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Flow equivalence of translations on compact metric abelian groups

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1. Introduction.

Let Γ be a countable discrete subgroup of the group $\boldsymbol{T}^1=\{z\in\boldsymbol{C}\mid |z|=1\}$. The character group $\Gamma^\wedge$ of Γ is a compact metric abelian group. Let χ_Γ be an element of $\Gamma^\wedge$ determined by $\langle z, \chi_\Gamma\rangle=z$ for $z\in\Gamma$, and $R(\Gamma)$ a homeomorphism of $\Gamma^\wedge$ defined by $R(\Gamma)\chi=\chi\chi_\Gamma$ for $\chi\in\Gamma^\wedge$. $R(\Gamma)$ is called the translation of $\Gamma^\wedge$. The notion of flow equivalence of homeomorphisms was introduced by W. Parry and D. Sullivan [2]. In this article we are concerned with flow equivalence of translations $R(\Gamma)$. This is closely related with stable isomorphism of irrational rotation C^*-algebras (N. Riedel [3], M. Rieffel [4], S. Kawamura and H. Takemoto [1]). We prove the following

THEOREM. *For countable subgroups Γ_1 and Γ_2 of $\boldsymbol{T}^1$, translations $R(\Gamma_1)$ and $R(\Gamma_2)$ are mutually flow equivalent if and only if there exists a positive constant c such that $K_1=cK_2$, where K_j are subgroups of $\boldsymbol{R}$ defined by $K_j=\{x\in\boldsymbol{R}\mid \exp(2\pi ix)\in\Gamma_j\}$, $j=1, 2$.*

As an application we shall give necessary and sufficient conditions for flow equivalence of n-dimensional irrational rotations, adding machine transformations and solenoidal transformations respectively in the following examples.

EXAMPLE 1. Let $\lambda(1), \lambda(2), \cdots, \lambda(n)$ be rationally independent irrational numbers and $\Gamma=\{\exp(2\pi i\sum_{j=1}^n m(j)\lambda(j))\mid m(j)\in\boldsymbol{Z},\ j=1, 2, \cdots, n\}$. The translation $R(\Gamma)$ is topologically conjugate with an n-dimensional irrational rotation $T=T(\lambda(1), \lambda(2), \cdots, \lambda(n))$ defined by $T(x_1, x_2, \cdots, x_n)=(x_1+\lambda(1), x_2+\lambda(2), \cdots, x_n+\lambda(n))$ for $(x_1, x_2, \cdots, x_n)\in\boldsymbol{R}^n/\boldsymbol{Z}^n$. Our theorem implies that irrational rotations $T(\lambda(1), \lambda(2), \cdots, \lambda(n))$ and $T(\mu(1), \mu(2), \cdots, \mu(n))$ are mutually flow equivalent if and only if there exist a positive constant c and a matrix $A\in SL(n+1, \boldsymbol{Z})$ such that

$$(1, \lambda(1), \lambda(2), \cdots, \lambda(n))=c(1, \mu(1), \mu(2), \cdots, \mu(n))A.$$

EXAMPLE 2. Let $r=(r_n)_{n\geq 1}$ be a sequence of integers≥ 2, and $\Gamma=\{\exp(2\pi ik/$

$(r_1 r_2 \cdots r_n)) | k \in \boldsymbol{Z}, n \geqq 1\}$. An infinite direct product $X(r)$ of sets $\{0, 1, \cdots, r_n - 1\}$, $n \geqq 1$, becomes a compact metric group admitting coordinatewise addition with right carry with respect to the product of discrete topologies on coordinate sets. The translation $R(\Gamma)$ is topologically conjugate with an adding machine transformation $T = T(r)$ of $X(r)$ defined by $T(x_n) = (x_n) + (1, 0, 0, \cdots)$ for $(x_n) \in X(r)$. Our theorem implies that adding machine transformations $T((r_n))$ and $T((s_n))$ are mutually flow equivalent if and only if there exist positive integers L and M such that $\{k/(L r_1 r_2 \cdots r_n) | k \in \boldsymbol{Z}, n \geqq 1\} = \{k/(M s_1 s_2 \cdots s_n) | k \in \boldsymbol{Z}, n \geqq 1\}$. For a positive integer n and a prime number p, let $r_n(p)$ be the maximal non-negative integer m such that r_n is divisible by p^m. Then the last condition is also equivalent to the condition that, for any prime number p, $r_n(p) \geqq 1$ for infinitely many n if and only if $s_n(p) \geqq 1$ for infinitely many n and that $\sum_{n=1}^{\infty} r_n(p) = \sum_{n=1}^{\infty} s_n(p)$ except for a finite number of prime numbers p.

