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Diastases and real analytic functions on complex manifolds

By Masaaki UMEHARA

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Introduction.

Let M be a complex manifold with the complex structure $\boldsymbol{J}$. A $\boldsymbol{J}$-invariant symmetric tensor g is called a *Kaehler tensor* if the associated 2-form $\omega_g(X, Y) = g(X, \boldsymbol{J}Y)$ ($X, Y \in \mathrm{TM}$) is closed. In addition, if g is non-degenerate it is called an *indefinite Kaehler metric*. A Kaehler tensor is called *analytic* if it is real analytic. Let $\boldsymbol{C}^{r,s}$ be a complex linear space $\boldsymbol{C}^N$ ($N=r+s$) with the indefinite Kaehler metric:

$$g_{r,s} = 2\left\{\sum_{\sigma=1}^{r} d\xi^\sigma \otimes d\bar{\xi}^\sigma - \sum_{\sigma=1}^{s} d\xi^{\sigma+r} \otimes d\bar{\xi}^{\sigma+r}\right\},$$

where $(\xi^1, \cdots, \xi^N)$ denotes the canonical complex coordinate system.

E. Calabi [**1**] gave a necessary and sufficient condition for a Kaehler manifold to be locally immersed into a complex space form as a Kaehler submanifold, and showed the rigidity of such immersions. In this paper we discuss the existence and the local rigidity of a full holomorphic mapping of M into $\boldsymbol{C}^{r,s}$ preserving a Kaehler tensor, and give several applications, where "full" means that the image of the mapping does not lie in any complex hyperplane in $\boldsymbol{C}^{r,s}$.

In § 1, we generalize the concept of diastases (introduced by Calabi [**1**] for analytic Kaehler metrics) for analytic Kaehler tensors and prepare some basic facts. In § 2, we define the *rank* of an analytic Kaehler tensor. For a Kaehler tensor of finite rank, a pair of integers, called "*extended signature*", is introduced. The Calabi condition for a local existence of holomorphic and isometric immersions into $\boldsymbol{C}^{N,0}$ (which is said to be *resolvable of rank N*) coincides with the condition that the extended signature is $(N, 0)$. We prove the following:

THEOREM. *A simply connected complex manifold M with a Kaehler tensor g admits a full holomorphic mapping Φ into $\boldsymbol{C}^{r,s}$ such that $\Phi^* g_{r,s} = g$ if and only if g is analytic and the extended signature is (r, s). Moreover Φ is locally rigid.*

Furthermore we mention some facts about holomorphic mappings into the Hilbert space l^2.

In § 3, we investigate the set $\Lambda(M)$ of real analytic functions on a complex manifold M, which consists of $\boldsymbol{R}$-linear combinations of functions $h\bar{k}+k\bar{h}$ (h and k are holomorphic functions on M), and show that if M is simply connected, then for an arbitrarily fixed point $p\in M$ such a function $f\in\Lambda(M)$ is decomposed into the following form

$$f=\operatorname{Re}(\phi^0)+\sum_{\sigma=1}^{r}|\phi^\sigma|^2-\sum_{\sigma=1}^{s}|\phi^{\sigma+r}|^2,$$

$$\phi^\sigma(p)=0 \qquad (\sigma=1, \cdots, r+s),$$

where $\phi^0, \cdots, \phi^{r+s}$ are holomorphic functions such that $\phi^1, \cdots, \phi^{r+s}$ are $\boldsymbol{C}$-linearly independent. In this decomposition, (r, s), which is called the *type* of f, is uniquely determined and coincides with the extended signature of the Kaehler tensor corresponding to $-2\sqrt{-1}\partial\bar{\partial}f$. Furthermore, ϕ^0 and $\{\phi^1, \cdots, \phi^{r+s}\}$ are also uniquely determined up to a constant term and a $\boldsymbol{C}$-linear transformation in the unitary group $U(r, s)$ of type (r, s) respectively. It is easily seen that the preceding theorem is a geometrical restatement of this decomposition theorem. Accordingly, the rigidity of indefinite Kaehler submanifolds in $\boldsymbol{C}^{r,s}$ comes to the uniqueness of the decomposition as above. This suggests that $\Lambda(M)$ is deeply concerned with the geometry of complex submanifolds. In fact, (as we show in § 3), the theorem in [4], which asserts that any two of complex space forms of different types have no Kaehler submanifolds in common, is obtained from the following transcendental properties concerned with $\Lambda(M)$.

PROPOSITION. *Let $p\in M$ be a fixed point of a complex manifold M and let $h^1, \cdots, h^N$ be non-constant holomorphic functions on M such that $h^\sigma(p)=0$ ($\sigma=1, \cdots, N$). Then*

(1) $\exp\left(\sum_{\sigma=1}^{N}|h^\sigma|^2\right)\notin\Lambda(M)$,

(2) $\log\left(1-\sum_{\sigma=1}^{N}|h^\sigma|^2\right)\notin\Lambda(M)$,

(3) $\left(1-\sum_{\sigma=1}^{N}|h^\sigma|^2\right)^{-\alpha}\notin\Lambda(M) \qquad (\alpha>0)$.

These properties will be also applied in [5] to prove that every Einstein Kaehler submanifold of a complex linear or hyperbolic space is totally geodesic.

In § 4, we mention conditions on the existence and the rigidity of holomorphic mappings preserving a Kaehler tensor into a non-flat indefinite complex space form.

§ 1. Diastases of analytic Kaehler tensors.

The diastases of analytic Kaehler metrics were originally introduced by E. Calabi [1]. In this section diastases are introduced for analytic Kaehler tensors on complex manifolds.

Let M be a complex manifold with complex structure $\boldsymbol{J}$. A (real) covariant symmetric 2-tensor g on M is said to be *$\boldsymbol{J}$-invariant* if it satisfies $g(X, Y)=g(\boldsymbol{J}X, \boldsymbol{J}Y)$ $(X, Y \in TM)$. For a $\boldsymbol{J}$-invariant symmetric tensor g, we define the associated 2-form ω_g by

$$\omega_g(X, Y) = g(X, \boldsymbol{J}Y) \qquad (X, Y \in TM).$$

If ω_g is closed, then g is called a *Kaehler tensor*. In addition, if g is non-degenerate, it is called an *indefinite Kaehler metric*. A Kaehler tensor is called *analytic* if it is real analytic.

On the complex tangent space, the associated 2-form ω_g of a Kaehler tensor g is a self-conjugate closed (1, 1)-form, namely it satisfies the following:

(1) $\bar{\omega}_g = \omega_g$,

(2) $d\omega_g = 0$,

(3) $\omega_g(Z, W) = 0$ if Z and W are both of (1, 0) or (0, 1)-type.

Conversely, for a given self-conjugate closed (1, 1)-form ω, we can construct a Kaehler tensor g_ω by

$$g_\omega(X, Y) = \omega(\boldsymbol{J}X, Y) \qquad (X, Y \in TM).$$

Hence there is a one to one correspondence between Kaehler tensors and self-conjugate closed (1, 1)-forms.

Let g be an analytic Kaehler tensor on a complex n-manifold. Then locally, there exists a real analytic function f such that $\omega_g = -2\sqrt{-1}\partial\bar{\partial}f$, where f is called a *primitive function* of g. The primitive function f is determined up to the real part of a holomorphic function, that is, for any holomorphic function h, $f+h+\bar{h}$ is also a primitive function.

Now we introduce the multi-index defined as follows: We arrange all n-tuples of non-negative integers as the sequence $\{(m_{I,1}, \cdots, m_{I,n})\}_{I=0,1,2,\dots}$ such that

$$m_0 = (0, \cdots, 0),$$

$$|m_I| \leqq |m_{I+1}| \qquad (I=0, 1, 2, \cdots),$$

where $m_I = (m_{I,1}, \cdots, m_{I,n})$ and $|m_I| = \sum_{\alpha=1}^n m_{I,\alpha}$. Then we denote by $(z)^{m_I}$ the monomial $\prod_{\alpha=1}^n (z^\alpha)^{m_{I,\alpha}}$ in n-variables.

Let f be a primitive function of g. For a complex local coordinate system $(z^1, \cdots, z^n)$, f is expressed as a power series expansion:

$$f(q)=\sum_{I,K=0}^{\infty} b_{I\bar{K}}(z(q))^{m_I}(\overline{z(q)})^{m_K}.$$

Using this expression we define a complex-valued function F as follows:

$$F(p, q)=\sum_{I,K=0}^{\infty} b_{I\bar{K}}(z(p))^{m_I}(\overline{z(q)})^{m_K},$$

where p, q are points in the convergence domain of f. Now a functional element of a *diastasis* $D_g(p, q)$ is defined as follows:

$$D_g(p, q)=F(p, p)+F(q, q)-F(p, q)-F(q, p). \tag{1.1}$$

Since F is independent of local coordinate systems, so is D_g. Using the same discussion as in Calabi [1], the following properties are easily verified:

(1) D_g is independent of the choice of a primitive function, namely it is uniquely determined by g.

