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Genus one fibered knots in lens spaces

Dedicated to Professor Junzo Tao on his 60th birthday

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Let M be an orientable closed 3-manifold and K a tame knot in M. We say that K is a fibered knot if $\mathrm{Cl}(M-N(K))$ is a fiber bundle over S^1 whose fiber is an orientable closed surface with one hole and a fiber intersects a meridian of K in a single point, where $N(K)$ is a regular neighborhood of K and $\mathrm{Cl}(\cdot)$ is the closure. In particular, we say that K is a genus one fibered knot if the fiber is a torus with one hole. Hereafter we call it GOF-knot for brevity. Then it was showed in [**3**] and [**6**] by Burde, Zieschang and González-Acũna that S^3 contains just two GOF-knots, those are the trefoil knot and the figure eight knot.

In this paper we will determine GOF-knots in some lens spaces and show existences of lens spaces containing no GOF-knots. In fact, we have the following results.

PROPOSITION 1. *Let m be a non-negative integer and $L(m, 1)$ a lens space of type $(m, 1)$, where $L(0, 1)=S^2\times S^1$ and $L(1, 1)=S^3$. Then $L(m, 1)$ contains at least two* GOF-*knots K_1 and K_2 illustrated in Figure* 1 *with a fiber surface as drawn, where ${}_m\bigcirc$ means a surgery description of $L(m, 1)$. The orientation is given in Figure* 1. *The monodromy of K_1 is presented by $\begin{pmatrix} m+2 & -1 \\ 1 & 0 \end{pmatrix}$ and the*

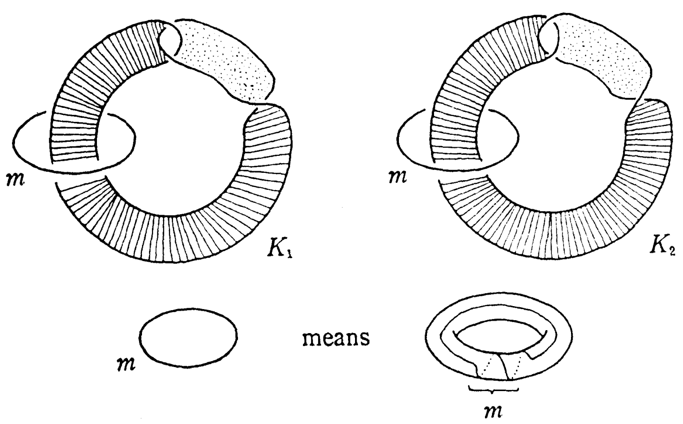

Figure 1.

monodromy of K_2 *is presented by* $\begin{pmatrix} -m+2 & -1 \\ 1 & 0 \end{pmatrix}$. *If* $m=1$, *then* K_1 *is the figure eight knot and* K_2 *is the trefoil knot. If* $m=0$, *then the knots* K_1 *and* K_2 *are equivalent. If* $m>0$, *then the knots* K_1 *and* K_2 *are not equivalent.*

THEOREM 1. $L(0, 1)$ *contains just one* GOF-*knot, which is* K_1 $(=K_2)$ *of Proposition* 1.

THEOREM 2. $L(1, 1)$, $L(2, 1)$ *and* $L(3, 1)$ *contain just two* GOF-*knots, which are* K_1 *and* K_2 *of Proposition* 1.

THEOREM 3. $L(4, 1)$ *contains just three* GOF-*knots, which are* K_1 *and* K_2 *of Proposition* 1 *and* K_3 *illustrated in Figure* 2 *with a fiber surface as drawn. The monodromy of* K_3 *is presented by* $\begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}$.

THEOREM 4. (1) $L(5, 1)$ *contains just two* GOF-*knots, which are* K_1 *and* K_2 *of Proposition* 1.

(2) $L(5, 2)$ *contains just one* GOF-*knot* K *illustrated in Figure* 3 *with a fiber surface as drawn. The monodromy of* K *is presented by* $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$.

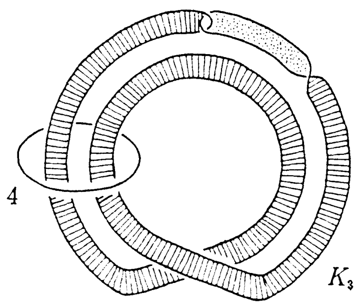

Figure 2.

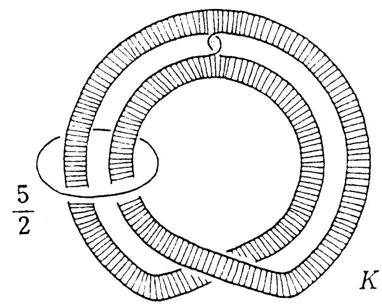

Figure 3.

REMARK 1. By the proof of Theorem 4, we will see that the knot K_2 in $L(5, 1)$ has two meridians. Then we can see that K_2 does not have "property P", where we say that a knot K in a 3-manifold M has "property P", if every non-trivial Dehn surgery along K yields a 3-manifold which is not homeomorphic to M.

THEOREM 5. (1) $L(19, 1)$ *contains just two* GOF-*knots, which are* K_1 *and* K_2 *of Proposition* 1.