EXAMPLE 3. Let λ be an irrational positive number, $r = (r_n)_{n \geqq 1}$ a sequence of integers $\geqq 2$, and $\Gamma = \{\exp(2\pi i k \lambda / (r_1 r_2 \cdots r_n)) | k \in \boldsymbol{Z}, n \geqq 1\}$. A set $X(\lambda, r)$ consisting of all sequences $(x_n)_{n \geqq 0}$ of elements of $\boldsymbol{R}/\boldsymbol{Z}$ such that $x_{n-1} = r_n x_n \pmod 1$ for $n \geqq 1$, becomes a compact subgroup of the infinite dimensional torus $\prod_{n=0}^{\infty} \boldsymbol{R}/\boldsymbol{Z}$. The translation $R(\Gamma)$ is topologically conjugate with a solenoidal transformation $T = T(\lambda, r)$ of $X(\lambda, r)$ defined by

$$T(x_n) = (x_n) + (\lambda, \lambda/r_1, \lambda/(r_1 r_2), \cdots, \lambda/(r_1 r_2 \cdots r_n), \cdots)$$

for $(x_n) \in X(\lambda, r)$. Our theorem implies that solenoidal transformations $T(\lambda, (r_n))$ and $T(\mu, (s_n))$ are mutually flow equivalent if and only if there exist positive integers L and M such that

$$\{k/(L r_1 r_2 \cdots r_n) | \ k \in \boldsymbol{Z}, n \geqq 1\} = \{k/(M s_1 s_2 \cdots s_n) | \ k \in \boldsymbol{Z}, n \geqq 1\}$$

and that $1/(L\lambda) + 1/(M\mu)$ or $1/(L\lambda) - 1/(M\mu)$ is in the above set.

2. Preliminaries.

We recall the definition of a flow built under function. For a homeomorphism T on a compact metric space X and a continuous positive function $f(x)$ defined on X, we denote by (X, f) the set $\{(x, u) \in X \times \boldsymbol{R} | x \in X, 0 \leqq u \leqq f(x)\}$. With the identification of a point $(x, f(x))$ with the point $(Tx, 0)$ for $x \in X$, the set (X, f) becomes a compact metric space under the relative topology induced from the product topology of $X \times \boldsymbol{R}$. Set for $(x, u) \in X \times \boldsymbol{Z}$

$$f(x, n) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=1}^{-n} f(T^{-i} x) & \text{if } n < 0. \end{cases}$$

A flow $((T, f)_t)_{t\in\boldsymbol{R}}$ on the space (X, f) defined by $(T, f)_t(x, u)=(T^n x, u+t-f(x, n))$ if $f(x, n)\leqq u+t<f(x, n+1)$, for $(x, u)\in(X, f)$ and $t\in\boldsymbol{R}$, is called a flow built under function.

Homeomorphisms T and S on compact metric spaces X and Y are said to be flow equivalent if flows $((T, f)_t)_{t\in\boldsymbol{R}}$ and $((S, g)_t)_{t\in\boldsymbol{R}}$ are topologically conjugate for some continuous positive functions $f(x)$ and $g(y)$, that is, if there exists a homeomorphism $\varphi:(X, f)\to(Y, g)$ such that $\varphi(T, f)_t=(S, g)_t\varphi$ for $t\in\boldsymbol{R}$.

Let K be a countable discrete subgroup of $\boldsymbol{R}$ and χ_t for $t\in\boldsymbol{R}$ be a character of K defined by $\langle x, \chi_t\rangle=\exp(2\pi ixt)$ for $x\in K$. Then we obtain a flow $(K_t)_{t\in\boldsymbol{R}}$ acting on the character group $K^\wedge$ of K defined by $K_t y=y\chi_t$ for $y\in K^\wedge$ and $t\in\boldsymbol{R}$.