(2) $D_g(p, q)=D_g(q, p)$.

(3) D_g is a real analytic function.

The diastasis $D_g(p, q)$ is defined on some neighborhood of the diagonal set $\{(p, p); p\in M\}$ of the product space $M\times M$. For $p\in M$ fixed, $D_g(p, q)$ is a primitive function of g. So we may regard the diastasis as a normalization of the primitive functions at the point p.

EXAMPLE 1. The space $\boldsymbol{C}^{r,s}$ $(r, s=0, \cdots, N, r+s=N)$ is the complex linear space $\boldsymbol{C}^N$ with the indefinite Kaehler metric:

$$g_{r,s}=2\left\{\sum_{\sigma=1}^{r} d\xi^\sigma\otimes d\bar{\xi}^\sigma-\sum_{\sigma=1}^{s} d\xi^{\sigma+r}\otimes d\bar{\xi}^{\sigma+r}\right\},$$

where $(\xi^1, \cdots, \xi^N)$ is the canonical complex coordinate system of $\boldsymbol{C}^N$. The associated 2-form $\omega_{r,s}$ is given by

$$\omega_{r,s}=-2\sqrt{-1}\left\{\sum_{\sigma=1}^{r} d\xi^\sigma\wedge d\bar{\xi}^\sigma-\sum_{\sigma=1}^{s} d\xi^{\sigma+r}\wedge d\bar{\xi}^{\sigma+r}\right\},$$

and the diastasis is given by

$$D_{r,s}(p, q)=\sum_{\sigma=1}^{r}|\xi^\sigma(p)-\xi^\sigma(q)|^2-\sum_{\sigma=1}^{s}|\xi^{\sigma+r}(p)-\xi^{\sigma+r}(q)|^2 \qquad (p, q\in\boldsymbol{C}^{r,s}). \tag{1.2}$$

EXAMPLE 2. The *indefinite complex projective space* $\boldsymbol{CP}_s^N(b)$ $(0\leqq s\leqq N)$ of constant holomorphic sectional curvature $b>0$ is the open submanifold $\{(\xi^0, \xi^1, \cdots, \xi^N)\in\boldsymbol{C}^{N+1}; \sum_{\sigma=0}^{N-s}|\xi^\sigma|^2-\sum_{\sigma=0}^{s-1}|\xi^{N-\sigma}|^2>0\}/\boldsymbol{C}^*$ of $\boldsymbol{CP}^N=(\boldsymbol{C}^{N+1}\setminus(0))/\boldsymbol{C}^*$. The associated 2-form of the indefinite Kaehler metric of $\boldsymbol{CP}_s^N(b)$ is defined by $(-4\sqrt{-1}/b)\partial\bar{\partial}\log(\sum_{\sigma=0}^{N-s}|\xi^\sigma|^2-\sum_{\sigma=0}^{s-1}|\xi^{N-\sigma}|^2)$. The diastasis is given by

$$(1.3) \qquad D(p, q) = (2/b)\log\Big(1+\sum_{\sigma=1}^{N-s}|\xi^{\sigma}(q)/\xi^{0}(q)|^{2}-\sum_{\sigma=0}^{s-1}|\xi^{N-\sigma}(q)/\xi^{0}(q)|^{2}\Big)$$

where $p=(1, 0, \cdots, 0)$ and $\xi^{0}(q)=0$. In case $s=0$, this space coincides with the ordinary complex projective space $\boldsymbol{CP}^{N}(b)$ with Fubini-Study metric.

EXAMPLE 3. The *indefinite complex hyperbolic space* $\boldsymbol{CH}_{s}^{N}(b)$ $(0\leqq s\leqq N)$ of constant holomorphic sectional curvature $b<0$ is obtained from $\boldsymbol{CP}_{N-s}^{N}(-b)$ by replacing the metric of $\boldsymbol{CP}_{N-s}^{N}(-b)$ by its negative. In case $s=0$, this space coinsides with the ordinary complex hyperbolic space $\boldsymbol{CH}^{N}(b)$.

The indefinite Kaehler manifolds $\boldsymbol{C}^{r,s}$, $\boldsymbol{CP}_{s}^{N}(b)$ and $\boldsymbol{CH}_{s}^{N}(b)$ are all called the *indefinite complex space forms with index* 2s.

The diastases have the following useful property.

PROPOSITION 1.1. *Let M and $\tilde{M}$ be complex manifolds with analytic Kaehler tensors g and $\tilde{g}$ respectively and Φ a holomorphic mapping of M into $\tilde{M}$. Then $\Phi^{*}\tilde{g}=g$ if and only if*

$$(1.4) \qquad D_{g}(p, q) = D_{\tilde{g}}(\Phi(p), \Phi(q)),$$

where $p, q\in M$ in the region of definition.

PROOF. We suppose that $\Phi^{*}\tilde{g}=g$. Then for a primitive function $\tilde{f}$ of $\tilde{g}$, we have

$$-2\sqrt{-1}\partial\bar{\partial}(\tilde{f}\circ\Phi) = -2\sqrt{-1}\Phi^{*}\partial\bar{\partial}\tilde{f} = \Phi^{*}\omega_{\tilde{g}} = \omega_{g}.$$

This implies that $\tilde{f}\circ\Phi$ is a primitive function of g. Since Φ is holomorphic, by the definition of the diastasis, we have the relation (1.4). The converse is easily shown by differentiating (1.4) with respect to the variable q. q.e.d.

§ 2. Extended signature of analytic Kaehler tensors.

First of all, we prepare some properties of infinite dimensional matrices.

DEFINITION. The *rank* of an infinite dimensional matrix $B=(b_{I\bar{K}})_{I,K=1,2,3,\dots}$ is defined by

$$\operatorname{rank}(B) = \lim_{m\to\infty}\operatorname{rank}(B_{mm}),$$

where $B_{mm}=(b_{i\bar{j}})_{i,j=1,\dots,m}$.

Let $a^{\sigma}=(a_{I}^{\sigma})_{I=1,2,3,\dots}$ $(\sigma=1, \cdots, N)$ be systems of sequences. If we set

$$b_{I\bar{K}} = \sum_{\sigma=1}^{r} a_{I}^{\sigma}\bar{a}_{K}^{\sigma}-\sum_{\sigma=1}^{s} a_{I}^{\sigma+r}\bar{a}_{K}^{\sigma+r} \qquad (r+s=N),$$

then $B=(b_{I\bar{K}})_{I,K=1,2,3,\cdots}$ is an infinite dimensional Hermitian matrix, that is $b_{I\bar{K}}=\overline{b_{K\bar{I}}}$. If we put $P_m=(a_i^\sigma)_{\sigma=1,\cdots,N,i=1,\cdots,m}$, then $B_{mm}={}^t\bar{P}_m I_{r,s} P_m$, where

$$I_{r,s}=\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}.$$

Hence we have

$$\operatorname{rank}(B)=\lim_{m\to\infty}\operatorname{rank}(B_{mm})\leqq\lim_{m\to\infty}\operatorname{rank}(P_m)\leqq N.$$

The equality holds if and only if $\lim_{m\to\infty}\operatorname{rank}(P_m)=N$, that is, $\{a^\sigma\}_{\sigma=1,\cdots,N}$ are $\boldsymbol{C}$-linearly independent.

