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Space curves of genus 7 and degree 8 on a non-singular cubic surface with stable normal bundle

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Introduction.

D. Perrin showed in [8] that the normal bundles of curves of degree s^2-1 which are linked to a line by two surfaces of degree s in $\boldsymbol{P}^3$ are semi-stable. In the case of $s=3$, the above curves have genus 7 and degree 8. In this paper, we shall show that the normal bundles of general non-singular curves of genus 7 and degree 8 on a non-singular cubic surface in $\boldsymbol{P}^3$ are stable (Theorem (2.3)).

In §1 we determine divisor classes of non-singular curves of genus 7 and degree 8 on a non-singular cubic surface in $\boldsymbol{P}^3$. In §2 we evaluate the number of isolated singular points of a cubic surface containing the above curve (Lemma (2.2)). This evaluation plays an important role in the proof of Theorem (2.3). In §3 we give examples of non-singular curves of genus 7 and degree 8 with non-stable normal bundle. In §4 we consider a few projectively normal curves on a non-singular cubic surface which are not contained in any quadric surface.

NOTATION. Throughout this paper we shall work over the ground field $\boldsymbol{C}$ and $\boldsymbol{C}^*$ denotes the multiplicative group of $\boldsymbol{C}$. Let X be a non-singular projective variety and let E be a vector bundle on X.

$h^i(X, E) := \dim_{\boldsymbol{C}} H^i(X, E)$; the dimension of $H^i(X, E)$,

$H^i(X, E)^\vee$; the dual vector space of $H^i(X, E)$,

$E^* := \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X)$; the dual vector bundle of E.

Moreover, if C is a curve on a surface S in $\boldsymbol{P}^3$, we use the same symbol C for the corresponding divisor class on S.

I_C; the ideal sheaf of C in $\boldsymbol{P}^3$,

N_C; the normal sheaf of C in $\boldsymbol{P}^3$,

$N_{C/S}$; the normal sheaf of C in S.

§ 1. Curves on a cubic surface.

Let S be a non-singular cubic surface in the projective space $\boldsymbol{P}^3$. Then S is obtained from $\boldsymbol{P}^2$ by blowing-up six points $p_1, \cdots, p_6$ which are not on a conic and no three of which are collinear. We denote by E_i the exceptional curve corresponding to $p_i (i=1, \cdots, 6)$, and $\tilde{L}$ the total transform of a line in $\boldsymbol{P}^2$. Let $e_i \in \operatorname{Pic} S (i=1, \cdots, 6)$ be the divisor class of E_i. Let $l \in \operatorname{Pic} S$ be the divisor class of $\tilde{L}$. Then $\operatorname{Pic} S$ is the free abelian group generated by $l, e_1, \cdots, e_6$ and the intersection pairing on $\operatorname{Pic} S$ is given by

$$l^2 = 1, \quad e_i^2 = -1, \quad l \cdot e_i = 0, \qquad e_i \cdot e_j = 0 \quad \text{for} \quad i \neq j.$$

For any divisor class $D = al - \sum b_i e_i$ where $a, b_1, \cdots, b_6$ are integers, we have

$$d = 3a - \sum b_i,$$
$$p_a(D) = (a-1)(a-2)/2 - \sum b_i(b_i - 1)/2$$

where $d = D \cdot H (H := 3l - \sum e_i$; the divisor class of a hyperplane section) and $p_a(D)$ is the arithmetic genus of D.

Definition (1.1). A divisor class $D = al - \sum b_i e_i$ on S is said to be of type $(a, b_1, b_2, b_3, b_4, b_5, b_6)$.

