $\chi(x)$ is a real-valued function satisfying either

$$\chi\in C_0^\infty(U(\Phi,2t)),\ \ \chi\equiv 1\qquad \text{in } U(\Phi,t)$$

or

$$\chi\in C_0^\infty(V(\Psi,2t)),\ \ \chi\equiv 1\qquad \text{in } V(\Psi,t),$$

we obtain the following facts:

1°. On the same assumption of Lemma 1.6, for any $t\geqq t_0$ and for any $s\in\boldsymbol{R}$ and any $N>0$, there are constants C_0, $C_1\geqq 0$ independent of $t\geqq t_0$ and $C=C(t)\geqq 0$ such that

$$C_0t^2\inf_{U(\Phi,t)}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_i}\Phi_{x_j}\Big)\|u\|_s+t\inf_{U(\Phi,t)}\Big(\sum_{i,j=1}^{d}a^{ij}\Phi_{x_ix_j}+\sum_{i=1}^{d}b^i\Phi_{x_i}\Big)\|u\|_s \quad (1.19)$$

$$\leqq C_1(\|Pu\|_s+\|u\|_s)+C\|u\|_{-N}\qquad (u\in C_0^\infty(U(\Phi,t))).$$

2°. On the same assumption of Lemma 1.7, for any $t\geqq t_0$ and for any $s\in\boldsymbol{R}$ and any $N>0$, there are constants $C_1\geqq 0$ independent of $t\geqq t_0$ and $C=C(t)\geqq 0$ such that

$$t \inf_{V(\Phi,t)} \left\{ \sum_{i=1}^{d} \left(b^i - \sum_{j=1}^{d} a^{ij}_{x_j} \right) \Phi_{x_i} \right\} \|u\|_s \tag{1.20}$$
$$\leqq C_1(\|Pu\|_s + \|u\|_s) + C\|u\|_{-N} \qquad (u \in C_0^\infty(V(\Phi, t))).$$

3°. If $\dim \mathrm{Lie}(x) \equiv d$ in Ω, then for any $\phi \in C_0^\infty(\Omega)$ and for any $s \in \boldsymbol{R}$ and any $N>0$, there is a function $\psi \in C_0^\infty(\Omega)$ with $\phi \Subset \psi$ and there are constants $C \geqq 0$ and $\kappa > 0$ such that

$$\|\phi u\|_s \leqq C(\|\psi Pu\|_{s-\kappa} + \|\psi u\|_{-N}) \qquad (u \in C^\infty(\Omega)). \tag{1.21}$$

2. A criterion of global hypoellipticity.

In this section, we shall show a criterion of global hypoellipticity (Proposition 2.1) which is a reformation of theorems given by Oleinik and Radkevich [**14**], Fedii [**3**] and Morimoto [**13**]. $\mathcal{F}$ denotes a set of all functions $f \in C_0^\infty(\Omega)$ such that $0 \leqq f \leqq 1$ in Ω and $f \equiv 1$ in a neighborhood of $S = \{x : \dim \mathrm{Lie}(x) < d\}$ in Ω. $\Lambda_{s,t,\varepsilon}$ is a pseudodifferential operator with symbol $\langle \xi \rangle^s (1 + \varepsilon \langle \xi \rangle)^{-t}$ ($s \in \boldsymbol{R}$, $t > 0$, $\varepsilon > 0$). $\|\cdot\|_{s,t,\varepsilon}$ stands for the norm $\|\cdot\|_{s,t,\varepsilon} = \|\Lambda_{s,t,\varepsilon} \cdot\|_0$. It is easy to show that

$$\|\Lambda^{(\alpha)}_{s,t,\varepsilon} u\|_0 \leqq C_\varepsilon \|\Lambda^{s-t-|\alpha|} u\|_0 \qquad (u \in \mathcal{S})$$

and

$$\|\Lambda^{(\alpha)}_{s,t,\varepsilon} u\|_0 \leqq C \|\Lambda^{s-|\alpha|} u\|_0 \qquad (u \in \mathcal{S}),$$

where $C_\varepsilon \geqq 0$ is a constant depending on ε and $C \geqq 0$ is a constant independent of ε. Unless otherwise specified, we use the same notation as in Section 0.

PROPOSITION 2.1. *Assume that S is a compact set, and assume that for any $\delta > 0$ and for any multi-index β $(1 \leqq |\beta| \leqq 2)$ and any $N > 0$, there is a bounded open neighborhood U of S in Ω and there is a nonnegative constant $C = C(\delta, \beta, N, U)$ such that*

$$\|u\|_0 \leqq C(\|Pu\|_0 + \|u\|_{-N}) \qquad (u \in C_0^\infty(U)) \tag{2.1}$$

and

$$\|P_{(\beta)} u\|_{-|\beta|} \leqq \delta \|Pu\|_0 + C\|u\|_{-N} \qquad (u \in C_0^\infty(U)). \tag{2.2}$$

Then for any $\phi \in \mathcal{F}$ and for any $s \in \boldsymbol{R}$ and any $N > 0$, there is a function $\psi \in \mathcal{F}$ with $\phi \Subset \psi$ and there is a nonnegative constant $C = C(\phi, s, N)$ such that

$$\|\phi u\|_s \leqq C(\|\psi Pu\|_s + \|\psi u\|_{-N}) \qquad (u \in H_{-N}(\Omega)). \tag{2.3}$$

COROLLARY. *Assume that S is a compact set, and assume that for any $\delta > 0$ and for any $N > 0$ there is a bounded open neighborhood U of S in Ω and there is a nonnegative constant $C = C(\delta, N, U)$ such that*

$$\|u\|_0 \leqq \delta \|Pu\|_0 + C\|u\|_{-N} \qquad (u \in C_0^\infty(U)). \tag{2.4}$$

Then for any $\phi\in\mathcal{F}$ *and for any* $s\in\boldsymbol{R}$ *and any* $N>0$, *there is a function* $\psi\in\mathcal{F}$ *with* $\phi\Subset\psi$ *and there is a nonnegative constant* $C=C(\phi, s, N)$ *such that*

$$\|\phi u\|_s\leqq C(\|\psi Pu\|_s+\|\psi u\|_{-N})\qquad(u\in H_{-N}(\Omega)).$$

PROOF OF COROLLARY. (1.5) and (2.4) imply (2.2). Hence, Corollary follows immediately from Proposition 2.1. ■

Before proving Proposition 2.1, we shall show two lemmas.