Conversely, every Hermitian matrix of finite rank is obtained in such a way. In fact, let $B=(b_{I\bar{K}})_{I,K=1,2,3,\cdots}$ be a Hermitian matrix of rank $N<\infty$. First, we suppose $\operatorname{rank}(B_{NN})=N$, then B_{NN} is known to be written as follows:

$$B_{NN}={}^t\bar{P}I_{r,s}P, \tag{2.1}$$

where r (resp. s) is the number of positive (resp. negative) eigenvalue of B_{NN} and $P=(p_\tau^\sigma)_{\sigma,\tau=1,\cdots,N}$ is a non-singular matrix. Since every column is a $\boldsymbol{C}$-linear combination of the first N-th columns, there exist $u_I^\sigma\in\boldsymbol{C}$ ($\sigma=1,\cdots,N$, $I=1,2,3,\cdots$) such that

$$b_{I\bar{K}}=\sum_{\sigma=1}^N b_{I\bar{\sigma}}u_K^\sigma \qquad (I,\ K=1,\ 2,\ 3,\ \cdots). \tag{2.2}$$

Now if we let $a_I^\sigma=\sum_{\tau=1}^N\bar{p}_\tau^\sigma\bar{u}_I^\tau$ ($\sigma=1,\cdots,N$, $I=1,2,3,\cdots$), then

$$b_{I\bar{K}}=\sum_{\sigma=1}^r a_I^\sigma\bar{a}_K^\sigma-\sum_{\sigma=1}^s a_I^{\sigma+r}\bar{a}_K^{\sigma+r}. \tag{2.3}$$

Next we consider the case $\operatorname{rank}(B_{NN})<N$. Since B is of rank$N$, there exists a positive integer m such that $\operatorname{rank}(B_{mm})=N$. By a suitable simultaneous permutation of the first m-th columns and lows, we can obtain a new Hermitian matrix $B'=(b'_{I\bar{K}})_{I,K=1,2,3,\cdots}$ such that $\operatorname{rank}(B'_{NN})=N$. We denote this permutation by

$$\iota=\begin{pmatrix} 1, & \cdots, & m \\ \iota(1), & \cdots, & \iota(m)\end{pmatrix}.$$

We have already shown that each component of B' is written as follows:

$$b'_{I\bar{K}}=\sum_{\sigma=1}^r c_I^\sigma\bar{c}_K^\sigma-\sum_{\sigma=1}^s c_I^{\sigma+r}\bar{c}_K^{\sigma+r}.$$

Hence if we put

$$a_i^\sigma = c_{\iota(i)}^\sigma \qquad (i=1, \cdots, m),$$
$$a_{I+m}^\sigma = c_{I+m}^\sigma \quad (I=1, 2, 3, \cdots),$$

then each component of B is also expressed as (2.3). The pair of integers (r, s) determined by (2.3) is called the *signature* of the Hermitian matrix B. For a sufficiently large integer m, r (resp. s) is the number of positive (resp. negative) eigenvalues of B_{mm}.

Moreover our construction (2.3) is determined up to a linear transformation. Let $U(r, s)$ $(r+s=N)$ be the group of linear transformations of $\boldsymbol{C}^{r,s}$ which preserve the indefinite Kaehler metric $g_{r,s}$. Each element of $U(r, s)$ is also regarded as an $N\times N$ complex matrix T which satisfies ${}^t\bar{T}I_{r,s}T=I_{r,s}$. We have the following:

LEMMA 2.1. *Let $B=(b_{I\bar{K}})_{I,K=1,2,3,\dots}$ be a Hermitian matrix of rank $N<\infty$ satisfying* (2.3) *with respect to a system of sequences $a^\sigma=\{a_I^\sigma\}_{I=1,2,3,\dots}$ $(\sigma=1, \cdots, N)$. Suppose that each component of B has another such decomposition:*

$$b_{I\bar{K}} = \sum_{\sigma=1}^{r'} c_I^\sigma \bar{c}_K^\sigma - \sum_{\sigma=1}^{s'} c_I^{\sigma+r'} \bar{c}_K^{\sigma+r'} \qquad (r'+s'=N), \tag{2.4}$$

for $I, K=1, 2, 3, \cdots$. Then $(r', s')=(r, s)$ and there exists a matrix $T=(t_{ij})\in U(r, s)$ $(r+s=N)$ such that

$$c_K^\sigma = \sum_{\tau=1}^{N} t_\tau^\sigma a_K^\tau \qquad (\sigma=1, \cdots, n,\ K=1, 2, 3, \cdots).$$

PROOF. Let

$$x_K = {}^t(a_K^1, \cdots, a_K^N),$$
$$y_K = {}^t(c_K^1, \cdots, c_K^N) \qquad (K=1, 2, 3, \cdots).$$

Then (2.3) and (2.4) give us

$$b_{I\bar{K}} = {}^t\bar{x}_K I_{r,s} x_I = {}^t\bar{y}_K I_{r,s} y_I. \tag{2.5}$$

Since the matrix B is of rank N, we can choose $\boldsymbol{C}$-linearly independent vectors $\{x_{i_\sigma}\}_{\sigma=1,\dots,N}$ and $\{y_{j_\sigma}\}_{\sigma=1,\dots,N}$. Then there exists an $N\times N$ complex matrix T such that $Tx_{i_\sigma}=y_{j_\sigma}$ $(\sigma=1, \cdots, N)$. Let X, Y be non-singular matrices $X=(a_{i_\tau}^\sigma)_{\sigma,\tau=1,\dots,N}$ and $Y=(c_{j_\tau}^\sigma)_{\sigma,\tau=1,\dots,N}$. From (2.5) it follows that

$${}^t\bar{X}I_{r,s}X = {}^t\bar{Y}I_{r,s}Y = {}^t\bar{X}({}^t\bar{T}I_{r,s}T)X.$$

Since X is non-singular, we have

$${}^t\bar{T}I_{r,s}T = I_{r,s}.$$

Hence $(r', s')=(r, s)$ and $T\in U(r, s)$. We also have

$$\begin{aligned} {}^t\bar{X}I_{r,s}x_K &= {}^t\bar{Y}I_{r,s}y_K \\ &= {}^t\bar{X}{}^t\bar{T}I_{r,s}y_K \\ &= {}^t\bar{X}({}^t\bar{T}I_{r,s}T)T^{-1}y_K \\ &= {}^t\bar{X}I_{r,s}(T^{-1}y_K), \end{aligned}$$

thus $x_K=T^{-1}y_K$ $(K=1, 2, 3, \cdots)$, which concludes our assertion. q.e.d.

Let g be a Kaehler tensor on a complex n-manifold M. For a complex local coordinate system $(z^1, \cdots, z^n)$, we put $g_{\alpha\bar{\beta}}=g(\partial/\partial z^\alpha, \partial/\partial\bar{z}^\beta)$, then $(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,\cdots,n}$ is a Hermitian matrix. At each point $p\in M$, the signature of the matrix $(g_{\alpha\bar{\beta}})$ is called the *signature* of the Kaehler tensor g. Obviously this definition is independent of local complex coordinate systems.

Now we define the *extended signature* of an analytic Kaehler tensor g on M. Let $p\in M$ be a point of a coordinate neighborhood $\{U;(z^1, \cdots, z^n)\}$. The diastasis $D_g(p, q)$ has the power series expansion at p as a function of variable q:

$$D_g(p, q)=\sum_{I,K=1}^{\infty} b_{I\bar{K}}(z(q)-z(p))^{m_I}(\overline{z(q)-z(p)})^{m_K}.$$

Then $B=(b_{I\bar{K}})_{I,K=1,2,3,\ldots}$ is considered as an infinite dimensional Hermitian matrix. The rank of the matrix B is called the *rank* of the analytic Kaehler tensor g at p. If the rank of B is finite, we call the signature of B the *extended signature* of g at p in this paper. Note that the extended signature is defined only for analytic Kaehler tensors of finite rank. We will show later that the extended signature is independent of local coordinate systems and the choice of a point in M.

REMARK. Let (r, s) and (r', s') be the signature and the extended signature of an analytic Kaehler tensor g at p respectively. Since $g_{\alpha\bar{\beta}}=b_{\alpha\bar{\beta}}$, the matrix $(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,\cdots,n}$ is considered as a submatrix of B. So we can easily check the inequalities $r'\geqq r$, $s'\geqq s$.

A holomorphic mapping of a complex manifold into $\boldsymbol{C}^{r,s}$ is called *full* if the image of the mapping does not lie in any complex hyperplane of $\boldsymbol{C}^{r,s}$. Now we prove the following:

THEOREM 2.2. *Let M be a complex manifold with a Kaehler tensor g. Suppose that there exists a full holomorphic mapping Φ of M into $\boldsymbol{C}^{r,s}$ such that $\Phi^*g_{r,s}=g$. Then g is analytic and its extended signature is (r, s) at every point with respect to any coordinate systems. Conversely, if g is analytic and its extended signature is (r, s) $(r+s=N<\infty)$ at $p\in M$ with respect to a coordinate system, then there exists a full holomorphic mapping Φ of some neighborhood of*

p into $C^{r,s}$ such that $\Phi^* g_{r,s}=g$.

To prove this, we prepare some lemmas.

LEMMA 2.3. *Let $\Phi=(\phi^1, \cdots, \phi^N)$ be a holomorphic mapping of a complex n-manifold M into C^N such that $\Phi(p)=0$ for fixed $p\in M$ and let $\phi^\sigma=\sum_{I=0}^{\infty} a_I^\sigma(z-z(p))^{m_I}$ $(\sigma=1, \cdots, N)$, where $(z^1, \cdots, z^n)$ is a local coordinate system of M. Then the system of sequences $a^\sigma=(a_I^\sigma)_{I=1,2,3,\dots}$ $(\sigma=1, \cdots, N)$ is C-linearly independent if and only if Φ is full.*

PROOF. For an open subset U, $\Phi|_U$ is full if and only if so is Φ by the analyticity of Φ. Now our assertion is immediate. q.e.d.