Lemma (1.2) ([6], p. 405). *Let $D = al - \sum b_i e_i$ be a divisor class on the cubic surface S and suppose that $b_1 \geqq b_2 \geqq \cdots \geqq b_6 > 0$ and $a \geqq b_1 + b_2 + b_5$. Then D is very ample.*

Let C be a non-singular irreducible curve of genus 7 and degree 8 in $\boldsymbol{P}^3$. We have an exact sequence

$$0 \longrightarrow I_C(3) \longrightarrow \mathcal{O}_{\boldsymbol{P}^3}(3) \longrightarrow \mathcal{O}_C(3) \longrightarrow 0.$$

This gives a long exact sequence of cohomology groups:

$$(1.a) \qquad 0 \longrightarrow H^0(\boldsymbol{P}^3, I_C(3)) \longrightarrow H^0(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(3)) \longrightarrow H^0(C, \mathcal{O}_C(3)) \longrightarrow \cdots.$$

Since $\deg \mathcal{O}_C(-3) \otimes w_C < 0$ where w_C is the canonical sheaf of C, we have $h^1(C, \mathcal{O}_C(3)) = 0$. Then $h^0(C, \mathcal{O}_C(3)) = 18$ by the Riemann-Roch theorem. By (1.a) we get

$$h^0(\boldsymbol{P}^3, I_C(3)) \geqq h^0(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(3)) - h^0(C, \mathcal{O}_C(3)) = 20 - 18 = 2.$$

Therefore there are two distinct irreducible cubic surfaces containing C. Let S', S'' be irreducible cubic surfaces containing C. Then the total intersection of S' and S'' is $C \cup L$, where L is a line. From now on we assume S'' is a non-singular cubic surface and replace S by S''. The divisor L on S has one of the following types:

$$(0, -1, 0, 0, 0, 0, 0), \quad (1, 0, 0, 0, 0, 1, 1), \quad (2, 0, 1, 1, 1, 1, 1).$$

On the other hand, the divisor $C+L$ is of type $(9, 3, 3, 3, 3, 3, 3)$. Therefore the divisor C on S is one of the following types:

$$\begin{array}{ll} (9, 4, 3, 3, 3, 3, 3) & \text{if } L \text{ is of type } (0, -1, 0, 0, 0, 0, 0), \\ (8, 3, 3, 3, 3, 2, 2) & \text{if } L \text{ is of type } (1, 0, 0, 0, 0, 1, 1), \\ (7, 3, 2, 2, 2, 2, 2) & \text{if } L \text{ is of type } (2, 0, 1, 1, 1, 1, 1). \end{array}$$

Since any of the other classes in the list can be transformed to the class $(7, 3, 2, 2, 2, 2, 2)$ by a change in the choice of $E_1, \cdots, E_6$, we shall take C to belong to the class $(7, 3, 2, 2, 2, 2, 2)$. We have $\mathcal{O}_S(C)$ is very ample by Lemma (1.2), and $\deg(C\cdot L)=4$.

LEMMA (1.3). *Let C be a non-singular irreducible curve of genus 7 and degree 8 on a non-singular cubic surface S in $\boldsymbol{P}^3$. Then it is nonhyperelliptic.*

PROOF. By the adjunction formula for C on S

$$w_C \cong w_S \otimes \mathcal{O}_S(C) \otimes \mathcal{O}_C \cong \mathcal{O}_S(-H+C) \otimes \mathcal{O}_C. \tag{1.b}$$

Since the divisor class $-H+C$ is of type $(4, 2, 1, 1, 1, 1, 1)$, it is very ample by Lemma (1.2) and so $\mathcal{O}_S(-H+C)\otimes\mathcal{O}_C$ is very ample on C. Therefore w_C is very ample by (1.b). Hence C is nonhyperelliptic.

§ 2. Stability of normal bundle N_C.

Let C be as in § 1. An effective divisor D of type $(7, 3, 2, 2, 2, 2, 2)$ is arithmetically Cohen-Macaulay by Watanabe's result [9] and so $\dim H^0(\boldsymbol{P}^3, I_D(3)) = \deg D - p_a(D)+1=2$. We consider the following exact sequence

$$0 \longrightarrow I_C^2(3) \longrightarrow I_C(3) \longrightarrow N_C^*(3) \longrightarrow 0. \tag{2.a}$$