LEMMA 2.2. *On the same assumption of Proposition* 2.1 *we have the following estimates: For any compact set* $K\subset\Omega$ *and for any* $\phi\in\mathcal{F}$, *any multi-index* $\beta\neq0$, *any* $N>0$ *and any* $\delta>0$, *there are constants* $C_0\geqq0$ *and* $\kappa>0$ *independent of* δ, *and there is a constant* $C\geqq0$ *such that*

$$\|u\|_0\leqq C_0(\|Pu\|_0+\|u\|_{-N})\qquad(u\in C_K^\infty(\Omega)),\tag{2.5}$$

$$\|P_{(\beta)}u\|_{-|\beta|}\leqq\delta\|Pu\|_0+C\|u\|_{-N}\qquad(u\in C_K^\infty(\Omega))\tag{2.6}$$

and

$$\|[P, \phi]u\|_0\leqq C_0(\|Pu\|_{-\kappa}+\|u\|_{-N})\qquad(u\in C_K^\infty(\Omega)).\tag{2.7}$$

PROOF. Unless otherwise specified, each C_k^0 denotes a nonnegative constant independent of δ, and each C_k denotes a nonnegative constant depending on δ.

Since we may assume that $P\in S^2$ and since

$$[P, \phi]=\sum_{|\alpha|=1}C_\alpha\phi_{(\alpha)}P^{(\alpha)}+\sum_{|\alpha|=2}C_\alpha\phi_{(\alpha)}\tag{2.8}$$

$$=\sum_{|\alpha|=1}C'_\alpha P^{(\alpha)}\phi_{(\alpha)}+\sum_{|\alpha|=2}C'_\alpha\phi_{(\alpha)},\tag{2.9}$$

where C_α, C'_α are constants, we have, by (2.9), (1.12) and (2.8),

$$\|[P, \phi]u\|_0\leqq C_1^0(\|Pu\|_{-\sigma}+\|u\|_{-\sigma}+\sum_{|\alpha|=1}\|\phi_{(\alpha)}u\|_\sigma+\sum_{|\alpha|=2}\|\phi_{(\alpha)}u\|_0)\quad(u\in C_K^\infty(\Omega))$$

for any $\sigma\geqq0$. By (1.21), there is a constant $\kappa_1>0$ independent of $\delta>0$ such that for any $\phi\in\mathcal{F}$

$$\sum_{1\leqq|\alpha|\leqq2}\|\phi_{(\alpha)}u\|_{\kappa_1}\leqq C_2^0(\|Pu\|_{-\kappa_1}+\|u\|_{-N})\qquad(u\in C_K^\infty(\Omega)).$$

Hence, we obtain

$$\|[P, \phi]u\|_0\leqq C_3^0(\|Pu\|_{-\kappa_1}+\|u\|_{-\kappa_1})\qquad(u\in C_K^\infty(\Omega))\tag{2.10}$$

for any $\phi\in\mathcal{F}$. It follows from (2.2) that there is a function $\phi_1\in\mathcal{F}$ such that

$$\|P_{(\beta)}\phi_1u\|_{-|\beta|}\leqq\frac{\delta}{3}\|P\phi_1u\|_0+C_4\|\phi_1u\|_{-N}\qquad(u\in C_K^\infty(\Omega))$$

for $1\leqq|\beta|\leqq2$. Let $\phi_2\in C_0^\infty(\Omega)$ be a function satisfying $0\leqq\phi_2\leqq1$ in Ω and $\phi_1+\phi_2=1$ in K. Then we have, by (1.5),

$$\|P_{(\beta)}\phi_2u\|_{-|\beta|}\leqq C_3^0(\|P\phi_2u\|_{-\sigma}+\|\phi_2u\|_\sigma)\qquad(u\in C_K^\infty(\Omega))$$

for any $\beta\neq 0$ and any $\sigma\geqq 0$. By (1.21), there is a constant $\kappa_2>0$ independent of $\delta>0$ such that

$$\|\psi_2 u\|_{\kappa_2}\leqq C_6^0(\|Pu\|_{-\kappa_2}+\|u\|_{-N})\qquad (u\in C_K^\infty(\Omega)).$$

Hence, by (1.12), (2.10) and the above three inequalities, we have

$$\|P_{(\beta)}u\|_{-|\beta|}\leqq\frac{\delta}{2}\|Pu\|_0+C_7\|u\|_{-\min(\kappa_1,\kappa_2,N)}\qquad (u\in C_K^\infty(\Omega))\tag{2.11}$$

for any $\beta\neq 0$. It follows from (2.1) that there is a function $\psi_1\in\mathcal{F}$ such that

$$\|\psi_1 u\|_0\leqq C_8^0(\|P\psi_1 u\|_0+\|\psi_1 u\|_{-N})\qquad (u\in C_K^\infty(\Omega)).$$

Let $\psi_2\in C_0^\infty(\Omega)$ be a function satisfying $0\leqq\psi_2\leqq 1$ in Ω and $\psi_1+\psi_2=1$ in K. Then we have, by (1.18),

$$\|\psi_2 u\|_0\leqq C_9^0(\|Pu\|_0+\|u\|_{-N})\qquad (u\in C_K^\infty(\Omega)).$$

Hence, by (2.10),

$$\|u\|_0\leqq C_{10}^0(\|Pu\|_0+\|u\|_{-\min(\kappa_3,N)})\qquad (u\in C_K^\infty(\Omega)),\tag{2.12}$$

where $\kappa_3>0$ is a constant independent of δ. Combining (2.11) and (2.12), we obtain (2.5) and (2.6).