LEMMA 2.4. *Let M be a complex manifold with a Kaehler tensor g and Φ a holomorphic mapping of M into $C^{r,s}$ $(r+s=N)$ such that $\Phi^* g_{r,s}=g$. Then g is analytic and its rank is less than or equal to N at each point. Moreover if Φ is full then the rank of g is everywhere N and its extended signature is (r, s).*

PROOF. Since $g_{r,s}$ is analytic, so is g. Let $p\in M$ be an arbitrarily fixed point. Without loss of generality, we may put $\Phi(p)=0$ and take a coordinate system $(z^1, \cdots, z^n)$ with the origin p. Then by Proposition 1.1,

$$D_g(p, q)=D_{r,s}(0, \Phi(q)) = \sum_{\sigma=1}^{r} |\phi^\sigma(z(q))|^2-\sum_{\sigma=1}^{s} |\phi^{\sigma+r}(z(q))|^2. \tag{2.6}$$

Substituting $\phi^\sigma=\sum_{I=1}^{\infty} a_I^\sigma(z)^{m_I}$ into (2.6), we have

$$D_g(p, q)=\sum_{I,K=1}^{\infty}\left\{\sum_{\sigma=1}^{r} a_I^\sigma \bar{a}_K^\sigma-\sum_{\sigma=1}^{s} a_I^{\sigma+r}\bar{a}_K^{\sigma+r}\right\}(z(q))^{m_I}(\overline{z(q)})^{m_K}.$$

Now our assertion follows immediately from Lemma 2.3 and the discussion of infinite dimensional matrices in this section. q.e.d.

PROOF OF THEOREM 2.2. By Lemma 2.4, the first assertion is ovbious. We prove the converse. By using a complex local coordinate system $(z^1, \cdots, z^n)$ with the origin $p\in M$, D_g has the power series expansion:

$$D_g(p, q)=\sum_{I,K=1}^{\infty} b_{I\bar{K}}(z(q))^{m_I}(\overline{z(q)})^{m_K}. \tag{2.7}$$

Let $\rho_1, \cdots, \rho_n$ be positive numbers such that the power series (2.7) converges absolutely in $|z^\alpha|<\rho_\alpha$ $(\alpha=1, \cdots, n)$. By the assumption, the Hermitian matrix $B=(b_{I\bar{K}})_{I,K=1,2,3,\dots}$ determined by (2.7) is of rank N. Without loss of generality, we may suppose that $\mathrm{rank}(B_{NN})=N$ and that B satisfies the relations (2.1) and (2.2) for some nonsingular matrix $P=(p_\tau^\sigma)_{\sigma,\tau=1,\dots,N}$ and $u_I^\tau\in C$ $(\sigma=1, \cdots, N, I=1, 2, 3, \cdots)$. Now let

$$a_I^\sigma = \sum_{\tau=1}^N \bar{p}_\tau^\sigma \bar{u}_I^\tau \qquad (\sigma=1, \cdots, N,\ I=1, 2, 3, \cdots),$$

then we have already shown that each component of B is expressed by

$$b_{I\bar{K}} = \sum_{\sigma=1}^r a_I^\sigma \bar{a}_K^\sigma - \sum_{\sigma=1}^s a_I^{\sigma+r} \bar{a}_K^{\sigma+r}.$$

By setting $\phi^\sigma = \sum_{I=1}^\infty a_I^\sigma (z)^{m_I}$, from (2.7) and (2.8), we have the relation (2.6). Therefore it suffices to show that each ϕ^σ converges absolutely on some neighborhood of p. Let B_I denote the I-th column ${}^t(b_{1\bar{I}}, \cdots, b_{N\bar{I}})$ of the matrix $B_{N\infty} = (b_{\sigma\bar{I}})_{\sigma=1,\cdots,N,\,I=1,2,3,\cdots}$ Then it is regarded as an element of $\boldsymbol{C}^N$. Let $\{e_1, \cdots, e_N\}$ be an orthonormal basis of $\boldsymbol{C}^N$. Since (2.2) implies $B_I = \sum_{\sigma=1}^N B_\sigma u_I^\sigma$, we have

$$u_I^\sigma = \sum_{\sigma=1}^N h^{\sigma\tau} \langle e_\tau, B_I \rangle,$$

where $(h^{\sigma\tau})_{\sigma,\tau=1,\cdots,N}$ is the inverse matrix of $(\langle e_\sigma, B_\tau \rangle)_{\sigma,\tau=1,\cdots,N}$. We set $d_1 = \max_{\sigma,\tau=1,\cdots,N}\{|p_\tau^\sigma|\}$ and $d_2 = \max_{\sigma,\tau=1,\cdots,N}\{|h_\tau^\sigma|\}$. Then we have the following estimates:

$$\begin{aligned} |a_I^\sigma| &\leqq \sum_{\tau=1}^N |p_\tau^\sigma|\,|u_I^\tau| \\ &\leqq d_1 \sum_{\tau=1}^N |u_I^\tau| \\ &\leqq d_1 \sum_{\sigma,\tau=1}^N |h^{\tau\sigma}|\,|\langle e, B_I \rangle| \\ &\leqq N^2 d_1 d_2 \Big(\sum_{\sigma=1}^N |b_{\sigma I}|^2 \Big)^{1/2}. \end{aligned}$$

Taking numbers ρ'_α $(\alpha=1, \cdots, n)$ such that $0<\rho'_\alpha<\rho_\alpha$ and putting $R = \sum_{I,K=1}^\infty |b_{I\bar{K}}|(\rho')^{m_I}(\rho')^{m_K}$, we have $|b_{I\bar{K}}| \leqq R/(\rho')^{m_I+m_K}$. This, together with the convention of our multi-indices $|m_\sigma| \leqq |m_N|$ $(\sigma=1, \cdots, N)$, yields

$$\begin{aligned} |a_I^\sigma| &\leqq N^2 d_1 d_2 \Big(\sum_{\sigma=1}^N R^2/(\rho')^{2(m_\sigma+m_I)} \Big)^{1/2} \\ &\leqq N^2 d_1 d_2 \Big\{ \sum_{\sigma=1}^N R^2/\delta^{2|m_N|}(\rho')^{2m_I} \Big\}^{1/2} \\ &= (N^{5/2} d_1 d_2 R/\delta^{|m_N|})(1/(\rho')^{m_I}), \end{aligned}$$

where $\delta = \min\{1, \rho'_1, \cdots, \rho'_n\}$. Hence each ϕ^σ $(\sigma=1, \cdots, N)$ converges absolutely in the region $\{|z^\alpha|<\rho'_\alpha, \alpha=1, \cdots, n\}$. By Lemma 2.3, the mapping $\Phi = (\phi^1, \cdots, \phi^N)$ is full. q.e.d.

REMARK. In the proof of Theorem 2.2, we assumed that the matrix B

satisfies the relations (2.1) and (2.2). But this normalization of the infinite dimensional Hermitian matrix is obtained by finite exchanges of columns and lows. So these assumptions give no influence on our estimates.

PROPOSITION 2.5. *Let g be an analytic Kaehler tensor on a connected complex manifold M. If the rank of g is $N<\infty$ at some point, then so is everywhere.*

PROOF. Let A be the subset of points at which the rank is N. It is sufficient to show that A is open and closed. By Theorem 2.2, A is obviously open. Now we take a sequence $\{x_l\}_{l=1,2,3,\dots}$ of points in A, which converges to a point $x_\infty \in M$. On a complex local coordinate neighborhood $\{U:(z^1, \cdots, z^n)\}$ around x_∞, the diastasis is expressed by

$$D_g(p, q) = \sum_{I,K=1}^{\infty} b_{I\bar{K}}(p)(z(q)-z(p))^{m_I}(\overline{z(q)-z(p)})^{m_K}.$$

Since each $b_{I\bar{K}}(p)$ is a continuous function of the variable p, we have $\infty > N = \operatorname{rank}\{(b_{I\bar{K}}(x_l))\} \geqq \operatorname{rank}\{(b_{I\bar{K}}(x_\infty))\}$.

By Theorem 2.2, the rank of g is locally constant unless it is infinite. Hence $x_\infty \in A$, which implies that A is closed. q.e.d.

COROLLARY 2.6. *The rank and the extended signature of a Kaehler tensor on a connected complex manifold are invariant under the change of complex local coordinate systems and the choice of a point.*

The rigidity holds for full holomorphic mappings into $\boldsymbol{C}^{r,s}$ which preserve a Kaehler tensor.