LEMMA 1. *Let Γ be a countable discrete subgroup of $\boldsymbol{T}^1$ and c a positive number, then the flow $((R(\Gamma), 1/c)_t)_{t\in\boldsymbol{R}}$ is topologically conjugate with the flow $((cK)_t)_{t\in\boldsymbol{R}}$, where K is a subgroup of $\boldsymbol{R}$ defined by $K=\{x\in\boldsymbol{R}\mid\exp(2\pi ix)\in\Gamma\}$.*

PROOF. Set $Y=(cK)^\wedge$ and $Y_0=\{y\in Y\mid\langle c, y\rangle=1\}$, then the closed subgroup Y_0 is a cross section for the flow $((cK)_t)_{t\in\boldsymbol{R}}$ with return time $1/c$. Therefore the flow $((cK)_t)_{t\in\boldsymbol{R}}$ is topologically conjugate with the flow $(((cK)_{1/c}, 1/c)_t)_{t\in\boldsymbol{R}}$ acting on $(Y_0, 1/c)$. Moreover the latter is topologically conjugate with the flow $((R(\Gamma), 1/c)_t)_{t\in\boldsymbol{R}}$ acting on $(\Gamma^\wedge, 1/c)$ under a conjugacy map $\varphi:(Y_0, 1/c)\to(\Gamma^\wedge, 1/c)$ defined by $\varphi(y, u)=(\chi, u)$, where $\chi\in\Gamma^\wedge$ such that $\langle cx, y\rangle=\langle\exp(2\pi ix), \chi\rangle$ for $x\in K$. Q.E.D.

We recall Schwartzman's winding number [5] which plays an important role in the sequel. Let $(F_t)_{t\in\boldsymbol{R}}$ be a flow on a compact metric space X and $C(X, \boldsymbol{T}^1)$ ($C(X, \boldsymbol{R}^1)$) the set of all $\boldsymbol{T}^1$-valued (resp. $\boldsymbol{R}^1$-valued) continuous functions defined on X. We take for a $\xi\in C(X, \boldsymbol{T}^1)$ and a point $x\in X$ a function $\rho_x\in C(\boldsymbol{R}^1, \boldsymbol{R}^1)$ satisfying

$$\xi(F_t x)/\xi(x)=\exp(2\pi i\rho_x(t)) \qquad \text{for } t\in\boldsymbol{R}, \text{ and } \rho_x(0)=0.$$

A winding number $W((F_t), x, \xi)$ is defined by

$$W((F_t), x, \xi)=\lim_{t\to\infty}\rho_x(t)/t$$

if the limit exists.

One can easily see the following properties:

(1) If flows $(F_t)_{t\in\boldsymbol{R}}$ and $(F_t')_{t\in\boldsymbol{R}}$ are topologically conjugate under a conjugacy map $\varphi: X\to X'$ then $W((F_t'), \varphi(x), \xi(\varphi^{-1}\cdot))=W((F_t), x, \xi)$, for $\xi\in C(X, \boldsymbol{T}^1)$ and $x\in X$.

(2) If ξ and $\eta\in C(X, \boldsymbol{T}^1)$ are homotopic with each other, that is, if $\xi(x)/\eta(x)=\exp(2\pi ir(x))$, $x\in X$, for some $r\in C(X, \boldsymbol{R}^1)$, then $W((F_t), x, \xi)=W((F_t), x, \eta)$, $x\in X$.

LEMMA 2. *Let T be a homeomorphism on a compact metric space X and $f(x)$ a positive continuous function on X. If T is uniquely ergodic, that is, if T has a unique invariant probability measure μ, then we have*

$$W(((T, 1)_t), (x, 0)) = \int_X f(x)d\mu(x) \times W(((T, f)_t), (x, 0)), \qquad x \in X.$$

PROOF. Let $\xi \in C((X, f), \boldsymbol{T}^1)$ and $x \in X$ and assume that the limit $\lim_{t\to\infty} \rho_{(x,0)}(t)/t$ exists, where $\rho_{(x,0)} \in C(\boldsymbol{R}^1, \boldsymbol{R}^1)$, $\rho_{(x,0)}(0)=0$ and $\xi((T, f)_t(x, 0))/\xi(x, 0)=\exp(2\pi i \rho_{(x,0)}(t))$ for $t \in \boldsymbol{R}$. We define a homeomorphic map $\varphi: (X, f) \to (X, 1)$ by $\varphi(z, u)=(z, u/f(z))$. Then we have for $n \leqq t < n+1$

$$\begin{aligned}\xi(\varphi^{-1}(T, 1)_t(x, 0))/\xi(\varphi^{-1}(x, 0)) &= \xi((T^n x, (t-n)f(T^n x))/\xi(x, 0) \\ &= \xi((T, f)_s(x, 0))/\xi(x, 0) \\ &= \exp(2\pi i \rho_{(x,0)}(s)),\end{aligned}$$

where $s=f(x, n)+(t-n)f(T^n x)$. Since T is uniquely ergodic.

$$\lim_{t\to\infty} s/n = \lim_{n\to\infty} f(x, n)/n = \int_X f(x)d\mu(x).$$

Hence we have

$$\begin{aligned}W(((T, 1)_t), (x, 0), \xi(\varphi^{-1}\cdot)) &= \lim_{t\to\infty} \rho_{(x,0)}(s)/t \\ &= \lim_{s\to\infty} \rho_{(x,0)}(s)/s \times \lim_{t\to\infty} s/n \times \lim_{t\to\infty} n/t \\ &= W(((T, f)_t), (x, 0), \xi) \times \int_X f(x)d\mu(x).\end{aligned}$$

This implies $W(((T, 1)_t), (x, 0)) = \int_X f(x)d\mu(x) \times W(((T, f)_t), (x, 0))$. Q.E.D.