Since (2.6) gives

$$\|P_{(\beta)}u\|_{s-|\beta|}\leqq\delta\|Pu\|_s+\delta\|[P,\Lambda^s]u\|_0+\|[P_{(\beta)},\Lambda^s]u\|_{-|\beta|}+C_{11}\|u\|_{-N}\qquad (u\in C_K^\infty(\Omega)),$$

we can prove the following estimates: For any $\delta>0$ and for any $r\,(1\leqq r\leqq s+2+N)$ and any $s\in\boldsymbol{R}$

$$\sum_{r\leq|\beta|\leq s+2+N}\|P_{(\beta)}u\|_{s-|\beta|}\leqq\delta\|Pu\|_s+C_{12}^0\Big(\delta\sum_{1\leq|\beta|\leq s+2+N}\|P_{(\beta)}u\|_{s-|\beta|}+\sum_{r+1\leq|\beta|\leq s+2+N}\|P_{(\beta)}u\|_{s-|\beta|}\Big)+C_{12}\|u\|_{-N}\qquad (u\in C_K^\infty(\Omega));$$

this implies that

$$\|P_{(\beta)}u\|_{s-|\beta|}\leqq\delta\|Pu\|_s+C_{13}\|u\|_{-N}\qquad (u\in C_K^\infty(\Omega))\tag{2.13}$$

for any $\beta\neq 0$. On the other hand, since

$$\|u\|_s\leqq C_{14}^0(\|Pu\|_s+\|[P,\Lambda^s]u\|_0+\|u\|_{-N})\qquad (u\in C_K^\infty(\Omega)),$$

by (2.5), we obtain

$$\|u\|_s\leqq C_{14}^0\|Pu\|_s+C_{15}^0\Big(\sum_{1\leq|\beta|\leq s+2+N}\|P_{(\beta)}u\|_{s-|\beta|}+\|u\|_{-N}\Big)\quad (u\in C_K^\infty(\Omega))\tag{2.14}$$

for any $s\in\boldsymbol{R}$. It follows from (2.13) and (2.14) that

$$\|u\|_s \leqq C_{16}^0(\|Pu\|_s+\|u\|_{-N}) \qquad (u\in C_K^\infty(\Omega)). \tag{2.15}$$

Combining (2.10) with (2.15), we have (2.7). ■

LEMMA 2.3. *On the same assumption of Proposition* 2.1 *we have the following estimates: For any compact set $K\subset\Omega$ and for any $\phi\in\mathcal{S}$, any multi-index $\beta\neq 0$, any s, $t\in\boldsymbol{R}$, any $N>0$, any ε $(0<\varepsilon<1)$ and any $\delta>0$, there are constants $C_0\geqq 0$ and $\kappa>0$ independent of ε and δ, and there is a constant $C\geqq 0$ independent of ε such that*

$$\|u\|_{s,t,\varepsilon} \leqq C_0(\|Pu\|_{s,t,\varepsilon}+\|u\|_{-N}) \qquad (u\in C_K^\infty(\Omega)), \tag{2.16}$$

$$\|P_{(\beta)}u\|_{s-|\beta|,t,\varepsilon} \leqq \delta\|Pu\|_{s,t,\varepsilon}+C\|u\|_{-N} \qquad (u\in C_K^\infty(\Omega)) \tag{2.17}$$

and

$$\|[P,\phi]u\|_{s,t,\varepsilon} \leqq C_0(\|Pu\|_{s-\kappa,t,\varepsilon}+\|u\|_{-N}) \qquad (u\in C_K^\infty(\Omega)). \tag{2.18}$$

PROOF. Unless otherwise specified, each C_k^0 denotes a nonnegative constant independent of ε and δ, and each C_k denotes a nonnegative constant independent of ε.

By (1.11), we have

$$\begin{aligned}\|[P,\phi]u\|_{s,t,\varepsilon} &\leqq C_1^0\sum_{|\alpha|=1}(\|P^{(\alpha)}\Lambda_{s,t,\varepsilon}\phi_{(\alpha)}u\|_0+\|[P^{(\alpha)},\Lambda_{s,t,\varepsilon}]\phi_{(\alpha)}u\|_0)\\ &\quad +C_2^0\sum_{|\alpha|=2}\|\phi_{(\alpha)}u\|_{s,t,\varepsilon}\\ &\leqq C_3^0\sum_{|\alpha|=1}(\|P\Lambda_{s,t,\varepsilon}\phi_{(\alpha)}u\|_{-\sigma}+\|\phi_{(\alpha)}u\|_{s+\sigma,t,\varepsilon}+\|u\|_{-N})\\ &\quad +C_2^0\sum_{|\alpha|=2}\|\phi_{(\alpha)}u\|_{s,t,\varepsilon} \qquad (u\in C_K^\infty(\Omega))\end{aligned}$$

for any $\sigma\geqq 0$. Direct computation gives

$$\begin{aligned}\|P\Lambda_{s,t,\varepsilon}\phi_{(\alpha)}u\|_{-\sigma} &\leqq \|\phi_{(\alpha)}\Lambda_{s,t,\varepsilon}Pu\|_{-\sigma}+\|\phi_{(\alpha)}[P,\Lambda_{s,t,\varepsilon}]u\|_{-\sigma}\\ &\quad +\|\Lambda_{s,t,\varepsilon}[P,\phi_{(\alpha)}]u\|_{-\sigma}+\|[\Lambda_{s,t,\varepsilon},[P,\phi_{(\alpha)}]]u\|_{-\sigma}\\ &\quad +\|[\Lambda_{s,t,\varepsilon},\phi_{(\alpha)}]Pu\|_{-\sigma}+\|[P,[\Lambda_{s,t,\varepsilon},\phi_{(\alpha)}]]u\|_{-\sigma}\\ &\leqq C_4^0(\|Pu\|_{s-\sigma,t,\varepsilon}+\sum_{|\alpha|=1}\|P^{(\alpha)}u\|_{s-\sigma,t,\varepsilon}\\ &\quad +\sum_{|\beta|=1}\|P_{(\beta)}u\|_{s-\sigma-|\beta|,t,\varepsilon}+\|u\|_{s-\sigma,t,\varepsilon}) \quad (u\in C_K^\infty(\Omega)),\end{aligned}$$

and (1.11) gives

$$\begin{aligned}\|P^{(\alpha)}u\|_{s-\sigma,t,\varepsilon} &\leqq C_5^0(\|Pu\|_{s-\sigma,t,\varepsilon}+\sum_{|\beta|=1}\|P_{(\beta)}u\|_{s-\sigma-|\beta|,t,\varepsilon}\\ &\quad +\|u\|_{s-\sigma,t,\varepsilon}+\|u\|_{-N}) \qquad (u\in C_K^\infty(\Omega)).\end{aligned}$$