THEOREM 2.7. *Let Φ_i $(i=1, 2)$ be full holomorphic mappings of a connected complex manifold M with an analytic Kaehler tensor g into $\boldsymbol{C}^{r_i,s_i}$ such that $\Phi_i^* g_{r_i,s_i} = g$. Then (r_i, s_i) $(i=1, 2)$ coincide with the extended signature (r, s) of g. In addition, Φ_1 and Φ_2 differ by a motion in $\boldsymbol{C}^{r,s}$.*

PROOF. By Theorem 2.2, we have $(r_1, s_1)=(r_2, s_2)=(r, s)$. We may assume $\Phi_i(p)=0$ $(i=1,2)$ for some fixed point $p \in M$. Each component of $\Phi_i=(\phi_i^1, \cdots, \phi_i^N)$ has a power series expansion on a coordinate neighborhood $\{U;(z^1, \cdots, z^n)\}$ with the origin p:

$$\phi_1^q = \sum_{I=1}^{\infty} a_I^q(z)^{m_I},$$

$$\phi_2^q = \sum_{I=1}^{\infty} c_I^q(z)^{m_I}.$$

On the other hand, $D_g(p, q)$ has a power series expansion:

$$D_g(p, q)=\sum_{I,K=1}^{\infty} b_{I\bar{K}}(z(q))^{m_I}(\overline{z(q)})^{m_K} \qquad (q\in U).$$

In the proof of Lemma 2.4, we have already seen that

$$\begin{aligned} b_{I\bar{K}} &= \sum_{\sigma=1}^{r} a_I^\sigma \bar{a}_K^\sigma - \sum_{\sigma=1}^{s} a_I^{\sigma+r}\bar{a}_K^{\sigma+r} \\ &= \sum_{\sigma=1}^{r} c_I^\sigma \bar{c}_K^\sigma - \sum_{\sigma=1}^{s} c_I^{\sigma+r}\bar{c}_K^{\sigma+r}. \end{aligned}$$

By applying Lemma 2.1, there exists a linear transformation $T\in U(r, s)$ such that $T\circ\Phi_1=\Phi_2$. This proves our assertion. q.e.d.

REMARK. Without the assumption 'full', the rigidity is not expected. In fact, the holomorphic mapping of $\boldsymbol{C}^{1,0}$ into $\boldsymbol{C}^{2,1}$ defined by $\Phi(z)=(z, 0, 0)$ $(z\in\boldsymbol{C}^{1,0})$ is not rigid. Though $\Psi(z)=(z, z^2, -z^2)$ also satisfies $\Psi^*g_{2,1}=g_{1,0}$, its components cannot be expressed by $\boldsymbol{C}$-linear combinations of the components of Φ. But in case $s=0$, our assumption 'full' can be omitted, because the restriction of the metric $g_{r,0}$ to any subspace in $\boldsymbol{C}^{N,0}$ is also non-degenerate.

Now we suppose that M is connected and simply connected. Since the extended signature is independent of the choice of a point, Theorem 2.2 and Theorem 2.7 imply that every full holomorphic mapping of an open subset of M into $\boldsymbol{C}^{r,s}$, which preserves a Kaehler tensor, is uniquely extended to the whole of M. So we obtain the following:

THEOREM 2.8. *Let M be a connected and simply connected complex manifold. Then for a Kaehler tensor g on M, the following two conditions are equivalent:*

(1) *g is analytic and its extended signature is (r, s).*

(2) *There exists a full holomorphic mapping Φ of M into $\boldsymbol{C}^{r,s}$ $(r+s=N)$ such that $\Phi^*g_{r,s}=g$.*

Isometric mappings of Kaehler manifolds into infinite dimensional spaces have been investigated in [1] and [4]. We consider holomorphic mappings of a complex manifold with a Kaehler tensor into the Hilbert space l^2. The space l^2 consists of the points with the coordinates $(\xi^1, \xi^2, \xi^3, \cdots)$ such that $\sum_{\sigma=1}^{\infty}|\xi^\sigma|^2<\infty$ $(\xi^\sigma\in\boldsymbol{C})$. The Hermitian innerproduct $\langle\,,\rangle$ is defined by $\langle p, q\rangle=\sum_{\sigma=1}^{\infty}\xi^\sigma(p)\overline{\xi^\sigma(q)}$ $(p, q\in l^2)$.

DEFINITION. Let M be a complex manifold and Φ a mapping of M into l^2. Then Φ is said to be *holomorphic* if it satisfies the following two conditions:

(1) $\phi^\sigma=\xi^\sigma\circ\Phi$ is holomorphic for all $\sigma=1, 2, 3, \cdots$.

(2) Φ is locally bounded, that is, for every $p\in M$, there exist a neighborhood U of p and a positive number m such that $|\Phi|=\langle\Phi, \Phi\rangle^{1/2}<m$ on U.

If Φ and Ψ are holomorphic mappings into l^2, then the function $f(p, q)=\langle\Phi(p), \Psi(q)\rangle$ $(p, q\in M)$ defined on $M\times\overline{M}$ is also holomorphic, where $\overline{M}$ denotes the conjugate manifold of M (cf. [**4**; Lemma 1.4]). Now we define a diastasis of l^2 by

$$D^{\infty}(p, q)=|p-q|^2 \qquad (p, q\in l^2).$$

LEMMA 2.9. *Let M be a complex manifold with an analytic Kaehler tensor and Φ a holomorphic mapping of M into l^2. Then the following two assertions are equivalent to each other.*

(1) *Φ preserves the diastasis, that is, $D_g(p, q)=D^{\infty}(\Phi(p), \Phi(q))$ $(p, q\in M)$.*

(2) *$|\Phi|^2$ is a primitive function of g.*

PROOF. We have

$$(2.9)\quad D^{\infty}(\Phi(p), \Phi(q))=\langle\Phi(p), \Phi(p)\rangle+\langle\Phi(q), \Phi(q)\rangle-\langle\Phi(p), \Phi(q)\rangle-\langle\Phi(q), \Phi(p)\rangle.$$

If $|\Phi|^2$ is a primitive function of g, then by (1.1) and (2.9), $D^{\infty}(\Phi(p), \Phi(q))$ coincides with $D_g(p, q)$. On the other hand, for a fixed $p\in M$, (2.9) implies that $D^{\infty}(\Phi(p), \Phi(q))$ and $|\Phi|^2$ differ by the real part of a holomorhic function with respect to the variable q. Hence the converse is obvious. q.e.d.

The following is an extension of [**4**; Proposition 1.8].

PROPOSITION 2.10. *Let M be a complex manifold with an analytic Kaehler tensor of extended signature (r, s) $(r+s=N)$ and Φ a holomorphic mapping of M into l^2 which preserves the diastasis. Then $s=0$ and $\Phi(M)$ lies in some complex N-plane in l^2.*

PROOF. Let $(\xi^1, \cdots, \xi^N)$ be the canonical complex coordinate system of $\boldsymbol{C}^N$ $(r+s=N)$. We define a real bilinear form $\beta_{r,s}$ on $\boldsymbol{C}^N$ by

$$\beta_{r,s}(p, q)=\mathrm{Re}\left\{\sum_{\sigma=1}^{r}\xi^{\sigma}(p)\overline{\xi^{\sigma}(q)}-\sum_{\sigma=1}^{s}\xi^{\sigma+r}(p)\overline{\xi^{\sigma+r}(q)}\right\} \qquad (p, q\in\boldsymbol{C}^N).$$

Obviously the following identity holds:

$$(2.10)\ \beta_{r,s}(\overrightarrow{pq_1}, \overrightarrow{pq_2})=(1/2)\{D_{r,s}(p, q_1)+D_{r,s}(p, q_2)+D_{r,s}(q_1, q_2)\}\ (p, q_1, q_2\in\boldsymbol{C}^{r,s}),$$

where $\overrightarrow{pq}=q-p$ $(p, q\in\boldsymbol{C}^N)$.