(2) $L(19, 3)$ *contains just one* GOF-*knot, which is* K *illustrated in Figure* 4 *with a fiber surface as drawn. The monodromy of* K *is presented by* $\begin{pmatrix} -16 & 3 \\ 5 & -1 \end{pmatrix}$.

(3) $L(19, 2)$, $L(19, 4)$ *and* $L(19, 7)$ *do not contain* GOF-*knots.*

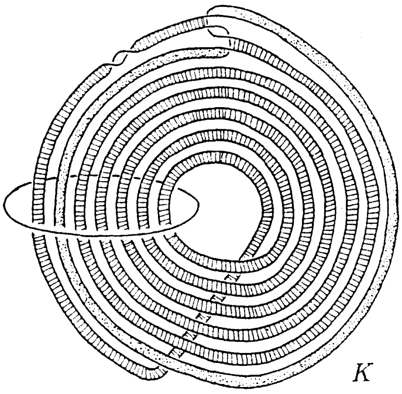

Figure 4.

In § 1 we will express 3-manifolds containing GOF-knots. It will be denoted by $M_{A(a)}$, where A is a matrix in $SL_2(Z)$ and a is an integer. In § 2 we will introduce some sufficient condition following Casson-Gordon in [**4**] for Heegaard splittings to be irreducible. In § 3 we will give a presentation of the fundamental group of $M_{A(a)}$. In § 4 we will describe how to draw a Heegaard diagram of a Heegaard splitting of $M_{A(a)}$. In § 5 we will show some lemmas to prove theorems. Finally in § 6 we will prove theorems.

In this paper we will work in the piecewise linear category. For the standard terms in the three dimensional topology and knot theory, we refer to [8] and [**13**].

REMARK 2. Let K be a fibered knot in an orientable closed 3-manifold M. If K is contained in a 3-ball in M then $\mathrm{Cl}(M-N(K))$ is homeomorphic to $M\#E$, where E is a knot exterior in S^3 and # means a connected sum. Since $\mathrm{Cl}(M-N(K))$ is surface bundle over S^1, it is irreducible and the connected sum is trivial. Then M is homeomorphic to S^3. Thus we see that any fibered knot in M except S^3 is not a local knot. But by Theorem of [7], any fibered knot in an orientable closed 3-manifold with an abelian fundamental group is null homotopic. Hence any GOF-knot in a lens space is null homotopic.

§ 1. Preliminaries.

Let T be a torus with one hole. We denote the orientation preserving homotopy group of T by $\mathcal{H}(T)$. Namely $\mathcal{H}(T)$ is the group of all orientation preserving self-homeomorphisms of T modulo the subgroup consisting of those homeomorphisms which are isotopic to the identity. Then $\mathcal{H}(T)$ is isomorphic to $SL_2(Z)$. Let ϕ be a self-homeomorphism of T and A a matrix in $SL_2(Z)$ with $[\phi]=A$, where $[\phi]$ is the element of $\mathcal{H}(T)$ containing ϕ. Then we denote the T-bundle over S^1 with monodromy ϕ by M_A. Let $u\,(\in\partial M_A)$ be the oriented boundary of a fiber as in Figure 5 and t an oriented simple loop in ∂M_A intersecting u in a single point as in Figure 5. In this paper we often regard oriented loops (with a common base point) as elements of a fundamental group. Then we have $\pi_1(\partial M_A)=(t, u \mid tut^{-1}u^{-1}=1)$. Let V be a solid torus and μ a meridian in ∂V. Let ϕ be a homeomorphism of ∂V to ∂M_A. Then there exist coprime integers a and b with $\phi(\mu)=u^a t^b$ in $\pi_1(\partial M_A)$, and by $M_{A(a,b)}$ we denote the closed 3-manifold obtained from M_A and V by identifying the boundaries by ϕ.

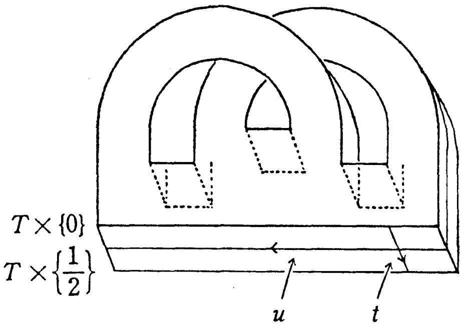

Figure 5.

Since there are infinitely many isotopy classes of the loops which intersect u in a single point, a depends on the choice of t. But by the arguments in §4, 5 and 6, we will see that there are no ambiguities in the proofs of theorems.

PROPOSITION 2. *$M_{A(a,b)}$ has a Heegaard splitting of genus three.*

PROOF. Recall $M_A=T\times[0, 1]/(x, 1)\sim(\phi(x), 0)$, where $x\in T$. Put $H_1=T\times[0, 1/2]$ and $H_2=T\times[1/2, 1]$. Let D be a small disk in T such that $D\cap\partial T=\partial D\cap\partial T$ is a subarc as in Figure 6-1. Put $N=D\times[1/2, 1]$ and put $V_1=H_1\cup N$. Then V_1 is a genus three handlebody. Put $V_2=\mathrm{Cl}(H_2-N)\cup V$. Since $\mathrm{Cl}(H_2-N)\cap V=\mathrm{Cl}(\partial T-\partial D)\times[1/2, 1]$ is a disk, V_2 is also a genus three handlebody. Thus $M_{A(a,b)}=V_1\cup V_2$ is a Heegaard splitting of genus three.