This gives rise to a homomorphism

$$f : H^0(\boldsymbol{P}^3, I_C(3)) \longrightarrow H^0(C, N_C^*(3)).$$

LEMMA (2.1). *The homomorphism f is isomorphic. Moreover,*

$$\dim H^0(C, N_C^*(3)) = \dim H^0(\boldsymbol{P}^3, I_C(3)) = 2.$$

PROOF. No cubic can be singular at every point of C (see (2.*)). Hence $H^0(\boldsymbol{P}^3, I_C^2(3))=0$, and the homomorphism f is injective. To compute $h^0(C, N_C^*(3))$, we consider the following exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_C \longrightarrow N_{S/\boldsymbol{P}^3}|C \longrightarrow 0.$$

By tensoring $\mathcal{O}_C(3)$ the dual sequence of the above, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow N_C^*(3) \longrightarrow N_{C/S}^*(3) \longrightarrow 0.$$

From the above sequence, we have

$$h^0(C, N_C^*(3)) \leqq h^0(C, \mathcal{O}_C)+h^0(C, N_{C/S}^*(3)). \tag{2.b}$$

On the other hand, $N_{C/S}^*(3) \cong \mathcal{O}_C(L|C)$ where L is $3H-C$. Next, we consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_S(L-C) \longrightarrow \mathcal{O}_S(L) \longrightarrow \mathcal{O}_C(L|C) \longrightarrow 0.$$

We get an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(S, \mathcal{O}_S(L-C)) \longrightarrow H^0(S, \mathcal{O}_S(L)) \longrightarrow H^0(C, \mathcal{O}_C(L|C)) \\ &\longrightarrow H^1(S, \mathcal{O}_S(L-C)) \longrightarrow \cdots. \end{aligned}$$

Since $-(L-C)$ is of type (5, 3, 1, 1, 1, 1, 1), this divisor is very ample by Lemma (1.2). Hence we have $h^i(S, \mathcal{O}_S(L-C))=0$ $(i=0, 1)$ by the Kodaira vanishing theorem. Therefore we have

$$H^0(S, \mathcal{O}_S(L)) \xrightarrow{\sim} H^0(C, \mathcal{O}_C(L|C)). \tag{2.c}$$

Since $h^0(S, \mathcal{O}_S(L))=1$, we get $h^0(C, \mathcal{O}_C(L|C))=1$ by (2.c). Hence $h^0(C, N_C^*(3)) \leqq 1+1=2$ by (2.b), which implies the surjectivity of f.

We shall consider cubic surfaces containing C. By the homomorphism f, any homogeneous polynomial of degree 3 which vanishes on C defines a section s of $N_C^*(3)$. It follows from (2.a) that a section s is zero precisely at the singular points of the corresponding cubic surface S' which lie on C. We have the following geometric lemma.

LEMMA (2.2). *Let C be a general irreducible non-singular curve of genus 7 and degree 8 on a non-singular cubic surface S in $\boldsymbol{P}^3$. Then irreducible cubic surfaces containing C have isolated singular points at most one.*

PROOF. The following fact is well-known:

(2.*) (*every irreducible cubic surface has either only isolated singular points or a singular line.*)

By the above fact we have only to consider a family of irreducible cubic surfaces with either only isolated singular points or a singular line.

Define a subvariety I as follows:

$$I \subset G \times |\mathcal{O}_{P^3}(3)|,$$
$$I = \{(\Lambda, S') | \Lambda \subset S'\},$$

where $\boldsymbol{G}$ is the Grassmannian $\boldsymbol{G}(1, 3)$ of lines in $\boldsymbol{P}^3$ and $|\mathcal{O}_{\boldsymbol{P}^3}(3)|$ is the 19-dimensional projective space consisting of cubic surfaces. We shall consider the following diagram:

$$\begin{tikzcd} & I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \boldsymbol{G} & & |\mathcal{O}_{\boldsymbol{P}^3}(3)|\,. \end{tikzcd}$$

Let Δ be a family of irreducible cubic surfaces with isolated singular points at least 2. By J. W. Bruce and C. T. C. Wall's result (see [**2**]) we can see

$$\operatorname{codim} \Delta = 2 \qquad \text{in} \quad |\mathcal{O}_{\boldsymbol{P}^3}(3)|\,.$$

Put $P_\Lambda = p_2(p_1^{-1}(\Lambda))$ for $\Lambda \in \boldsymbol{G}$. Then P_Λ is a 15-dimensional linear subvariety of $|\mathcal{O}_{\boldsymbol{P}^3}(3)|$. Here we must show the following fact:

(2.d) *For any* $\Lambda \in \boldsymbol{G}$, $\operatorname{codim}(P_\Lambda \cap \Delta) \geqq 2$ *in* P_Λ.