Therefore, we obtain

$$\|[P, \phi]u\|_{s,t,\varepsilon} \leqq C_6^0(\|Pu\|_{s-\sigma,t,\varepsilon} + \sum_{|\beta|=1} \|P_{(\beta)}u\|_{s-\sigma-|\beta|,t,\varepsilon} + \|u\|_{s-\sigma,t,\varepsilon}$$

$$+ \|u\|_{-N} + \sum_{|\alpha|=1} \|\phi_{(\alpha)}u\|_{s+\sigma,t,\varepsilon} + \sum_{|\alpha|=2} \|\phi_{(\alpha)}u\|_{s,t,\varepsilon}) \tag{2.19}$$

$$(u \in C_K^\infty(\Omega))$$

for any $\sigma \geqq 0$. Since, by (2.6),

$$\|P_{(\beta)}u\|_{s-|\beta|,t,\varepsilon} \leqq \delta\|Pu\|_{s,t,\varepsilon} + \delta\|[P, \Lambda_{s,t,\varepsilon}]u\|_0$$

$$+ \|[P_{(\beta)}, \Lambda_{s,t,\varepsilon}]u\|_{-|\beta|} + C_7\|u\|_{-N} \qquad (u \in C_K^\infty(\Omega)),$$

we can prove the following fact: For any $\delta > 0$ and for any r ($1 \leqq r \leqq s+2+N$)

$$\sum_{r \leq |\beta| \leq s+2+N} \|P_{(\beta)}u\|_{s-|\beta|,t,\varepsilon} \leqq \delta\|Pu\|_{s,t,\varepsilon} + C_8^0\Big(\delta \sum_{1 \leq |\beta| \leq s+2+N} \|P_{(\beta)}u\|_{s-|\beta|,t,\varepsilon}$$

$$+ \sum_{r+1 \leq |\beta| \leq s+2+N} \|P_{(\beta)}u\|_{s-|\beta|,t,\varepsilon}\Big) + C_8\|u\|_{-N}$$

$$(u \in C_K^\infty(\Omega));$$

this implies that

$$\|P_{(\beta)}u\|_{s-|\beta|,t,\varepsilon} \leqq \delta\|Pu\|_{s,t,\varepsilon} + C_9\|u\|_{-N} \qquad (u \in C_K^\infty(\Omega)) \tag{2.20}$$

for any $\beta \neq 0$. It follows from (2.5) that

$$\|u\|_{s,t,\varepsilon} \leqq \|Pu\|_{s,t,\varepsilon} + C_9^0\Big(\sum_{1 \leq |\beta| \leq s+2+N} \|P_{(\beta)}u\|_{s-|\beta|,t,\varepsilon} + \|u\|_{-N}\Big) \tag{2.21}$$

$$(u \in C_K^\infty(\Omega)).$$

(1.21) shows that there is a constant κ_1 ($0 < \kappa_1 < 1/2$) independent of ε and δ such that

$$\|\phi_{(\alpha)}u\|_{s+\kappa_1,t,\varepsilon} \leqq C_{10}^0(\|\phi_{(\alpha)}\Lambda_{0,t,\varepsilon}u\|_{s+\kappa_1} + \|u\|_{s+\kappa_1-1,t,\varepsilon})$$

$$\leqq C_{11}^0(\|P\Lambda_{0,t,\varepsilon}u\|_{s-\kappa_1} + \|u\|_{s+\kappa_1-1,t,\varepsilon})$$

$$\leqq C_{12}^0(\|Pu\|_{s-\kappa_1,t,\varepsilon} + \sum_{|\beta|=1} \|P_{(\beta)}u\|_{s-\kappa_1+|\beta|,t,\varepsilon}$$

$$+ \|u\|_{s-\kappa_1,t,\varepsilon} + \|u\|_{s+\kappa_1-1,t,\varepsilon}) \qquad (u \in C_K^\infty(\Omega))$$

for any $\alpha \neq 0$; this gives

$$\sum_{1 \leq |\alpha| \leq 2} \|\phi_{(\alpha)}u\|_{s+\kappa_1}$$

$$\leqq C_{13}^0\Big(\|Pu\|_{s-\kappa_1,t,\varepsilon} + \sum_{|\beta|=1} \|P_{(\beta)}u\|_{s-\kappa_1-|\beta|,t,\varepsilon} + \|u\|_{s-\kappa_1,t,\varepsilon}\Big) \quad (u \in C_K^\infty(\Omega)). \tag{2.22}$$

Combining (2.19), (2.20), (2.21) and (2.22), we obtain (2.16), (2.17) and (2.18). ■

PROOF OF PROPOSITION 2.1. Let $\psi \in \mathcal{F}$ be a function satisfying $\phi \Subset \psi$ and let $\{\psi_j\}_{j=0}^{J}$ be a finite sequence of functions belonging to the class $\mathcal{F}$ such that $s+2-J\kappa<-N$ and $\phi=\psi_0\Subset\psi_1\Subset\cdots\Subset\psi_J=\psi$. Since $C_0^\infty(\Omega)$ is dense in $H_{-N}(\Omega)$, we have, by Lemma 2.3,

$$\|\psi_0 u\|_{s,s+2+N,\varepsilon}\leqq C_1(\|P\psi_0 u\|_{s,s+2+N,\varepsilon}+\|u\|_{-N}),$$

$$\|P\psi_0 u\|_{s,s+2+N,\varepsilon}\leqq C_{2,0}(\|\psi_0 Pu\|_{s,s+2+N,\varepsilon}+\|[P,\psi_0]\psi_1 u\|_{s,s+2+N,\varepsilon})$$

$$\leqq C_{3,0}(\|\psi_0 Pu\|_{s,s+2+N,\varepsilon}+\|P\psi_1 u\|_{s-\kappa,s+2+N,\varepsilon})$$

and

$$\|P\psi_j u\|_{s-j\kappa,s+2+N,\varepsilon}\leqq C_{2,j}(\|\psi_j Pu\|_{s-j\kappa,s+2+N,\varepsilon}+\|[P,\psi_j]\psi_{j+1}u\|_{s-j\kappa,s+2+N,\varepsilon})$$

$$\leqq C_{3,j}(\|\psi_j Pu\|_{s-j\kappa,s+2+N,\varepsilon}+\|P\psi_{j+1}u\|_{s-(j+1)\kappa,s+2+N,\varepsilon}),$$

where C_1, $C_{2,j}$, $C_{3,j}\geqq 0$ and $\kappa>0$ are constants independent of ε. Hence we have

$$\|\phi u\|_{s,s+2+N,\varepsilon}\leqq C_4(\|\psi Pu\|_{s,s+2+N,\varepsilon}+\|\psi u\|_{-N}),$$

where $C_4\geqq 0$ is a constant independent of ε. Letting $\varepsilon\to 0$, we obtain the desired estimate. ■