Now, suppose that M contains a GOF-knot. Then M is homeomorphic to some $M_{A(a,b)}$. Moreover from the definition of fibered knots, we may assume $b=1$. Hereafter we denote $M_{A(a,1)}$ by $M_{A(a)}$.

PROPOSITION 3. *$M_{A(a)}$ has a Heegaard splitting of genus two.*

The proof of Proposition 3 is contained in the arguments in §4.

§2. Rectangle condition.

In this section, we introduce some sufficient condition following Casson-Gordon in [4] for Heegaard splittings to be irreducible.

Let F be a genus $g\,(>1)$ orientable closed surface and P and Q two pants embedded in F, where a pants is a disk with two holes. Put $\partial P=l_1\cup l_2\cup l_3$ and $\partial Q=m_1\cup m_2\cup m_3$. We suppose that ∂P and ∂Q intersect transversely. Then we say that P and Q are tight if:

(1) there is no 2-gon B in F such that $\partial B=\alpha\cup\beta$, where α is a subarc of ∂P and β is a subarc of ∂Q,

(2) for any two components l_r and l_s of ∂P and for any two components m_t and m_u of ∂Q there is a rectangle R embedded in P and Q such that Int $R\cap(\partial P\cup\partial Q)=\emptyset$ and the four edges of ∂R are subarcs of l_r, l_s, m_t and m_u.

Let $\mathcal{L}=\{l_1, l_2, \cdots, l_{3g-3}\}$ ($\mathcal{M}=\{m_1, m_2, \cdots, m_{3g-3}\}$ resp.) be a collection of mutually disjoint simple loops on F such that $\mathcal{L}$ ($\mathcal{M}$ resp.) cuts F into $2g-2$ pants $P_1, P_2, \cdots, P_{2g-2}$ ($Q_1, Q_2, \cdots, Q_{2g-2}$ resp.). Then we say that $\mathcal{L}$ and $\mathcal{M}$ are tight if any pair P_i and Q_j are tight.

Let $(V_1, V_2\colon F)$ be a Heegaard splitting of a closed 3-manifold M. We say that $(V_1, V_2\colon F)$ satisfies a rectangle condition if there exist two collections of mutually disjoint simple loops $\mathcal{L}$ and $\mathcal{M}$ on F such that $\mathcal{L}$ and $\mathcal{M}$ are tight and each l_i (m_j resp.) bounds a disk in V_1 (V_2 resp.). We say that $(V_1, V_2\colon F)$ has a cancelling pair if there exists a non-separating disk D_i properly embedded in V_i ($i=1, 2$) such that ∂D_1 and ∂D_2 intersect transversely in a single point. Then Casson-Gordon proved in [**4**]:

THEOREM A. *If $(V_1, V_2\colon F)$ satisfies a rectangle condition, then $(V_1, V_2\colon F)$ has no cancelling pairs.*

Here, we note that the lens spaces (including $S^2\times S^1$ and S^3) are exactly the 3-manifolds with Heegaard splittings of genus one. Bonahon-Otal and Waldhausen proved in [**1**] and [**14**] that any two Heegaard splittings of a lens space L (including $S^2\times S^1$ and S^3) of the same genus (>1) are isotopic each other. Therefore any Heegaard splitting of genus $g\,(>1)$ of L has a cancelling pair. Thus we have:

COROLLARY B. *If an orientable closed 3-manifold M has a Heegaard splitting of genus g (>1) which satisfies a rectangle condition, then M is not homeomorphic to a lens space, $S^2\times S^1$ or S^3.*

§3. A presentation of the fundamental group of $M_{A(a)}$.

Let x^* and y^* be two oriented simple loops in T as in Figure 6-1. And let x and y be two oriented loops based at P as in Figure 6-2. Then we have $\pi_1(T)=(x, y\mid\ )$. Let f be a Dehn twist along x^* and g a Dehn twist along y^*, and let f_* and g_* be induced self-isomorphisms of $\pi_1(T)$. Then $f_*(x)=x$, $f_*(y)$ yx, $g_*(x)=xy$ and $g_*(y)=y$. Hence we have $[f]=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}=X$ and $[g]=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=Y$, since $(1, 0)X=(1, 0)$, $(0, 1)X=(1, 1)$, $(1, 0)Y=(1, 1)$ and $(0, 1)Y=(0, 1)$.

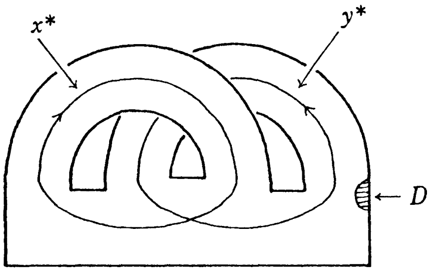

Figure 6-1.