Assume $\operatorname{codim}(P_\Lambda \cap \Delta) \leqq 1$ in P_Λ. For any $\Lambda' \in \boldsymbol{G}$, there is a projective transformation Ψ of $\boldsymbol{P}^3$ such that $\Psi(\Lambda) = \Lambda'$. And also we have $\Psi(S') \in P_{\Lambda'} \cap \Delta$ for any $S' \in P_\Lambda \cap \Delta$. By the aboves,

$$\dim(P_\Lambda \cap \Delta) = \dim(P_{\Lambda'} \cap \Delta) \qquad \text{for any} \quad \Lambda' \in \boldsymbol{G}.$$

Hence $\dim p_2^{-1}(\Delta) \geqq \dim(P_\Lambda \cap \Delta) + \dim \boldsymbol{G} \geqq 14 + 4 = 18$. On the other hand, $p_2^{-1}(S')$ is a finite set of lines on S' for any $S' \in \Delta$ and hence $\dim p_2^{-1}(\Delta) = 17$. This is a contradiction.

Next, we consider a family of irreducible cubic surfaces with a singular line. Let $(x_0 : \cdots : x_3)$ be a system of homogeneous coordinates of $\boldsymbol{P}^3$. Take a line Λ in $\boldsymbol{P}^3$. By a change of coordinates, we may assume that Λ is defined by the equation $x_0 = x_1 = 0$. Let S' be an irreducible cubic surface with the singular line Λ. Then, it is easy to show that S' is defined by the following equation:

$$F_3(x_0, x_1) + x_2 F_2(x_0, x_1) + x_3 G_2(x_0, x_1) = 0$$

where F_i (resp. G_i) is a homogeneous polynomial of degree i in x_0, x_1 ([**2**], p. 252). And the cubic forms $F_3(x_0, x_1) + \cdots + x_3 G_2(x_0, x_1)$ have 10 coefficients. Let F_Λ be a family of irreducible cubic surfaces with the singular line Λ. By the above fact, we have $\dim F_\Lambda \leqq 9$. Let F be a family of irreducible cubic surfaces with a singular line, i.e., $F := \bigcup_{\Lambda \in \boldsymbol{G}} F_\Lambda$. By a similar argument to the one in (2.d), we have

$$\dim F_\Lambda = \dim F_{\Lambda'} \qquad \text{for any} \quad \Lambda, \Lambda' \in \boldsymbol{G}.$$

By the aboves we get

$$\dim F\cap P_\Lambda \leqq \dim F_\Lambda + \dim \boldsymbol{G} \leqq 9+4 = 13.$$

Therefore we obtain

(2.d′) $$\operatorname{codim} F\cap P_\Lambda \geqq 2 \quad \text{in} \quad P_\Lambda.$$

Let D be an effective divisor of type (7, 3, 2, 2, 2, 2, 2), and L be a line of type (2, 0, 1, 1, 1, 1, 1) on S. Then, for any $S'(\neq S)\in H^0(\boldsymbol{P}^3, I_D(3))-\{0\}/\boldsymbol{C}^*$, we have $S\cap S'=D\cup L$. Define a mapping Φ as follows:

$$\begin{array}{ccc} \Phi : |D| & \longrightarrow & P_L^* \\ \Cup & & \Cup \\ D' & \longrightarrow & D'^* \end{array}$$