REMARK. Modifying the proof of Proposition 2.1, we can prove the following fact: Let P^δ $(0<\delta\leqq 1)$ be a degenerate elliptic-parabolic operator of the form

$$P^\delta=\sum_{i,j=1}^{d}a_\delta^{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_\delta^i(x)\frac{\partial}{\partial x_i}+c_\delta(x)$$

with real coefficients satisfying a_δ^{ij}, b_δ^i, $c_\delta\in C^\infty(\Omega)$ and

$$\sup_{0<\delta\leq 1}\sup_{x\in U_\delta}\max_{i,j}\{|\partial_x^\alpha a_\delta^{ij}(x)|,\ |\partial_x^\beta b_\delta^i(x)|,\ |\partial_x^\gamma c_\delta(x)|\}<\infty$$

for any multi-indices α, β, γ. Here U_δ is an open neighborhood of S_δ in Ω. We define a subset S_δ of Ω by

$$S_\delta=\{x\in\Omega:\dim\mathrm{Lie}(X_0^\delta, X_1^\delta,\cdots,X_d^\delta)(x)<d\},$$

where $X_0^\delta=\sum_{i=1}^d(b_\delta^i-\sum_{j=1}^d\partial_{x_j}a_\delta^{ij})\partial_{x_i}$ and $X_i^\delta=\sum_{j=1}^d a_\delta^{ij}\partial_{x_j}$ $(1\leqq i\leqq d)$. Assume that each S_δ is a compact set and $\overline{\bigcup_{0<\delta\leq 1}S_\delta}\subset\Omega$, and assume that for any $\delta>0$ and for any multi-index β $(1\leqq|\beta|\leqq 2)$ and any $N>0$, there is a nonnegative constant C_0 independent of δ and $C(\delta)=C(\delta,\beta,N,U_\delta)$ such that

$$\|u\|_0\leqq C_0\|P^\delta u\|_0+C(\delta)\|u\|_{-N}\qquad(u\in C_0^\infty(U_\delta))\tag{2.23}$$

and

$$\|P_{(\beta)}^\delta u\|_{-|\beta|}\leqq\delta\|P^\delta u\|_0+C(\delta)\|u\|_{-N}\qquad(u\in C_0^\infty(U_\delta)).\tag{2.24}$$

Then for any $\phi\in\mathcal{F}$ and for any $s\in\boldsymbol{R}$ and any $N>0$, there is a number $\delta>0$, there is a function $\psi\in\mathcal{F}$ with $\phi\Subset\psi$ and there is a nonnegative constant $C=C(\phi,s,N)$

such that

$$\|\phi u\|_s \leqq C(\|\phi P^\delta u\|_s + \|\phi u\|_{-N}) \qquad (u\in H_{-N}(\Omega)). \tag{2.25}$$

We can replace (2.23) and (2.24) by

$$\|u\|_0 \leqq \delta\|P^\delta u\|_0 + C(\delta)\|u\|_{-N} \qquad (u\in C_0^\infty(U_\delta)). \tag{2.26}$$

3. Proof of Theorems 1, 2 and 3.

In this section, we shall prove Theorems 1, 2 and 3. It is to be noted that, by virture of Proposition 2.1, we have only to show either (2.1)-(2.2) or (2.4).

PROOF OF THEOREM 1. *Step* 1. We put

$$\Phi^{(0)} = \Phi, \qquad \Phi^{(k)} = X^{(k)}\cdots X^{(1)}\Phi \qquad (k\geqq 1)$$

and

$$S_0 = S, \qquad S_k = \emptyset \qquad (k>2N),$$

$$S_{2k+1} = \left\{x\in S_{2k}:\ \sum_{i=1}^d |X_i\Phi^{(k)}(x)| = 0\right\},$$

$$S_{2k+2} = \{x\in S_{2k+1}:\ \Phi^{(k+1)}(x)=0\} \qquad (0\leqq k\leqq N-1)$$

and further, we put

$$S^{(0)} = S_0\diagdown S_1, \qquad S^{(k)} = S_k\diagdown S_{k+1} \qquad (0\leqq k\leqq 2N).$$

It is easy to show, by (1.2), that

$$\Phi^{(k)} = 0, \qquad \sum_{i,j=1}^d a^{ij}\Phi^{(k)}_{x_i}\Phi^{(k)}_{x_j} > 0 \qquad \text{in } S^{(2k)}$$

and

$$\sum_{i,j=1}^d a^{ij}\Phi^{(k)}_{x_i}\Phi^{(k)}_{x_j} = 0, \quad X_0\Phi^{(k)} \neq 0 \qquad \text{in } S^{(2k+1)}.$$

Let $\{U^{(k)}\}_{k=0}^{2N}$ and $\{U^{(k)}(t_k)\}_{k=0}^{2N}$ be families of open sets such that

$$S^{(k)} \Subset U^{(k)} \Subset \Omega$$

and

$$U^{(2k)}(t_{2k}) = \left\{x\in U^{(2k)}:\ |\Phi^{(k)}(x)| < t_{2k}^{-1},\ \Big(\sum_{i,j=1}^d a^{ij}\Phi^{(k)}_{x_i}\Phi^{(k)}_{x_j}\Big)^{1/2}(x) > t_{2k}^{-\gamma}\right\}, \tag{3.1}$$

$$\begin{aligned} U^{(2k+1)}(t_{2k+1}) = \Big\{x\in U^{(2k+1)}:\ &\Big(\sum_{i,j=1}^d a^{ij}\Phi^{(k)}_{x_i}\Phi^{(k)}_{x_j}\Big)^{1/2}(x) + |\Phi^{(k)}(x)| < t_{2k+1}^{-1}, \\ &|X_0\Phi^{(k)}(x)| > t_{2k+1}^{-\gamma}\Big\}, \end{aligned} \tag{3.2}$$

where $t_k\geqq 1$ are parameters and $\gamma>0$ is a sufficiently small constant. Here