Let $p\in M$ be an arbitrary point. Since the extended signature of g is (r, s), there exists a full holomorphic mapping Ψ of some neighborhood U of p into $\boldsymbol{C}^{r,s}$ such that $\Psi^*g_{r,s}=g$. Now we suppose that $\Phi(U)$ does not lie in any real $2N$-plane. Then there exist points $p, q_1, \cdots, q_{2N+1}\in U$ such that $\{\overrightarrow{\Phi(p)\Phi(q_j)}\}_{j=1,\cdots,2N+1}$ are $\boldsymbol{R}$-linearly independent. On the other hand, since $\{\overrightarrow{\Psi(p)\Psi(q_j)}\}_{j=1,\cdots,2N+1}$ are $\boldsymbol{R}$-linearly dependent, there exist a $(2N+1)$-tuple of real numbers $(a^1, \cdots, a^{2N+1})\neq 0$

such that $\sum_{j=1}^{2N+1} a^j\overrightarrow{\Psi(p)\Psi(q_j)}=0$. By (2.10) and Proposition 1.1, we have

$$\begin{aligned}0 &= \beta_{r,s}\Big(\sum_{j=1}^{2N+1} a^j\overrightarrow{\Psi(p)\Psi(q_j)}, \sum_{k=1}^{2N+1} a^k\overrightarrow{\Psi(p)\Psi(q_k)}\Big) \\ &= (1/2)\sum_{j,k=1}^{2N+1} a^j a^k \{D_g(p, q_j)+D_g(p, q_k)-D_g(q_j, q_k)\} \\ &= \mathrm{Re}\Big\langle \sum_{j=1}^{2N+1} a^j\overrightarrow{\Phi(p)\Phi(q_j)}, \sum_{k=1}^{2N+1} a^k\overrightarrow{\Phi(p)\Phi(q_k)}\Big\rangle.\end{aligned}$$

Hence $\sum_{j=1}^{2N+1} a^j\overrightarrow{\Phi(p)\Phi(q_j)}=0$, which yields a contradiction. So $\Phi(U)$ lies in some real $2N$-plane. Since p is arbitrary, $\Phi(M)$ spans a finite dimensional complex plane. Thus, by Theorem 2.7, $s=0$ and $\Phi(M)$ lies in a complex N-plane in l^2.

q.e.d.

§3. Real analytic functions on complex manifolds.

A real analytic function on a connected complex manifold M is said to be of *finite rank* if the Kaehler tensor g associated to the form $-2\sqrt{-1}\partial\bar{\partial}f$ is of finite rank. We call the extended signature (r, s) of g the *type* of f. Suppose that f has the power series expansion at a fixed point $p\in M$

$$f = \sum_{I,K=0}^{\infty} b_{I\bar{K}}(z)^{m_I}(\bar{z})^{m_K},$$

where $(z^1, \cdots, z^n)$ is a local coordinate system with the origin p. Then f is of finite rank if and only if the rank of the matrix $(b_{I\bar{K}})_{I,K=0,1,2,\dots}$ is finite. The type of f coincides with the signature of the Hermitian matrix $(b_{I\bar{K}})_{I,K=1,2,3,\dots}$. Let $\Lambda(M)$ denote the set of $\boldsymbol{R}$-linear combinations of real analytic functions $h\bar{k}+k\bar{h}$ (h and k are holomorphic functions on M). Obviously $\Lambda(M)$ forms an associative algebra. If M is compact, $\Lambda(M)$ is nothing but the set of constant functions.

PROPOSITION 3.1. *If $f\in\Lambda(M)$, then f is of finite rank.*

PROOF. By using the identity $h\bar{k}+k\bar{h}=|h+k|^2-|h|^2-|k|^2$, f is expressed as

$$f = \sum_{\sigma=1}^{r} |\phi^\sigma|^2 - \sum_{\sigma=1}^{s} |\phi^{\sigma+r}|^2,$$

where ϕ^σ ($\sigma=1, \cdots, N$) are holomorphic functions. We define a holomorphic mapping Φ of M into $\boldsymbol{C}^{r,s}$ by $\Phi=(\phi^1, \cdots, \phi^N)$. Then $-2\sqrt{-1}\partial\bar{\partial}f=\Phi^*\omega_{r,s}$, this implies that f is of finite rank by Lemma 2.4. q.e.d.

The converse is also true.

THEOREM 3.2. *Let M be a connected and simply connected complex manifold and f a real analytic function of finite rank defined on M. Then for a fixed point $p\in M$, there exist a pair of non-negative integers (r, s) and holomorphic functions $\phi^0, \phi^1, \cdots, \phi^N$ $(r+s=N)$ on M such that*

$$f=\mathrm{Re}(\phi^0)+\sum_{\sigma=1}^{r}|\phi^\sigma|^2-\sum_{\sigma=1}^{s}|\phi^{\sigma+r}|^2,$$

$$\phi^\sigma(p)=0 \qquad (\sigma=1, \cdots, N),$$

where $\phi^1, \cdots, \phi^N$ are C-linearly independent. In addition, if f has the decomposition as above, then (r, s) coincides with the type of f, and ϕ^0 is uniquely determined up to a constant term. And $\Phi=(\phi^1, \cdots, \phi^N)$ are also uniquely determined up to a complex linear transformation in $U(r, s)$.

PROOF. Let f be a real analytic function of type (r, s) $(r+s=N<\infty)$. Since the extended signature of the Kaehler tensor g associated to the form $-2\sqrt{-1}\partial\bar{\partial}f$ is (r, s), by Theorem 2.8 there exists a full holomorphic mapping $\Phi=(\phi^1, \cdots, \phi^N)$ of M into $C^{r,s}$ such that $\Phi^* g_{r,s}=g$ and $\Phi(p)=0$. This implies that

$$D_g(p, q)=\sum_{\sigma=1}^{r}|\phi^\sigma(q)|^2-\sum_{\sigma=1}^{s}|\phi^{\sigma+r}(q)|^2 \qquad (q\in M).$$

Since f is a primitive function of g, there exists a holomorphic function ϕ^0 such that

$$f=\mathrm{Re}(\phi^0)+\sum_{\sigma=1}^{r}|\phi^\sigma|^2-\sum_{\sigma=1}^{s}|\phi^{\sigma+r}|^2. \tag{3.1}$$

Now we prove the uniqueness of this decomposition. Suppose that we have another such decomposition:

$$f=\mathrm{Re}(\theta^0)+\sum_{\sigma=1}^{r'}|\psi^\sigma|^2-\sum_{\sigma=1}^{s'}|\psi^{\sigma+r'}|^2, \tag{3.2}$$

$$\psi^\sigma(p)=0 \qquad (\sigma=1, \cdots, N';\ N'=r'+s').$$

Then we obtain a full holomorphic mapping Ψ of M into $C^{r',s'}$ defined by $\Psi=(\psi^1, \cdots, \psi^{N'})$. Hence $(r', s')=(r, s)$, and by Theorem 2.7, Φ and Ψ differ by a linear transformation of $C^{r,s}$ which preserves the metric $g_{r,s}$. Moreover, since $\Phi(p)=\Psi(p)=0$, by Proposition 1.1, we have

$$\begin{aligned} f(q)-\mathrm{Re}(\phi^0(q)) &= D_{r,s}(0, \Phi(q))=D_g(p, q) \\ &= D_{r,s}(0, \Psi(q))=f(q)-\mathrm{Re}(\psi^0(q)). \end{aligned}$$

Hence $\mathrm{Re}(\phi^0)=\mathrm{Re}(\psi^0)$, which implies that ϕ^0 and ψ^0 differ by a constant term.

q.e.d.

COROLLARY 3.3. *Let M be a simply connected complex manifold. Then $\Lambda(M)$ coincides with the set of real analytic functions of finite rank.*

Next we show some transcendental properties concerned with $\Lambda(M)$. The following lemma is a direct consequence of Proposition 2.10.

LEMMA 3.4. *Let Φ be a holomorphic mapping of a complex manifold M into l^2. If $\Phi(M)$ does not lie in any finite dimensional subspace in l^2. Then $|\Phi|^2 \notin \Lambda(M)$.*

PROOF. Let g be a Kaehler tensor associated with the 2-form $-2\sqrt{-1}\partial\bar{\partial}|\Phi|^2$. Then by Lemma 2.9, Φ preserves the diastasis D_g. Since $\Phi(M)$ does not lie in any finite dimensional subspace, the rank of g is infinite by Proposition 2.10. Hence $|\Phi|^2 \notin \Lambda(M)$. q.e.d.

PROPOSITION 3.5. *Let $p \in M$ be a fixed point of a complex manifold M and let $h^1, \cdots, h^N$ be non-constant holomorphic function on M such that $h^\sigma(p)=0$ $(\sigma=1, \cdots, N)$. Then*

(1) $\exp\left(\sum_{\sigma=1}^{N} |h^\sigma|^2\right) \notin \Lambda(M)$,

(2) $\log\left(1-\sum_{\sigma=1}^{N} |h^\sigma|^2\right) \notin \Lambda(M)$,

(3) $\left(1-\sum_{\sigma=1}^{N} |h^\sigma|^2\right)^{-\alpha} \notin \Lambda(M) \qquad (\alpha>0)$.