where P_L^* is a projective space consisting of lines in P_L through the point S of $|\mathcal{O}_{P^3}(3)|$, and $D'^*=H^0(\boldsymbol{P}^3, I_{D'}(3))-\{0\}/\boldsymbol{C}^*$ is a line through the point S. Then Φ is an isomorphism between projective spaces. Since $|D|$ is very ample, there is a non-empty Zariski open set U consisting of non-singular curves in $|D|$. Let U^* be $\Phi(U)$. Then U^* is a non-empty Zariski open subset of P_L^*. Put $\mathrm{Co}^*=\{D'^*\in P_L^* \mid D'^*\cap(\Delta\cup F)\neq\emptyset\}$. Then $\operatorname{codim}\mathrm{Co}^*\geqq 1$ in P_L^* by (2.d) and (2.d′). Hence $P_L^*-\overline{\mathrm{Co}^*}$ is a non-empty Zariski open subset of P_L^*, where $\overline{\mathrm{Co}^*}$ is the Zariski closure of Co^*. Therefore $U^*\cap(P_L^*-\overline{\mathrm{Co}^*})$ is a non-empty Zariski open subset of P_L^*. By the above construction, we have

$$C^*\cap(\Delta\cup F) = \emptyset$$

for any non-singular curve $C\in U\cap\Phi^{-1}(P_L^*-\overline{\mathrm{Co}^*})\subset |D|$, i.e.,

$$S' \notin \Delta\cup F$$

for any $S'\in H^0(\boldsymbol{P}^3, I_C(3))-\{0\}/\boldsymbol{C}^*$. Therefore we get the required result.

Theorem (2.3). *Let C be a general non-singular curve of genus 7 and degree 8 lying on a non-singular cubic surface S in $\boldsymbol{P}^3$. Then the normal bundle of C in $\boldsymbol{P}^3$ is stable.*

Proof. In order to show that N_C is stable, it is sufficient to show that $N_C^*(3)$ has no line subbundle of degree 2 or greater, because we have

$$\begin{aligned}(\deg N_C^*(3))/2 &= \deg \mathcal{O}_C(3)-(\deg N_C)/2 \\ &= \deg \mathcal{O}_C(3)-(\deg w_C\otimes w_{P^3}^*)/2 \\ &= 24-(12+32)/2 = 2.\end{aligned}$$

Let C be as in Lemma (2.2). Since $\mathcal{O}_S(C)$ is very ample, we may assume that

(2.e) $$C\cap L = \{\text{distinct 4 points}\}.$$

Suppose that $N_C^*(3)$ has a line subbundle E of degree 2 or greater. Then the canonical homomorphism

$$g: E \longrightarrow N_{C/S}^*(3)$$

is injective. Therefore we have

$$h^0(C, E) \leqq h^0(C, N_{C/S}^*(3)) = 1.$$

Hence we shall consider the following two cases.

Case (1). Suppose that $h^0(C, E)=0$. Let E' be the quotient line bundle $N_C^*(3)/E$. By the above assumption, the homomorphism

$$\varphi: \quad H^0(C, N_C^*(3)) \longrightarrow H^0(C, E')$$

is injective. Since $h^0(C, N_C^*(3))=2$, the dimension of the vector space $\mathrm{Im}(\varphi)$ is 2. Therefore the dimension of a linear system on C corresponding to the subspace $\mathrm{Im}(\varphi) \subseteqq H^0(C, E')$ is 1. Hence, E' has at least degree 3, since C is neither rational nor hyperelliptic by Lemma (1.3). So we obtain

$$\deg E = \deg(N_C^*(3)) - \deg E' \leqq 4-3 = 1.$$

This is a contradiction.