$U^{(k)}=U^{(k)}(t_k)=\emptyset$ if $S^{(k)}=\emptyset$. It is easy to show, by (0.1), (3.1) and (3.2), that

$$S\subset\bigcup_{k=0}^{2N}U^{(k)}(t_k) \tag{3.3}$$

for all sufficiently large t_k. Then there is a family of C^∞ functions $\{\phi_{k,t_k}\}_{k=0}^{2N}$ satisfying the following conditions:

$$0\leqq\phi_{k,t_k}\leqq 1 \quad \text{in } \Omega, \qquad \operatorname{supp}\phi_{k,t_k}\subset U^{(k)}(t_k),$$

$$\phi_{k,t_k}\equiv 1 \quad \text{in } U^{(k)}(t_k)\setminus\{x\in U^{(k)}(t_k): \operatorname{dist}(x, \partial U^{(k)}(t_k))<t_k^{-A}\} \tag{3.4}$$

and

$$\sup\{t_k^{-B}|\partial_x^\alpha\phi_{k,t_k}(x)| : x\in U^{(k)}(t_k),\ t_k\geqq 1\}<+\infty \qquad (1\leqq|\alpha|\leqq 3), \tag{3.5}$$

where A and B are sufficiently large positive constants independent of $x\in U^{(k)}$, $t_k\geqq 1$ and $k=0,\cdots,2N$. Here $\phi_{k,t_k}\equiv 0$ if $U^{(k)}(t_k)=\emptyset$. In fact, we have only to consider functions $f(t)\in C_0^\infty(\boldsymbol{R})$ and $\chi\in C_0^\infty(\boldsymbol{R}^d)$ such that $0\leqq f(t)\leqq 1$ $(t\in\boldsymbol{R})$, $f(t)=0$ $(|t|\leqq 1/3)$, $f(t)=1$ $(|t|\geqq 2/3)$, $0\leqq\chi(x)\leqq 1$ $(x\in\boldsymbol{R}^d)$, $\operatorname{supp}\chi\subset\{x:|x|\leqq 1\}$ and $\int\chi(x)dx=1$, and put

$$\phi_{k,t_k}(x)=\int\chi_{\varepsilon_k}(x-y)f(t_k^A\operatorname{dist}(y, \partial U^{(k)}(t_k)))dy$$

$$=\int\chi_{\varepsilon_k}(y)f(t_k^A\operatorname{dist}(x-y, \partial U^{(k)}(t_k)))dy,$$

where $\chi_{\varepsilon_k}(x)=\varepsilon_k^{-d}\chi(\varepsilon_k^{-1}x)$ and $\varepsilon_k=3^{-1}t_k^{-A}$.

Step 2. Modifying the proof of Lemma 1.6, we have

$$t_{2k}^2\inf_{U^{(2k)}(t_{2k})}\Big(\sum_{i,j=1}^d a^{ij}\Phi_{x_i}^{(k)}\Phi_{x_j}^{(k)}\Big)\int e^{t_{2k}\Phi^{(k)}}(T-e^{t_{2k}\Phi^{(k)}})^{-4}u^2dx$$

$$+t_{2k}\inf_{U^{(2k)}(t_{2k})}\Big(\sum_{i,j=1}^d a^{ij}\Phi_{x_ix_j}^{(k)}+\sum_{i=1}^d b^i\Phi_{x_i}^{(k)}\Big)\int e^{t_{2k}\Phi^{(k)}}(T-e^{t_{2k}\Phi^{(k)}})^{-3}u^2dx$$

$$\leqq-\int(T-e^{t_{2k}\Phi^{(k)}})^{-2}Pu\cdot udx$$

$$+\sup_{U^{(2k)}(t_{2k})}\Big|\frac{1}{2}\sum_{i,j=1}^d a_{x_ix_j}^{ij}-\frac{1}{2}\sum_{i=1}^d b_{x_i}^i+c\Big|\int(T-e^{t_{2k}\Phi^{(k)}})^{-2}u^2dx$$

$$(u\in C_0^\infty(U^{(2k)}(t_{2k})));$$

this implies, by (3.1),

$$(t_{2k}^{2(1-\gamma)}-t_{2k}C_1-C_1)\|u\|_0\leqq C_1\|Pu\|_0 \qquad (u\in C_0^\infty(U^{(2k)}(t_{2k}))), \tag{3.6}$$

where $C_1\geqq 0$ is a constant independent of $t_{2k}\geqq 1$. Modifying the proof of Lemma 1.7, we obtain

$$t_{2k+1} \inf_{U^{(2k+1)}(t_{2k+1})} \Big\{\sum_{i=1}^{d}(b^i - \sum_{j=1}^{d} a^{ij}_{x_j})\Phi^{(k)}_{x_i}\Big\} \int e^{t_{2k+1}\Phi^{(k)}}(T-e^{t_{2k+1}\Phi^{(k)}})^{-3}u^2 dx$$

$$\leqq -\int (T-e^{t_{2k+1}\Phi^{(k)}})^{-2} Pu\cdot u dx$$

$$+t^2_{2k+1} \sup_{U^{(2k+1)}(t_{2k+1})} \Big(\sum_{i,j=1}^{d} a^{ij}\Phi^{(k)}_{x_i}\Phi^{(k)}_{x_j}\Big) \int e^{2t_{2k+1}\Phi^{(k)}}(T-e^{t_{2k+1}\Phi^{(k)}})^{-4}u^2 dx$$

$$+ \sup_{U^{(2k+1)}(t_{2k+1})} \Big|\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}_{x_i x_j} - \frac{1}{2}\sum_{i=1}^{d} b^i_{x_i} + c\Big| \int (T-e^{t_{2k+1}\Phi^{(k)}})^{-2}u^2 dx$$

$$(u \in C_0^\infty(U^{(2k+1)}(t_{2k+1})))\,;$$

this gives, by (3.2),

$$(t^{1-\gamma}_{2k+1} - C_2)\|u\|_0 \leqq C_2\|Pu\|_0 \qquad (u \in C_0^\infty(U^{(2k+1)}(t_{2k+1})))\,, \tag{3.7}$$

where $C_2 \geqq 0$ is a constant independent of $t_{2k+1} \geqq 1$. Here, it follows from (3.6) and (3.7) that

$$t_k^{1-\gamma}\|u\|_0 \leqq C_3\|Pu\|_0 \qquad (u \in C_0^\infty(U^{(k)}(t_k))) \tag{3.8}$$

for all sufficiently large $t_k \geqq 1$ ($k=0, \cdots, 2N$), where $C_3 \geqq 0$ is a constant independent of t_k.