LEMMA 3. *There exist for any $\xi \in C(\boldsymbol{T}^n, \boldsymbol{T}^1)$ a $r \in C(\boldsymbol{T}^n, \boldsymbol{R}^1)$ and integers $m_1, m_2, \cdots, m_n$ such that*

$$\xi(z_1, z_2, \cdots, z_n) = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \exp(2\pi i r(z_1, z_2, \cdots, z_n))$$

for $(z_1, z_2, \cdots, z_n) \in \boldsymbol{T}^n$.

PROOF. Since $\xi \in C(\boldsymbol{T}^n, \boldsymbol{T}^1)$, we obtain a $t \in C(\boldsymbol{R}^n, \boldsymbol{R}^1)$ such that

$$\xi(\exp(2\pi i u_1), \exp(2\pi i u_2), \cdots, \exp(2\pi i u_n)) = \exp(2\pi i t(u_1, u_2, \cdots, u_n))$$

for $(u_1, u_2, \cdots, u_n) \in \boldsymbol{R}^n$. Then for each $j=1, 2, \cdots, n$, $t(u_1, \cdots, u_j+1, \cdots, u_n) - t(u_1, \cdots, u_j, \cdots, u_n)$ is an integer-valued continuous function, and hence, a constant. We denote it by m_j. Set

$$r(z_1, z_2, \cdots, z_n) = t(u_1, u_2, \cdots, u_n) - \sum_{j=1}^{n} m_j u_j$$

for $(z_1, z_2, \cdots, z_n) \in \boldsymbol{T}^n$, where $z_j=\exp(2\pi i u_j)$, $j=1, 2, \cdots, n$, then r is well-defined,

$r\in C(\boldsymbol{T}^n, \boldsymbol{R}^1)$ and it satisfies the equation of the lemma. Q.E.D.

3. Proof of the theorem.

First we show that for any countable discrete subgroup K of $\boldsymbol{R}$, $W((K_t), y)=K$ for $y\in K^\wedge$. Let for $x\in K$ ξ_x be a function defined by $\xi_x(y)=\langle x, y\rangle$ for $y\in K^\wedge$. Since

$$\begin{aligned}\xi_x(K_t y) &= \langle x, y\chi_t\rangle \\ &= \langle x, y\rangle\langle x, \chi_t\rangle \\ &= \exp(2\pi ixt)\xi_x(y) \qquad \text{for} \quad y\in K^\wedge \text{ and } t\in\boldsymbol{R},\end{aligned}$$

we have $W((K_t), y, \xi_x)=x$. Therefore by the property (2) of Section 2 it suffices to show that each $\xi\in C(K^\wedge, \boldsymbol{T}^1)$ is homotopic with ξ_x for some $x\in K$.

For the group K, we can take a rationally independent sequence $\{\lambda(i)\,|\,i\in I\cup J\}$ of real numbers and sequences $(r(j, n))_{n\geq 1}$, $j\in J$, of integers $\geqq 2$ such that K is generated by $\bigcup_{i\in I}\lambda(i)Z\cup\bigcup_{j\in J}\lambda(j)K(j)$, where I and J are countable sets with $I\cap J=\emptyset$, and $K(j)=\{k/(r(j,1)r(j,2)\cdots r(j,n))\,|\,k\in\boldsymbol{Z}, n=1,2,\cdots\}$, $j\in J$. Then the character group $K^\wedge$ of K is isomorphic with a compact subgroup X of the infinite dimensional torus $\boldsymbol{T}^\infty$ defined by

$$\begin{aligned}X=\{&(z_1, z_2, \cdots, z_{10}, z_{11}, \cdots, z_{20}, z_{21}, \cdots, z_{j0}, z_{j1}, \cdots)\,| \\ &z_i\in\boldsymbol{T}^1 \text{ for } i\in I,\ z_{jn}\in\boldsymbol{T}^1 \text{ for } n=0, 1, \cdots \text{ and } j\in J, \\ &\text{and } z_{jn}^{r(j,n)}=z_{j(n-1)} \text{ for } n=1, 2, \cdots \text{ and } j\in J\}.\end{aligned}$$