Case (2). Suppose that $h^0(C, E)=1$. We consider the following diagram:

$$\begin{array}{ccc} H^0(C, E) & \hookrightarrow & H^0(C, N_C^*(3)) \\ & & \wr\uparrow \\ & & H^0(\boldsymbol{P}^3, I_C(3)). \end{array}$$

Take a non-zero section τ of E. It corresponds to an irreducible cubic surface S' containing C. Then we obtain

$$\{\text{zeroes of } \tau\} \subseteqq \{\text{the singular points of } S' \text{ which lie on } C\}.$$

Moreover, by the injective homomorphism $g: E \to N_{C/S}^*(3) \cong \mathcal{O}_C(L \mid C)$ and (2.e) we get

$$\{\text{zeroes of } \tau\} = \{\text{distinct } r \text{ points}\}$$

where $r = \deg E$. Hence we have

$$\deg E \leqq \sharp\{\text{the singular points of } S' \text{ which lie on } C\}$$

where $\sharp\{\ \}$ means the number of elements of sets. By virtue of Lemma (2.2) S' has only isolated singular points at most one. Therefore, we have $\deg E \leqq 1$. This is a contradiction.

§ 3. Examples of non-stable normal bundle.

In this section we shall give examples of curves of genus 7 and degree 8 with non-stable normal bundle.

First we consider a cubic surface S with two double points. See [5] for details. We take 6-points $p_1, \cdots, p_6$ of $\boldsymbol{P}^2$ as follows:

(a) the points p_2, p_3, p_4 lie on the line L_1,

(b) the points p_2, p_5, p_6 lie on the line $L_2 (\neq L_1)$,

(c) the points $\{p_i\}$ are in general position apart from the aboves.

Let X be the non-singular surface obtained by blowing-up of $\boldsymbol{P}^2$ at the points $p_1, \cdots, p_6$. The notation for the generators of $\operatorname{Pic} X \cong \boldsymbol{Z}^{\oplus 7}$, divisors on X and their intersection pairing are same as in § 1. Let K_X be the canonical divisor class of X. Then $|-K_X|$ is base-point free. Hence it defines a morphism $v: X \to \boldsymbol{P}^3$. Put $S = v(X)$. Then the morphism v has the following properties:

(1) $v(\tilde{L}_i) = x_i$ and $x_1 \neq x_2$, where $\tilde{L}_i$ is the strict transform of L_i.

(2) $v: X - \tilde{L}_1 \cup \tilde{L}_2 \to S - \{x_1, x_2\}$ is an isomorphism.

(3) Each point x_i is a double point of S.

Lemma (3.1). *Let C be a non-singular curve on X. If C meets each $\tilde{L}_i$ $(i=1, 2)$ transversely at only one point, then $v(C)$ is a non-singular curve through each singular point x_i $(i=1, 2)$.*

Proof. See [3].

Let D be the divisor class on X of type $(7, 3, 2, 2, 2, 2, 2)$. Then $p_a(D) = 7$ and $D \cdot H = 8$, where H is the anti-canonical divisor class $-K_X$. It is easy to show that there are non-singular curves in $|D|$.

Lemma (3.2). *Let C be a non-singular curve in $|D|$. Then $v(C)$ is a non-singular curve through each singular point x_i $(i=1, 2)$ of S.*

Proof. Since $\tilde{L}_1$ is of type $(1, 0, 1, 1, 1, 0, 0)$ and $\tilde{L}_2$ is of type $(1, 0, 1, 0, 0, 1, 1)$, we have $C \cdot \tilde{L}_i = 1$. Therefore the statement is obvious from Lemma (3.1).

Proposition (3.3). *Let C be as in the above lemma and $N_{v(C)}$ be the normal bundle of $v(C)$ in $\boldsymbol{P}^3$. Then $N_{v(C)}$ is not stable.*

Proof. Since $v(C)$ is a non-singular curve, the normal sheaf $N_{v(C)/S}$ is locally free, and so $N_{v(C)/S}$ is a line subbundle of $N_{v(C)}$. We have an exact sequence

$$0 \longrightarrow v_*(N_{C/X}) \xrightarrow{\phi} N_{v(C)/S} \longrightarrow F \longrightarrow 0.$$

Since ϕ is an isomorphism outside singular points $\{x_1, x_2\}$, we get $\operatorname{Supp} F =$

$\{x_1, x_2\}$. Hence we get an inequality

$$\deg N_{v(C)/S} \geqq \deg v_*(N_{C/X})+2 = C^2+2 = 22.$$

On the other hand, we have

$$(\deg N_{v(C)})/2 = 2(C\cdot H)+p_a(C)-1 = 22.$$

Therefore $\deg N_{v(C)/S} \geqq (\deg N_{v(C)})/2$, i.e., $N_{v(C)}$ is not stable.