Step 3. By (3.8), we obtain

$$s_k^{1-\gamma}\|\phi_{k,s_k}\phi_{k\ t_k} u\|_0$$

$$\leqq C_3\Big[\|\phi_{k,s_k}\phi_{k,t_k}Pu\|_0 + \Big\|\Big\{\sum_{i,j=1}^{d} a^{ij}(\phi_{k,s_k}\phi_{k,t_k})_{x_i x_j} + \sum_{i=1}^{d} b^i(\phi_{k,s_k}\phi_{k,t_k})_{x_i}\Big\}u\Big\|_0$$

$$+\Big\|\Big\{\sum_{n,m=1}^{d} a^{nm}(\phi_{k,s_k}\phi_{k,t_k})_{x_n}(\phi_{k,s_k}\phi_{k,t_k})_{x_m}\Big\}\Big\{\sum_{i,j=1}^{d} a^{ij}u_{x_i}u_{x_j}\Big\}\Big\|^{1/2}_{L_1(\Omega)}\Big]$$

$$(u \in C_K^\infty(\Omega))\,,$$

where K is a compact set of Ω. Here we used the inequality

$$\Big\{\sum_{i,j=1}^{d} a^{ij}(\phi_{k,s_k}\phi_{k,t_k})_{x_i}u_{x_j}\Big\}^2$$

$$\leqq \sum_{n,m=1}^{d} a^{nm}(\phi_{k,s_k}\phi_{k,t_k})_{x_n}(\phi_{k,s_k}\phi_{k,t_k})_{x_m}\cdot \sum_{i,j=1}^{d} a^{ij}u_{x_i}u_{x_j}\,.$$

Integrating by parts, we have

$$\int \tilde{a}^{ij}u_{x_i}u_{x_j}dx = -\int(\tilde{a}^{ij}u_{x_i x_j} + \tilde{b}^i u_{x_i} + \tilde{c}u)u dx + \int\Big(\frac{1}{2}\tilde{a}^{ij}_{x_i x_j} - \frac{1}{2}\tilde{b}^i_{x_i} + \tilde{c}\Big)u^2 dx\,,$$

where

$$\tilde{a}^{ij} = \left\{\sum_{n,m=1}^{d} a^{nm}(\phi_{k,s_k}\phi_{k,t_k})_{x_n}(\phi_{k,s_k}\phi_{k,t_k})_{x_m}\right\} a^{ij},$$

$$\tilde{b}^{i} = \left\{\sum_{n,m=1}^{d} a^{nm}(\phi_{k,s_k}\phi_{k,t_k})_{x_n}(\phi_{k,s_k}\phi_{k,t_k})_{x_m}\right\} b^{i},$$

$$\tilde{c} = \left\{\sum_{n,m=1}^{d} a^{nm}(\phi_{k,s_k}\phi_{k,t_k})_{x_n}(\phi_{k,s_k}\phi_{k,t_k})_{x_m}\right\} c;$$

this gives

$$\left\|\left\{\sum_{n,m=1}^{d} a^{nm}(\phi_{k,s_k}\phi_{k,t_k})_{x_n}(\phi_{k,s_k}\phi_{k,t_k})_{x_m}\right\}\left\{\sum_{i,j=1}^{d} a^{ij}u_{x_i}u_{x_j}\right\}\right\|_{L_1(\Omega)}$$

$$\leqq C_4\Big[\|Pu\|_0^2+\left\|\left\{\sum_{n,m=1}^{d} a^{nm}(\phi_{k,s_k}\phi_{k,t_k})_{x_n}(\phi_{k,s_k}\phi_{k,t_k})_{x_m}\right\}u\right\|_0^2$$

$$+\left\|\left(\frac{1}{2}\sum_{i,j=1}^{d}\tilde{a}^{ij}_{x_ix_j}-\frac{1}{2}\sum_{i=1}^{d}\tilde{b}^{i}_{x_i}+\tilde{c}\right)u\right\|_0^2\Big] \qquad (u\in C_K^\infty(\Omega)),$$

where $C_4\geqq 0$ is a constant independent of s_k and t_k. (3.5) shows that

$$\|\phi_{k,s_k}(\partial_x^\alpha\phi_{k,t_k})u\|_0 \leqq t_k^B C_5\|u\|_0 \qquad (1\leqq|\alpha|\leqq 3),$$

where $C_5\geqq 0$ is a constant independent of s_k and t_k. Hence, we obtain

$$s_k^{1-\gamma}\|\phi_{k,s_k}\phi_{k,t_k}u\|_0 \leqq C_6(\|Pu\|_0+t_k^B\|u\|_0+\sum_{\substack{|\alpha+\beta|\leqq 3\\ |\alpha|\neq 0}}\|(\partial_x^\alpha\phi_{k,s_k})(\partial_x^\beta\phi_{k,t_k})u\|_0) \qquad (u\in C_K^\infty(\Omega)), \tag{3.9}_k$$

where $C_6\geqq 0$ is a constant independent of s_k and t_k. (3.4) shows that

$$(\mathrm{supp}(\partial_x^\alpha\phi_{k,s_k})(\partial_x^\beta\phi_{k,t_k}))\cap S_k = \emptyset \qquad (\alpha\neq 0) \tag{3.10}_k$$

for all sufficiently large t_k and $s_k=e^{t_k}$. Here we note that, by virture of $(3.10)_k$, we can estimate the last term of $(3.9)_k$ by $(3.9)_{i<k}$ [resp. (1.18)] when $k>0$ [resp. $k=0$]. Therefore, (2.4) follows from (1.18), (3.3), (3.4), (3.9) and (3.10). ∎

PROOF OF THEOREM 2. Throughout the proof, each C_k $(k\in\boldsymbol{N})$ denotes a nonnegative constant independent of $\gamma>0$. Without loss of generality, we may assume, by Remark of Lemma 1.3, that $a^{i1}(x)=a^{1i}(x)\equiv 0$ $(1\leqq i\leqq d)$ and $b^1(x)\neq 0$ in S. We may suppose that $\Omega=\boldsymbol{R}^d$ and $b^1(x)\equiv 1$ in $\boldsymbol{R}^d$. We put

$$P^\gamma = \sum_{i,j=1}^{d} a_\gamma^{ij}(x)\partial_{x_i}\partial_{x_j}+\sum_{i=1}^{d} b_\gamma^i(x)\partial_{x_i}+c_\gamma(x),$$

where $a_\gamma^{ij}=a^{ij}$, $b_\gamma^i=b^i-2\gamma a^{i1}$, $c_\gamma=c-\gamma+\gamma^2a^{11}$ and $\gamma>0$. Modifying the proof of Lemma 1.6, we can show that for any sufficiently large $\gamma>0$ and for any bounded open neighborhood U of S