Here an isomorphism map $\varphi: K^\wedge\to X$ is given by

$$\varphi x=(z_1, z_2, \cdots, z_{10}, z_{11}, \cdots, z_{20}, z_{21}, \cdots, z_{j0}, z_{j1}, \cdots) \qquad \text{for} \quad x\in K^\wedge,$$

where $z_i=\langle\lambda(i), x\rangle$ for $i\in I$ and $z_{jn}=\langle\lambda(j)/(r(j,1)r(j,2)\cdots r(j,n)), x\rangle$ for $n=0, 1, 2, \cdots$ and $j\in J$. By the Stone-Weierstrass theorem there exists for any $\xi\in C(K^\wedge, \boldsymbol{T}^1)$ a function $\tilde{\xi}\in C(X, \boldsymbol{T}^1)$ whose values depend only on a finite number of coordinates z_i, $i\in I'$ and $z_{jn(j)}$, $j\in J'$, such that $|\xi(\varphi^{-1}z)-\tilde{\xi}(z)|$, $z\in X$, are uniformly small, say

$$|\xi(\varphi^{-1}z)-\tilde{\xi}(z)|<2 \qquad \text{for} \quad z\in X,$$

where I' and J' are finite subsets of I and J respectively and $n(j)$, $j\in J'$, are positive integers. Here we note that for $j\in J$ $z_{j0}, z_{j1}, \cdots, z_{jn(j)-1}$ are determined by $z_{jn(j)}$. Therefore $\tilde{\xi}$ can be considered to be a function on $\boldsymbol{T}^k$, where k is the cardinality of the set $I'\cup J'$. Then by Lemma 3 $\tilde{\xi}\in C(\boldsymbol{T}^k, \boldsymbol{T}^1)$ is homotopic with a function $\prod_{i\in I'}z_i^{m(i)}\times\prod_{j\in J'}z_{jn(j)}^{m(j)}$ for some integers $m(i)$, $i\in I'\cup J'$. Since

$$\begin{aligned}\prod_{i\in I'}z_i^{m(i)}\times\prod_{j\in J'}z_{jn(j)}^{m(j)} &= \prod_{i\in I'}\langle\lambda(i), \chi\rangle^{m(i)}\times\prod_{j\in J'}\langle\lambda(j)/(r(j,1)\cdots r(j,n(j))), \chi\rangle^{m(j)} \\ &= \langle x, \chi\rangle,\end{aligned}$$

where $x=\sum_{i\in I'}m(i)\lambda(i)+\sum_{j\in J'}m(j)\lambda(j)/(r(j,1)\cdots r(j,n(j)))$, $\tilde{\xi}(\varphi\cdot)$ is homotopic with ξ_x. From the above inequality ξ is homotopic with $\tilde{\xi}(\varphi\cdot)$, and hence with ξ_x.

Next we let Γ_1 and Γ_2 be countable discrete subgroups of $\boldsymbol{T}^1$ and assume that there exist positive continuous functions f_1 and f_2 on character groups $\Gamma_1^{\wedge}$ and $\Gamma_2^{\wedge}$ such that the flows $((R(\Gamma_1), f_1)_t)_{t\in\boldsymbol{R}}$ and $((R(\Gamma_2), f_2)_t)_{t\in\boldsymbol{R}}$ are topologically conjugate. Since the translations $R(\Gamma_1)$ and $R(\Gamma_2)$ are uniquely ergodic, from property (1) of Section 2, Lemma 1, Lemma 2 and the above result we have

$$\left(1\Big/\int_{\Gamma_1^{\wedge}}f_1(z)d\mu_1(z)\right)\times K_1=\left(1\Big/\int_{\Gamma_2^{\wedge}}f_2(z)d\mu_2(z)\right)\times K_2,$$

where μ_j is the normalized Haar measure on $\Gamma_j^{\wedge}$, $j=1, 2$.

Conversely if $K_1=cK_2$ for some positive constant c then by Lemma 1 the flow $((R(\Gamma_1), 1)_t)_{t\in\boldsymbol{R}}$ is topologically conjugate with $((R(\Gamma_2), 1/c)_t)_{t\in\boldsymbol{R}}$ and hence $R(\Gamma_1)$ and $R(\Gamma_2)$ are mutually flow equivalent. We complete the proof of the theorem. Q.E.D.

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