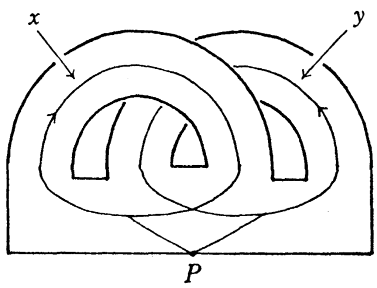

Figure 6-2.

Let ϕ be a self-homeomorphism of T with $[\phi]=A=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ in $SL_2(Z)$. Then $\phi_*(x)$ and $\phi_*(y)$ are words of x and y. Since $SL_2(Z)$ is generated by X and Y, A is decomposed into a word of X and Y. Then by the above correspondence, $[\phi]$ is decomposed into a word of $[f]$ and $[g]$. Therefore the exponent sum of x in the word $\phi_*(x)$ is p, that of y in $\phi_*(x)$ is q, that of x in $\phi_*(y)$ is r and that of y in $\phi_*(y)$ is s. Then we have:

$$\pi_1(M_{A(a)})=\left\langle \begin{matrix} x, y \\ t, u \end{matrix} \middle| \begin{matrix} t^{-1}xt=\phi_*(x), & t^{-1}yt=\phi_*(y) \\ u^a t=1, & u=xyx^{-1}y^{-1} \end{matrix} \right\rangle,$$

$$H_1(M_{A(a)})=(x, y \mid x=x^p y^q,\ y=x^r y^s,\ xy=yx)$$

$$=\left\langle \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} p-1 & q \\ r & s-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\rangle,$$

and

$$|H_1(M_{A(a)})| = \left|\det\begin{pmatrix} p-1 & q \\ r & s-1 \end{pmatrix}\right| = |p+s-2|.$$

§ 4. Heegaard diagrams of $M_{A(a)}$.

In this section we describe how to draw a Heegaard diagram of $M_{A(a)}$ and study conditions for the Heegaard splitting to satisfy a rectangle condition.

Let ϕ be a self-homeomorphism of T with $[\phi]=A$. Recall $M_A=T\times[0, 1]/(x, 1)\sim(\phi(x), 0)$ and $M_{A(a)}=M_A\cup_\psi V$, where ψ is the attaching homeomorphism of ∂V to ∂M_A. Put $V_1=T\times[0, 1/2]$ and $V_2=(T\times[1/2, 1])\cup_\psi V$. Then V_1 is a genus two handlebody. Since $b=1$, $\psi^{-1}(\partial T\times[1/2, 1])$ is an annulus in ∂V whose core is homotopic to the core of V, and V_2 is a genus two handlebody. Thus $M_{A(a)}=V_1\cup V_2$ is a genus two Heegaard splitting of $M_{A(a)}$. Furthermore we can regard V_2 as $T\times[1/2, 1]$. Therefore the attaching homeomorphism h of ∂V_2 to ∂V_1 is as follows:

$h|_{T\times\{1/2\}}: T\times\{1/2\}\to T\times\{1/2\}$ is the identity,

$h|_{T\times\{1\}}: T\times\{1\}\to T\times\{0\}$ is ϕ,

$h|_{\partial T\times[1/2,1]}: \partial T\times[1/2, 1]\to\partial T\times[0, 1/2]$ is a-times Dehn twists along u, where $u=\partial T\times\{1/4\}$ is the oriented simple loop illustrated in Figure 5.

Let α, β and γ be three oriented arcs properly embedded in T as in Figure 7.

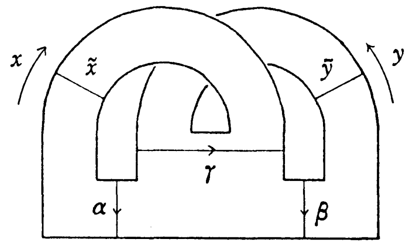

Figure 7.

Since ϕ is orientation preserving, we may assume that $\phi|_{\partial T}$ is the identity. Then $\phi(\alpha)$ is an oriented arc with $\partial\phi(\alpha)=\partial\alpha$, similarly $\partial\phi(\beta)=\partial\beta$ and $\partial\phi(\gamma)=\partial\gamma$. Put $D_\alpha^1=\alpha\times[0, 1/2]$, $D_\beta^1=\beta\times[0, 1/2]$, $D_\gamma^1=\gamma\times[0, 1/2]$, $D_\alpha^2=\alpha\times[1/2, 1]$, $D_\beta^2=\beta\times[1/2, 1]$ and $D_\gamma^2=\gamma\times[1/2, 1]$. Then D_α^i, D_β^i and D_γ^i are three mutually disjoint non-separating disks properly embedded in $V_i\,(i=1, 2)$. Hence a Heegaard diagram of the Heegaard splitting $M_{A(a)}=V_1\cup V_2$ is obtained by drawing $\alpha\times\{1/2\}$, $\beta\times\{1/2\}$, $\gamma\times\{1/2\}$, $\phi(\alpha)\times\{0\}$, $\phi(\beta)\times\{0\}$, $\phi(\gamma)\times\{0\}$ and the image of $(\partial\alpha\cup\partial\beta\cup\partial\gamma)\times[0, 1/2]$ by a-times Dehn twists along u.