$$\int (T-e^{tx_1})^{-2} P^\gamma u \cdot u\, dx$$

$$\leqq -t^2 \int e^{tx_1}(T-e^{2tx_1})(T-e^{tx_1})^{-4} a_\gamma^{11} u^2 dx - t\int e^{tx_1}(T-e^{tx_1})^{-3} b_\gamma^1 u^2 dx$$

$$-\int (T-e^{tx_1})^{-2}\Big(\frac{1}{2}(a_\gamma^{ij})_{x_i x_j} - \frac{1}{2}(b_\gamma^i)_{x_i} + c_\gamma\Big) u^2 dx \qquad (u\in C_0^\infty(U)),$$

where $T>0$ is a sufficiently large constant and $t>0$ is a parameter; since $a^{1i}=a^{i1}\equiv 0$ in S and $b^1\equiv 1$, this implies that

$$\|u\|_0 \leqq \gamma^{-1} C_4 \|P^\gamma u\|_0 \qquad (u\in C_0^\infty(U))$$

for any $\gamma>0$ and for some open neighborhood U of S. Hence, Remark of the proof of Proposition 2.1 shows that for any $s\in \boldsymbol{R}$ there is a number $\gamma>0$ such that $P^\gamma u\in C^\infty(\boldsymbol{R}^d)$ implies $u\in H_s^{\mathrm{loc}}(\boldsymbol{R}^d)$.

Let $u\in \mathcal{D}'(\boldsymbol{R}^d)$ be a distribution satisfying $Pu\in C^\infty(\boldsymbol{R}^d)$. We put $v=e^{\gamma x_1}u$ ($\gamma>0$). Direct computation gives

$$Pu = P(e^{-\gamma x_1}v) = e^{-\gamma x_1}(P^\gamma v).$$

Hence, for any $s\in\boldsymbol{R}$ there is a number $\gamma>0$ such that $v\in H_s^{\mathrm{loc}}(\boldsymbol{R}^d)$, i.e., $u\in H_s^{\mathrm{loc}}(\boldsymbol{R}^d)$. Therefore, P is globally hypoelliptic. ■

PROOF OF THEOREM 3. Throughout the proof, each C_k ($k\in\boldsymbol{N}$) denotes a nonnegative constant. Modifying the proofs of Lemmas 1.3 and 1.4, we can show that for any δ ($0<\delta\leqq 1$) and for any bounded open neighborhood U of S with $U\Subset\Omega$

$$\sum_{|\beta|=1}\|P_{(\beta)}u\|_0^2 \leqq C_1\Big\{\sup(|\partial_x^\gamma a^{ij}(x)| : x\in U,\ 1\leqq i,\ j\leqq d,\ 1\leqq|\gamma|\leqq 2)$$

$$\times\Big(\sum_{k=1}^d \int a^{ij} u_{x_k x_i} u_{x_k x_j} dx + \|u\|_1^2\Big) + \|u\|_0^2\Big\} \qquad (u\in C_0^\infty(U))$$

and

$$\int a^{ij} u_{x_k x_i} u_{x_k x_j} dx = \int (a^{ij}u_{x_i x_j} + a^{ij}_{x_j} u_{x_i} + cu) u_{x_k x_k} dx$$

$$+\int (a^{ij}_{x_k} u_{x_i x_j} + a^{ij}_{x_j x_k} u_{x_i} + c_{x_k} u) u_{x_k} dx + \int c u_{x_k}^2 dx$$

$$(u\in C_0^\infty(U));$$

these imply, by (0.3), that

$$\|P_{(\beta)}u\|_{-|\beta|} \leqq \delta(\|Pu\|_0 + \|u\|_0) + C_2\|u\|_{-1} \qquad (u\in C_0^\infty(U),\ 1\leqq|\beta|\leqq 2) \quad (3.11)$$

for any δ ($0<\delta\leqq 1$) and for some open neighborhood U of S.

Let V and W be open sets of Ω and let $\phi(t, x)$ $(0\leqq t\leqq T,\ x\in V)$ be integral curves in Ω such that $V\Subset\Omega\setminus S$, $W\Subset\Omega$ and

$$\phi_t = \sum_{i=1}^{d} \lambda_i(\phi)X_i(\phi), \qquad \phi(0, x)=x,$$

where $\lambda_i\in C^\infty(\Omega)$, and such that $\phi(\cdot)=\phi(T, \cdot)$ is a C^∞ diffeomorphism from V onto W. Since

$$\begin{aligned} u(\phi(x))-u(x) &= \int_0^T \frac{\partial}{\partial t}\{u(\phi(t, x))\}\,dt \\ &= \int_0^T \sum_{i=1}^{d} (\lambda_i X_i u)(\phi(t, x))\,dt \qquad (u\in C_0^\infty(U)) \end{aligned}$$

we have

$$\int_W |u(y)|^2 dy \leqq C_3\Big(\int_V |u(x)|^2 dx + \sum_{i=1}^{d}\int_U |X_i u(x)|^2 dx\Big),$$

i. e.,

$$\|u\|_{L_2(W)} \leqq C_4\Big(\|u\|_{L_2(V)} + \sum_{i=1}^{d} \|X_i u\|_{L_2(U)}\Big) \qquad (u\in C_0^\infty(U)).$$

Direct computation gives, by Lemma 1.2 and (0.3),

$$\|X_i u\|_{L_2(U)}^2 \leqq C_5|(Pu, u)| \qquad (u\in C_0^\infty(U)).$$

Since $V\Subset\Omega\setminus S$, we have, by Lemma 1.8,

$$\|u\|_{L_2(V)} \leqq C_6(\|Pu\|_{-\kappa}+\|u\|_{-N}) \qquad (u\in C_0^\infty(U))$$

for any $N>0$, where $\kappa>0$. Combining the above estimates, we obtain

$$\|u\|_{L_2(W)} \leqq C_6(\|Pu\|_{-\kappa}+\|u\|_{-N})+\varepsilon^{-1}C_7\|Pu\|_0+\varepsilon\|u\|_0 \qquad (u\in C_0^\infty(U))$$

for any $\varepsilon>0$. Hence, by the weak controllability of (0.4),

$$\|u\|_0 \leqq C_8(\|Pu\|_0+\|u\|_{-N}) \qquad (u\in C_0^\infty(U)). \tag{3.12}$$

By Proposition 2.1, it follows from (3.11) and (3.12) that the operator P is globally hypoelliptic in Ω. ■

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