§4. Some comments.

Let $D_s^0(g)$ be the first integer d such that there is a non-singular irreducible curve C in $\boldsymbol{P}^3$ of genus g, degree d with stable normal bundle and with $H^1(C, N_C)=0$ ([4]).

First we shall claim $D_s^0(7)=8$. It is known that $8 \leqq D_s^0(7) \leqq 10$ (see [4]). Let C be a general non-singular irreducible curve of genus 7 and degree 8 on a non-singular cubic surface S in $\boldsymbol{P}^3$. From Theorem (2.3) it is sufficient to show that $H^1(C, N_C)=0$. We consider the following exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_C \longrightarrow N_{S/\boldsymbol{P}^3}|C \longrightarrow 0. \tag{4.a}$$

This gives an exact sequence of cohomology groups:

$$\cdots \longrightarrow H^1(N_{C/S}) \longrightarrow H^1(N_C) \longrightarrow H^1(N_{S/\boldsymbol{P}^3}|C) \longrightarrow \cdots.$$

By Serre duality $H^1(N_{C/S}) \cong H^0(N_{C/S}^* \otimes w_C)^\vee$ and $H^1(N_{S/\boldsymbol{P}^3}|C) \cong H^0(N_{S/\boldsymbol{P}^3}^* \otimes w_C)^\vee$. Since $\deg N_{C/S}^* \otimes w_C = \deg \mathcal{O}_C(-1) < 0$ and $\deg N_{S/\boldsymbol{P}^3}^* \otimes w_C = \deg w_C(-3) < 0$, we have $H^1(N_{C/S}) = H^1(N_{S/\boldsymbol{P}^3}|C) = 0$. Hence $H^1(N_C)=0$.

Next we shall consider projectively normal curves on a non-singular cubic surface in $\boldsymbol{P}^3$. Let C be a non-singular curve of genus g and degree d on a non-singular cubic surface such that C is not contained in any quadric surface. Moreover we assume that C is projectively normal and that $g \leqq d$. The second condition is a necessary condition for the stability of N_C. This is due to (4.a). We consider the following exact sequence

$$0 \longrightarrow I_C(2) \longrightarrow \mathcal{O}_{\boldsymbol{P}^3}(2) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0.$$

We have an exact sequence of cohomology groups:

$$0 \longrightarrow H^0(\boldsymbol{P}^3, I_C(2)) \longrightarrow H^0(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(2)) \longrightarrow H^0(C, \mathcal{O}_C(2)) \longrightarrow H^1(\boldsymbol{P}^3, I_C(2)) \longrightarrow \cdots.$$

By hypothesis we obtain $h^0(\boldsymbol{P}^3, I_C(2)) = h^1(\boldsymbol{P}^3, I_C(2)) = 0$, and so $h^0(C, \mathcal{O}_C(2)) = h^0(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(2)) = 10$. By the Riemann-Roch theorem we have $h^0(C, \mathcal{O}_C(2)) = 2d-g+1$, and so $g=2d-9$. Under the condition that $g \leqq d$, we have the following integral solutions

$$(g, d) = (1, 5), (3, 6), (5, 7), (7, 8), (9, 9).$$

But a quintic curve of genus 1 isn't projectively normal. Therefore we shall exclude (1, 5). Conversely, there are such curves with above (g, d) (cf. [**9**]).

From the results of [**3**], [**1**], [**7**] and Theorem (2.3), normal bundles of general (resp. all) projectively normal curves on a non-singular cubic surface with above (g, d) are stable. By the results of [**4**] and $D_s^0(7)=8$, we have $D_s^0(g)=d$ for above (g, d).

g	3	5	7	9
D_s^0	6	7	8	9

Finally, we claim $D_s^0(8) \leqq 10$. It follows from Theorem 2 (e) in [**4**] immediately.

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