We assume that $\phi_*(x)$ and $\phi_*(y)$ are reduced words of x and y. We say that ϕ is of type I if $\phi_*(x)=zwx^{-1}y^{-1}$ $(z=x^{-1}, y$ or $y^{-1})$ and $\phi_*(y)=yxy^{-1}wy^{-1}$, ϕ is of type II if $\phi_*(x)=xyx^{-1}wx^{-1}$ and $\phi_*(y)=zwy^{-1}x^{-1}$ $(z=x, x^{-1}$ or $y^{-1})$ where w is a word of x and y.

LEMMA 1. *If ϕ is of type* I *(type* II *resp.), then the Heegaard splitting (V_1, V_2) of $M_{A(a)}$ satisfies a rectangle condition for any a with $a>0\,(a<0$ resp.).*

PROOF. Put $l_1=\partial D_\alpha^1$, $l_2=\partial D_\beta^1$, $l_3=\partial D_\gamma^1$, $m_1=\partial D_\alpha^2$, $m_2=\partial D_\beta^2$ and $m_3=\partial D_\gamma^2$, and put $\mathcal{L}=\{l_1, l_2, l_3\}$ and $\mathcal{M}=\{m_1, m_2, m_3\}$. Put $F=V_1\cap V_2=\partial V_1=\partial V_2$. Then $\mathcal{L}$ ($\mathcal{M}$ resp.) cuts F into two pants P_1 and P_2 (Q_1 and Q_2 resp.). See Figure 8.

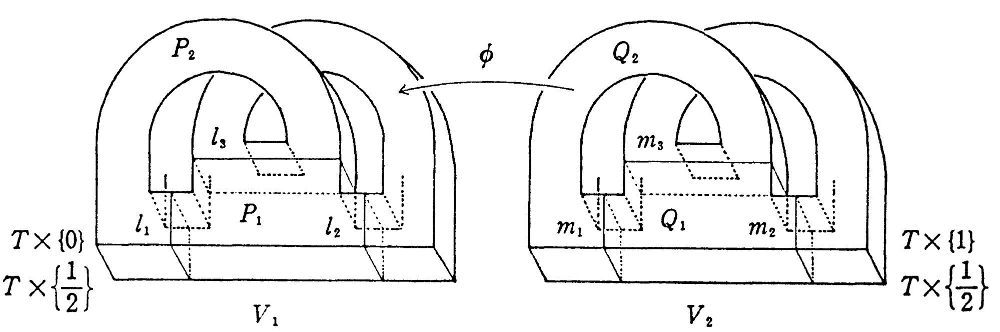

Figure 8.

Let $\tilde{x}$ and $\tilde{y}$ be two arcs properly embedded in T as in Figure 7. By $\phi_*(\alpha)$ ($\phi_*(\beta)$ resp.) we denote the word of x and y obtained by counting the intersections of $\phi(\alpha)$ ($\phi(\beta)$ resp.) and $\tilde{x}\cup\tilde{y}$. Since we may assume that $\phi|_N$ is the identity for a small neighborhood N of ∂T in T, we have $\phi_*(y)=y\phi_*(\alpha)$ from the deformation in Figure 9. Then $\phi_*(\alpha)=y^{-1}\phi_*(y)$. Similarly we have $\phi_*(\beta)=x^{-1}\phi_*(x)$.

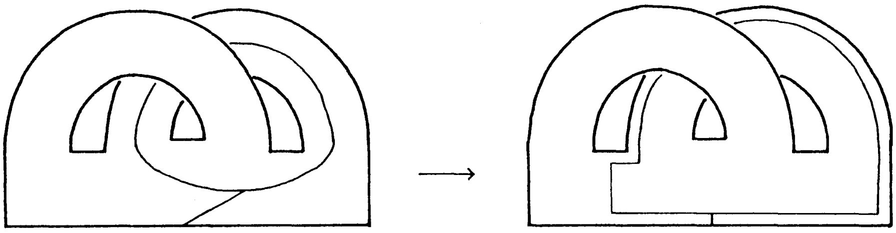

Figure 9.

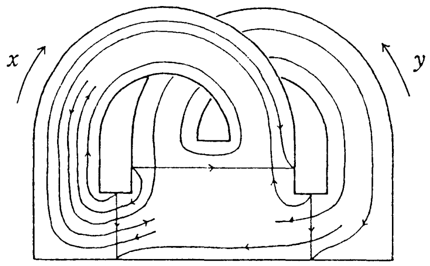

Figure 10.

Suppose that ϕ is of type I. Then $\phi_*(\alpha)=xy^{-1}wy^{-1}$ and $\phi_*(\beta)=x^{-1}zwx^{-1}y^{-1}$ are reduced words, for $z=x^{-1}$, y or y^{-1}. Therefore we may draw the images $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ as in Figure 10. Note that $\phi_*(\gamma)=\phi_*(\alpha)\phi_*(\beta)$. Moreover we may assume that there are no 2-gons in components of $\mathrm{Cl}(T-(\alpha\cup\beta\cup\gamma\cup\phi(\alpha)\cup\phi(\beta)\cup\phi(\gamma)))$. Therefore for $i=1$ or 2 and $j=1$ or 2, P_i and Q_j are tight for any a with $a>0$. Thus (V_1, V_2) satisfies a rectangle condition for any a with $a>0$.