PROOF. We obtain the series expansions

$$\exp\left(\sum_{\sigma=1}^{N} |h^\sigma|^2\right)-1 = \sum_{i_1+\cdots+i_N \geq 1} |(h^1)^{i_1}\cdots(h^N)^{i_N}/\sqrt{i_1!\cdots i_N!}|^2, \tag{3.3}$$

$$-\log\left(1-\sum_{\sigma=1}^{N} |h^\sigma|^2\right) = \sum_{i_1+\cdots+i_N \geq 1} \left|\sqrt{\frac{(i_1+\cdots+i_N-1)!}{i_1!\cdots i_N!}}(h^1)^{i_1}\cdots(h^N)^{i_N}\right|^2. \tag{3.4}$$

Since $h^\sigma(p)=0$ $(\sigma=1, \cdots, N)$, those series converge in a sufficiently small neighborhood U of p. Thus we rewrite those series as follows:

$$\exp\left(\sum_{\sigma=1}^{N} |h^\sigma|^2\right)-1 = \sum_{l=1}^{\infty} |\phi^l|^2, \tag{3.5}$$

$$-\log\left(1-\sum_{\sigma=1}^{N} |h^\sigma|^2\right) = \sum_{l=1}^{\infty} |\psi^l|^2, \tag{3.6}$$

where ϕ^l and ψ^l $(l=1, 2, 3, \cdots)$ are holomorphic functions which are determined by the series expansions (3.3) and (3.4) respectively. Then we may define holomorphic mappings $\Phi=(\phi^1, \phi^2, \phi^3, \cdots)$ and $\Psi=(\psi^1, \psi^2, \psi^3, \cdots)$ of U into l^2. Since $\{\phi^l\}$ and $\{\psi^l\}$ have the $\boldsymbol{C}$-linearly independent subsequences $\{(h^1)^m/m!\}_{m=1,2,3,\ldots}$ and $\{(h^1)^m/m\}_{m=1,2,3,\ldots}$ respectively, by Lemma 3.4, we have

$$\exp\Big(\sum_{\sigma=1}^{N}|h^{\sigma}|^{2}\Big)=1+\sum_{l=1}^{\infty}|\phi^{l}|^{2}\notin\Lambda(M),$$

$$\log\Big(1-\sum_{\sigma=1}^{N}|h^{\sigma}|^{2}\Big)=-\sum_{l=1}^{\infty}|\phi^{l}|^{2}\notin\Lambda(M).$$

This proves (1) and (2). Next we prove (3). Using the expression (3.6), we have

$$\begin{aligned}\Big(1-\sum_{\sigma=1}^{N}|h^{\sigma}|^{2}\Big)^{-\alpha}&=\exp\Big\{-\alpha\log\Big(1-\sum_{\sigma=1}^{N}|h^{\sigma}|^{2}\Big)\Big\}\\&=\exp\Big(\alpha\sum_{l=1}^{\infty}|\phi^{l}|^{2}\Big)\\&=1+\sum_{k=1}^{\infty}\sum_{i_{1}+\cdots+i_{k}\geq 1}^{\infty}|(\sqrt{\alpha})^{k}\phi^{i_{1}}\cdots\phi^{i_{k}}|^{2}.\end{aligned}$$

If we arrange the holomorphic functions $\{(\sqrt{\alpha})^{k}\phi^{i_{1}}\cdots\phi^{i_{k}}\}$ as a sequence $\{\tilde{\phi}^{l}\}_{l=1,2,3,\ldots}$, we have the following:

$$\Big(1-\sum_{\sigma=1}^{N}|h^{\sigma}|^{2}\Big)^{-\alpha}=1+\sum_{l=1}^{\infty}|\tilde{\phi}^{l}|^{2}.$$

So we obtain a holomorphic mapping of U into l^{2} defined by $\tilde{\phi}=(\tilde{\phi}^{1},\tilde{\phi}^{2},\tilde{\phi}^{3},\cdots)$. Now we can easily take a subsequence of $\{\tilde{\phi}^{l}\}_{l=1,2,3,\ldots}$ which is linearly independent. Hence, by Lemma 3.4, we have

$$\Big(1-\sum_{\sigma=1}^{N}|h^{\sigma}|^{2}\Big)^{-\alpha}=1+\sum_{l=1}^{\infty}|\tilde{\phi}^{l}|^{2}\notin\Lambda(M).\qquad\text{q.e.d.}$$

In [4], the author showed that any two of complex space forms of different types have no Kaehler submanifolds in common. Now we show that this fact is a corollary of Proposition 3.5. We prepare the following basic lemma:

LEMMA 3.6 ([4; Lemma 1.2]). *Let (M, g) be an analytic Kaehler manifold and $p\in M$ an arbitrarily fixed point. Then a neighborhood U of p is holomorphically and isometrically immersed into a complex space form of holomorphic sectional curvature $2b\neq 0$ if and only if there exist holomorphic functions $\phi^{1},\cdots,\phi^{N}$ defined on U such that*

$$\exp\{bD_{g}(p,q)\}=1+\mathrm{sgn}(b)\sum_{\sigma=1}^{N}|\phi^{\sigma}(q)|^{2}\qquad(q\in U),$$

$$\phi^{\sigma}(p)=0\qquad(\sigma=1,\cdots,N).$$

PROOF. See [4], or this lemma is easily obtained from Theorem 4.1 in the next section. q.e.d.

COROLLARY 3.7 ([4; Proposition 2.1]). *Let (M, g) be a Kaehler n-submanifold* *of* $\boldsymbol{C}^{N,0}$. *Then any open subset of* M *can not be a Kaehler submanifold of* $\boldsymbol{CP}^{N'}(b')$ *for any* N' *and* $b'>0$.

PROOF. We suppose that an open subset U of M is a Kaehler submanifold of $\boldsymbol{CP}^{N'}(2b)$ $(b'=2b)$. For a fixed $p\in U$, $\exp\{bD_g(p, *)\}\in\Lambda(M)$ by Lemma 3.6. Since $b>0$ and M is a Kaehler submanifold of $\boldsymbol{C}^{N,0}$, by Theorem 3.2 there exist holomorphic functions $h^1, \cdots, h^N$ such that

$$bD_g(p, q)=\sum_{\sigma=1}^{N}|h^\sigma(q)|^2 \qquad (q\in U),$$

$$h^\sigma(p)=0 \qquad (\sigma=1, \cdots, N).$$

Hence $\exp\left(\sum_{\sigma=1}^{N}|h^\sigma|^2\right)\in\Lambda(M)$. But this contradicts (1) of Proposition 3.5.

q.e.d.

COROLLARY 3.8 ([4; Proposition 2.2]). *Let* (M, g) *be a Kaehler n-submanifold of* $\boldsymbol{C}^{N,0}$. *Then any open subset of* M *can not be a Kaehler submanifold of* $\boldsymbol{CH}^{N'}(b')$ *for any* N' *and* $b'<0$.

PROOF. We suppose that an open subset U of M is a Kaehler submanifold of $\boldsymbol{CH}^{N'}(2b)$ $(b'=2b)$. Let $p\in U$ be a fixed point. Then by Lemma 3.6, there exist holomorphic functions $h^1, \cdots, h^{N'}$ defined on U such that

$$D_g(p, q)=\frac{1}{b}\log\left(1-\sum_{\sigma=1}^{N'}|h^\sigma(q)|^2\right) \qquad (q\in U),$$

$$h^\sigma(p)=0 \qquad (\sigma=1, \cdots, N').$$

Since M is a Kaehler submanifold of $\boldsymbol{C}^{N,0}$, we have $(1/b)\log(1-\sum_{\sigma=1}^{N'}|h^\sigma|^2)\in\Lambda(M)$. But this contradicts (2) of Proposition 3.5. q.e.d.

COROLLARY 3.9 ([4; Proposition 2.3]). *Let* (M, g) *be a Kaehler n-submanifold of* $\boldsymbol{CH}^{N}(b')$. *Then any open subset of* M *can not be a Kaehler submanifold of* $\boldsymbol{CP}^{N'}(c')$ *for any* N' *and* $c'>0$.

PROOF. Since M is a Kaehler submanifold of $\boldsymbol{CH}^N(2b)$ $(b'=2b)$, for a fixed $p\in M$, there exist holomorphic functions $h^1, \cdots, h^N$ defined on some sufficiently small neighborhood U of p such that

$$D_g(p, q)=\frac{1}{b}\log\left(1-\sum_{\sigma=1}^{N}|h^\sigma(q)|^2\right) \qquad (q\in U), \tag{3.7}$$

$$h^\sigma(p)=0 \qquad (\sigma=1, \cdots, N).$$

Now we assume that U is a Kaehler submanifold of $\boldsymbol{CP}^{N'}(2c)$ $(c'=2c)$. By Lemma 3.6, there exist holomorphic functions $\phi^1, \cdots, \phi^{N'}$ such that

(3.8) $$D_g(p, q)=\frac{1}{c}\log\Bigl(1+\sum_{\sigma=1}^{N'}|\phi^\tau(q)|^2\Bigr) \qquad (q\in U).$$

From (3.7) and (3.8) we have

$$1+\sum_{\tau=1}^{N'}|\phi^\tau|^2=\Bigl(1-\sum_{\sigma=1}^{N}|h^\sigma|^2\Bigr)^{c/b}.$$

Since $c/b<0$, this contradicts (3) of Proposition 3.5. q.e.d

§4. Non-flat indefinite complex space forms.