The case when ϕ is of type II, we can similarly see that (V_1, V_2) satisfies a rectangle condition for any a with $a<0$. This completes the proof.

LEMMA 2. *Suppose that $\phi_*(x)$ does not take any form of $xyx^{-1}y^{-1}xw$, $wxy^{-1}x^{-1}$, yxw or $wxyx^{-1}y^{-1}$ and $\phi_*(y)$ does not take any form of xyw, $wyxy^{-1}x^{-1}$, $yxy^{-1}x^{-1}yw$ or $wyx^{-1}y^{-1}$, where w is a word of x and y. Then the Heegaard splitting (V_1, V_2) of $M_{A(a)}$ satisfies a rectangle condition for any a with $|a|>1$.*

PROOF. In the presentation of $\pi_1(M_{A(a)})$ in §3, put $tu=\tilde{t}$. Then we have

$$\pi_1(M_{A(a)})=\left\langle \begin{matrix} x, y \\ \tilde{t}, u \end{matrix} \middle| \begin{matrix} \tilde{t}^{-1}x\tilde{t}=\tilde{\phi}_*(x), & \tilde{t}^{-1}y\tilde{t}=\tilde{\phi}_*(y) \\ u^{a-1}\tilde{t}=1, & u=xyx^{-1}y^{-1} \end{matrix} \right\rangle,$$

where $\tilde{\phi}_*(x)=u^{-1}\phi_*(x)u$, $\tilde{\phi}_*(y)=u^{-1}\phi_*(y)u$ and $\tilde{\phi}$ is a self-homeomorphism of T which is isotopic to ϕ in T but not rel. ∂T. Then by the hypothesis, $\tilde{\phi}$ is of type I. So by Lemma 1, the Heegaard splitting (V_1, V_2) of $M_{A(a)}$ satisfies a rectangle condition for any a with $a-1>0$. Namely $a>1$.

Next, in the presentation of $\pi_1(M_{A(a)})$, put $tu^{-1}=\tilde{t}$. Then we can similarly see that the Heegaard splitting (V_1, V_2) of $M_{A(a)}$ satisfies a rectangle condition for any a with $a<-1$. This completes the proof.

§ 5. Some lemmas.

LEMMA 3. *Let A and B be two matrices in $SL_2(Z)$. Then M_A is homeomorphic to M_B if and only if A is conjugate to B or B^{-1} in $GL_2(Z)$.*

PROOF. This is proved by the same arguments as the proof of Proposition 2 of [**11**]. Note that in the proof of the sufficiency of Theorem of [**10**] there is no need to hypothesize that F is closed and the genus of F is greater than 1.

LEMMA 4 (Lemma of [**3**]). *Let $A=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be a matrix in $SL_2(Z)$ and put $T=p+s$. Then A is conjugate to the matrix taking the form of $\begin{pmatrix} T-s' & q' \\ r' & s' \end{pmatrix}$ in $GL_2(Z)$, where $0\leqq s'\leqq T/2$ if $T>0$, $0\geqq s'\geqq T/2$ if $T<0$ and $s'=0$ if $T=0$.*

PROOF. This is proved by using the following two formulas:

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}=\begin{pmatrix} p-xq & q \\ * & s+xq \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} p+xr & * \\ r & s-xr \end{pmatrix}.$$

For the detail, see Lemma of [**3**].

LEMMA 5. *Put $A=\begin{pmatrix} m+2 & -1 \\ 1 & 0 \end{pmatrix}$, where m is an integer. Then the following holds.*

(1) $\pi_1(M_{A(0)})\cong Z_{|m|}$.

(2) *$\pi_1(M_{A(-1)})$ is cyclic if and only if $m=-5$ or -7. Moreover $\pi_1(M_{A(-1)})\cong Z_5$ if $m=-5$ and $\pi_1(M_{A(-1)})\cong Z_7$ if $m=-7$.*

(3) *$\pi_1(M_{A(1)})$ is not cyclic.*

(4) *For any a with $|a|>1$, $M_{A(a)}$ does not admit a Heegaard splitting of genus one.*

PROOF. Let ϕ be a self-homeomorphism of T with $[\phi]=A$. Recall $X=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $Y=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $[f]=X$, $[g]=Y$, $f_*^{\pm 1}(x)=x$, $f_*^{\pm 1}(y)=yx^{\pm 1}$, $g_*^{\pm 1}(x)=xy^{\pm 1}$ and $g_*^{\pm 1}(y)=y$. We note that $A=XY^{-1}X^{-(m+1)}$ so that $\phi_*=f_*^{-(m+1)}\circ g_*^{-1}\circ f_*$. Then $\phi_*(x)=f_*^{-(m+1)}\circ g_*^{-1}\circ f_*(x)=f_*^{-(m+1)}\circ g_*^{-1}(x)=f_*^{-(m+1)}(xy^{-1})=f_*^{-m}(x^2y^{-1})=\cdots=x^{m+2}y^{-1}$

and $\phi_*(y)=f_*^{-(m+1)}\circ g_*^{-1}\circ f_*(y)=f_*^{-(m+1)}\circ g_*^{-1}(yx)=f_*^{-(m+1)}(yxy^{-1})=\cdots=yxy^{-1}$. Thus we have

$$\pi_1(M_{A(a)})=\left\langle \begin{matrix} x, y \\ t, u \end{matrix} \middle| \begin{matrix} t^{-1}xt=x^{m+2}y^{-1}, & t^{-1}yt=yxy^{-1} \\ u^a t=1, & u=xyx^{-1}y^{-1} \end{matrix} \right\rangle.$$

PROOF OF (1). If $a=0$, then $\pi_1(M_{A(0)})=(x, y\,|\,x=x^{m+2}y^{-1},\ y=yxy^{-1})=(x, y\,|\,x^{m+1}=y,\ x=y)=Z_{|m|}$.

PROOF OF (4). Since $\phi_*(x)=x^{m+2}y^{-1}$ and $\phi_*(y)=yxy^{-1}$, ϕ_* satisfies the hypothesis of Lemma 2, and the Heegaard splitting of (V_1, V_2) of $M_{A(a)}$ satisfies a rectangle condition for any a with $|a|>1$. Then by Corollary B, $M_{A(a)}$ does not admit a Heegaard splitting of genus one.

PROOF OF (2). If $a=-1$, then $\pi_1(M_{A(-1)})=(x, y, t\,|\,t^{-1}xt=x^{m+2}y^{-1},\ t^{-1}yt=yxy^{-1},\ t=xyx^{-1}y^{-1})=(x, y\,|\,yxy^{-1}xy=x^{m+3},\ yxy=xyx)=(x, y\,|\,yxy^{-1}xy=x^{m+3},\ yxyyxy=xyxyxy)=(x, y, a, b\,|\,yxy^{-1}xy=x^{m+3},\ a^2=b^3,\ a=yxy,\ b=xy)=(x, a, b\,|\,x^{-1}bxb^{-1}xb=x^{m+3},\ a^2=b^3,\ a=x^{-1}b^2)=(x, a, b\,|\,bxb^{-1}xb=x^{m+4},\ a^2=b^3,\ x=b^2a^{-1})=(a, b\,|\,b^3a^{-1}ba^{-1}b=(b^2a^{-1})^{m+4},\ a^2=b^3)=(a, b\,|\,aba^{-1}b(ab^{-2})^{m+4}=1,\ a^2=b^3)$. Add the relation $a^2=b^3=1$ to the last presentation. Then we have $\pi_1(M_{A(-1)})/a^2=b^3=1=(a, b\,|\,(ab)^{m+6}=a^2=b^3=1)$. By Satz 3 of [**12**], $\pi_1(M_{A(-1)})/a^2=b^3=1$ is not cyclic for any m with $|m+6|>1$. Therefore $\pi_1(M_{A(-1)})$ is not cyclic for any m with $|m+6|>1$. If $m+6=0$, then $\pi_1(M_{A(-1)})=(a, b\,|\,aba^{-1}b^3a^{-1}b^2a^{-1}=1,\ a^2=b^3)=(a, b\,|\,ab^3a^{-1}=1,\ a^2=b^3)=(a, b\,|\,a^2=b^3=1)\cong Z_2*Z_3$, is not cyclic.

Suppose $|m+6|=1$, then $m=-5$ or -7. If $m=-5$, then $\pi_1(M_{A(-1)})=(a, b\,|\,aba^{-1}b^3a^{-1}=1,\ a^2=b^3)=(a, b\,|\,ab=1,\ a^2=b^3)=Z_5$. If $m=-7$, then $\pi_1(M_{A(-1)})=(a, b\,|\,ab^3a^{-1}b^2a^{-1}=1,\ a^2=b^3)=(a, b\,|\,a=b^5,\ a^2=b^3)=Z_7$. This completes the proof of (2).

PROOF OF (3). If $a=1$, then $\pi_1(M_{A(1)})=(x, y, t\,|\,t^{-1}xt=x^{m+2}y^{-1},\ t^{-1}yt=yxy^{-1},\ t=yxy^{-1}x^{-1})=(x, y\,|\,xyx^{-1}y^{-1}xyxy^{-1}x^{-1}=x^{m+2}y^{-1},\ xyx^{-1}y^{-1}yyxy^{-1}x^{-1}=yxy^{-1})=(x, y\,|\,y^{-1}xyx^{-1}y^{-1}x^{-1}yxy^{-1}x^{m+1}=1,\ yxy^{-1}xyx^{-1}y^{-1}xy^{-1}x^{-1}=1)$.

If m is even, then $\pi_1(M_{A(1)})/x^2=y^2=1=(x, y\,|\,(yx)^5=x^2=y^2=1)$. By Satz 3 of [**12**], $\pi_1(M_{A(1)})$ is not cyclic. If $m=1$, then M_A is the figure eight knot exterior in S^3. By § 10 of [**6**], $\pi_1(M_{A(1)})$ is not cyclic. If $m=-1$, then M_A is the trefoil knot exterior in S^3. By [**3**], $\pi_1(M_{A(1)})$ is not cyclic (cf. [**9**]). Thus hereafter we assume $|m|>2$.