In this section we consider holomorphic mappings into nonflat indefinite complex space forms preserving Kaehler tensor. Since the metrics of indefinite complex hyperbolic and projective spaces differ by the sign, we may only discuss on the indefinite complex projective spaces. In §1, we defined the indefinite complex projective spaces as open subsets of ordinary complex projective spaces. A holomorphic mapping into $\boldsymbol{CP}_s^N(b)$ ($\subset\boldsymbol{CP}^N$) is called *full* if its image does not lie in any hyperplane of $\boldsymbol{CP}^N$.

THEOREM 4.1. *Let M be a complex manifold with an analytic Kaehler tensor g and $p\in M$ be an arbitrarily fixed point. Then there exists a full holomorphic mapping Φ of some sufficiently small neighborhood U into $\boldsymbol{CP}_s^N(2b)$ such that $\Phi^*g_0=g$ if and only if the function* $\exp\{bD_g(p, *)\}$ *is of type $(N-s, s)$, where g_0 is the metric of $\boldsymbol{CP}_s^N(2b)$.*

PROOF. Now we suppose that there is such a holomorphic mapping expressed by $\Phi=(\phi^0, \cdots, \phi^N)$ in terms of the homogeneous coordinate system. By a suitable motion in $\boldsymbol{CP}_s^N(2b)$ we may put $\Phi(p)=(1, 0, \cdots, 0)$. Then by (1.3) and Proposition 1.1, we have

$$D_g(p, *)=\frac{1}{b}\log\Bigl(1+\sum_{\sigma=1}^{N-s}|\phi^\sigma/\phi^0|^2-\sum_{\sigma=0}^{s-1}|\phi^{N-\sigma}/\phi^0|^2\Bigr),$$

that is,

$$\exp\{bD_g(p, *)\}=1+\sum_{\sigma=1}^{N-s}|\phi^\sigma/\phi^0|^2-\sum_{\sigma=0}^{s-1}|\phi^{N-\sigma}/\phi^0|^2.$$

Since Φ is full, $\{\phi^\sigma/\phi^0\}_{\sigma=1,\dots,N}$ are $\boldsymbol{C}$-linearly independent. Hence Theorem 3.2 implies that $\exp\{bD_g(p, *)\}$ is of type $(N-s, s)$. Conversely, we suppose that $\exp\{bD_g(p, *)\}$ is of type $(N-s, s)$. Then by Theorem 3.2, there exist holomorphic functions $\phi^0, \phi^1, \cdots, \phi^N$ on a simply connected neighborhood U of p such that

$$\exp\{bD_g(p, *)\}=\mathrm{Re}(\phi^0)+\sum_{\sigma=1}^{N-s}|\phi^\sigma|^2-\sum_{\sigma=0}^{s-1}|\phi^{N-\sigma}|^2,$$
$$\phi^\sigma(p)=0 \qquad (\sigma=1, \cdots, N),$$

where $\{\phi^1, \cdots, \phi^N\}$ are $\boldsymbol{C}$-linearly independent. Since $D_g(p, *)$ is the diastasis, we can easily check that $\phi^0=1$. Now if we define a holomorphic mapping of U into $\boldsymbol{CP}_s^N(2b)$ by $\Phi=(1, \phi^1, \cdots, \phi^N)$, then

$$D_g(p, *) = \frac{1}{b}\log\Big(1+\sum_{\sigma=1}^{N-s}|\phi^\sigma|^2-\sum_{\sigma=0}^{s-1}|\phi^{N-\sigma}|^2\Big).$$

By Proposition 1.1, we have $\Phi^*g_0=g$. q.e.d.

THEOREM 4.2. *Let M be a complex manifold with a Kaehler tensor and let Φ_i $(i=1, 2)$ be full holomorphic mappings of M into $\boldsymbol{CP}_{s_i}^{N_i}(2b)$ such that $\Phi_i^*g_i=g$, where g_i is the metric of $\boldsymbol{CP}_{s_i}^{N_i}(2b)$. Then $N_1=N_2$ and $s_1=s_2$. In addition Φ_1 and Φ_2 differ by a motion in $\boldsymbol{CP}_{s_1}^{N_1}(2b)$.*

PROOF. Obviously $N_1=N_2(=N)$ and $s_1=s_2(=s)$ (by Theorem 4.1). By a suitable motion $\boldsymbol{CP}_s^N(2b)$, we may put $\Phi_1(p)=\Phi_2(p)=(1, 0, \cdots, 0)$ for a fixed point $p\in M$. By using the homogeneous coordinate system $(\xi^0, \cdots, \xi^N)$, Φ_i $(i=1, 2)$ are expressed by $\Phi_i=(1, \phi_i^1, \cdots, \phi_i^N)$. Then we have

$$\exp\{bD_g(p, *)\} = 1+\sum_{\sigma=1}^{N-s}|\phi_i^\sigma|^2-\sum_{\sigma=0}^{s-1}|\phi_i^{N-\sigma}|^2 \qquad (i=1, 2),$$

$$\phi_i^\sigma(p) = 0 \qquad (\sigma=1, \cdots, N).$$

Since Φ_i $(i=1, 2)$ are full, by the uniqueness of such decompositions, there exists a matrix $(t_\tau^\sigma)_{\sigma,\tau=1,\cdots,N}\in U(N-s, s)$ such that

$$\phi_2^\sigma = \sum_{\tau=1}^{N} t_\tau^\sigma\phi_1^\tau \qquad (\sigma=1, \cdots, N).$$

The transformation T of $\boldsymbol{CP}_s^N(2b)$, which is defined by $T(q)=(\xi^0(q), \sum_{\sigma=1}^N t_\sigma^1\xi^\sigma(q), \cdots, \sum_{\sigma=1}^N t_\sigma^N\xi^\sigma(q))$ $(q\in \boldsymbol{CP}_s^N(2b))$, is ovbiously an isometry of $\boldsymbol{CP}_s^N(2b)$ and $T\circ\Phi_1 =\Phi_2$. q.e.d.

REMARK. After this paper submitted, the author received the paper [3], in which A. Romero independently showed that full holomorphic and isometric immersions of indefinite Kaehler manifolds into indefinite complex space forms are rigid. This result is also obtained from Theorem 2.7 and Theorem 4.2.

As an application of Theorem 4.2, we consider the following example.

EXAMPLE. The canonical mapping $\boldsymbol{C}^{n+1}\times\boldsymbol{C}^{m+1}\to\boldsymbol{C}^{n+1}\otimes\boldsymbol{C}^{m+1}$ induces the full holomorphic mapping $\Phi: \boldsymbol{CP}^n\times\boldsymbol{CP}^m\to\boldsymbol{CP}^{n+m+nm}$, which is called the *Segre imbedding*. It is easy to see that Φ maps $\boldsymbol{CP}_s^n(c)\times\boldsymbol{CP}_t^m(c)$ into $\boldsymbol{CP}_r^{n+m+nm}(c)$, where $r=s(m-t)+t(n-s)+s+t$ $(0\leqq s\leqq n,\ 0\leqq t\leqq m)$. The restricted mapping $\tilde{\Phi}: \boldsymbol{CP}_s^n(c)\times\boldsymbol{CP}_t^m(c)\to\boldsymbol{CP}_r^{n+m+nm}(c)$ is called the *indefinite Segre imbedding*. Using Proposition 1.1, we can easily show that $\tilde{\Phi}$ is isometric. By Theorem 4.2, $\tilde{\Phi}$ is rigid.

REMARK. The rigidity of the indefinite Segre imbedding was pointed out in T. Ikawa, H. Nakagawa and A. Romero [**2**]. For further properties of the indefinite Segre imbedding, see [**2**].

Now we suppose that M is connected and simply connected. Then Theorem 4.1 and Theorem 4.2 imply that every full holomorphic mapping of an open subset of M into $\boldsymbol{CP}_s^N(2b)$, which preserves a Kaehler tensor, is uniquely extended to the whole of M. So we have the following:

THEOREM 4.3. *Let M be a connected and simply connected complex manifold and $p \in M$ an arbitrary point. Then for a Kaehler tensor g on M, the following two conditions are equivalent:*

(1) *g is analytic and $\exp\{bD_g(p, *)\}$ is of type (r, s).*

(2) *There exists a full holomorphic mapping Φ of M into $\boldsymbol{CP}_s^N(2b)$ $(N=r+s)$ such that $\Phi^* g_0 = g$, where g_0 is the metric of $\boldsymbol{CP}_s^N(2b)$.*

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Masaaki UMEHARA
Institute of Mathematics
University of Tsukuba
Ibaraki, 305
Japan