Put $G=\pi_1(M_{A(1)})$, $r_1=y^{-1}xyx^{-1}y^{-1}x^{-1}yxy^{-1}x^{m+1}$ and $r_2=yxy^{-1}xyx^{-1}y^{-1}x\cdot y^{-1}x^{-1}$. Let F be a free group generated by x and y. Let $\gamma: F\to G$ be an onto homomorphism and $\alpha: G\to H$ an abelianaizer, where $H=G/xy=yx=(t\,|\,t^m=1)$.

Suppose $m>2$. Then by noting $\alpha(x)=\alpha(y)=t$ and performing the free calculus, we have the following:

$$\alpha\gamma\frac{\partial r_1}{\partial x}=-1+3t^{-1}-t^{-2}+(1+t+\cdots+t^{m-1}),$$

$$\alpha\gamma\frac{\partial r_1}{\partial y}=1-3t^{-1}+t^{-2},$$

$$\alpha\gamma\frac{\partial r_2}{\partial x}=-t^2+3t-1,$$

$$\alpha\gamma\frac{\partial r_2}{\partial y}=t^2-3t+1.$$

For the free calculus, we refer Ch. VII of [5]. Let A be the Alexander matrix of G, then

$$A=\begin{pmatrix}-1+3t^{-1}-t^{-2}+(1+t+\cdots+t^{m-1}) & 1-3t^{-1}+t^{-2}\\ -t^2+3t-1 & t^2-3t+1\end{pmatrix}$$

$$\sim\begin{pmatrix}1+t+\cdots+t^{m-1} & 0\\ 0 & t^2-3t+1\end{pmatrix}.$$

Let B be the Alexander matrix of H, then $B=(1+t+\cdots+t^{m-1})$. Let $E_k(A)$ ($E_k(B)$ resp.) be the kth elementary ideal of A (B resp.). Then $E_0(A)=E_0(B)$ $=(1+t+\cdots+t^{m-1})$ and $E_k(A)=E_k(B)=JH$ ($k>1$), where J is the integer ring and JH is the group ring on H. Furthermore $E_1(B)=JH$.

From now we will show that $E_1(A)\neq E_1(B)$. Suppose that $E_1(A)=E_1(B)$ $(=JH)$. Then since $E_1(A)=(1+t+\cdots+t^{m-1},\ t^2-3t+1)$, there exist two polynomials $p(t)$ and $q(t)$ in JH such that $p(t)(1+t+\cdots+t^{m-1})+q(t)(t^2-3t+1)=1$. Since $t^m=1$, we can put $p(t)=p_0+p_1t+\cdots+p_{m-1}t^{m-1}$ and $q(t)=q_0+q_1t+\cdots+q_{m-1}t^{m-1}$. Then by comparing the coefficients, we have the following equalities:

$$p(1)+q_{m-2}-3q_{m-1}+q_0=1,$$

$$p(1)+q_{m-1}-3q_0+q_1=0 \quad \text{and}$$

$$p(1)+q_{i-2}-3q_{i-1}+q_i=0 \quad (2\leqq i\leqq m-1).$$

Put $b_i=q_i-p(1)$ $(0\leqq i\leqq m-1)$. Then we have:

$$b_0=3b_{m-1}-b_{m-2}+1,$$

$$b_1=3b_0-b_{m-1} \quad \text{and}$$

$$b_i=3b_{i-1}-b_{i-2} \quad (2\leqq i\leqq m-1).$$

Let $\{c_n\}$ be the sequence defined inductively as follows: $c_1=1$, $c_2=3$ and $c_{n+2}=3c_{n+1}-c_n$.

Then the following equalities are easily checked inductively:

$$b_i=c_ib_1-c_{i-1}b_0 \quad (2\leqq i\leqq m-1),$$

$$b_0=c_mb_1-c_{m-1}b_0+1 \quad \text{and}$$

$$b_1=-c_{m-1}b_1+(3+c_{m-2})b_0.$$

From the last two equalities we have

$$b_0+\frac{c_{m-1}^2-c_{m-2}c_m-c_{m+1}}{1+c_{m-1}}b_0=1.$$

By the way, $c_n^2-c_{n-1}c_{n+1}=c_n^2-c_{n-1}(3c_n-c_{n-1})=c_n(c_n-3c_{n-1})+c_{n-1}^2=c_{n-1}^2-c_{n-2}c_n=\cdots=c_2^2-c_1c_3=1$. Then we have $(1-(c_{m+1}-1)/(c_{m-1}+1))b_0=1$. Since $c_{m+1}-1-2(c_{m-1}+1)=3(c_m-c_{m-1}-1)\geqq 3(c_2-c_1-1)>0$, we have $(c_{m+1}-1)/(c_{m-1}+1)>2$ and $|1-(c_{m+1}-1)/(c_{m-1}+1)|>1$.

Therefore $|b_0|<1$, and this contradicts to that b_0 is an integer $(\neq 0)$.

If $m<-2$, then $\alpha\gamma(\partial r_1/\partial x)=-1+3t^{-1}-t^{-2}-(1+t+\cdots+t^l)$, where $l=-m$. We have a contradiction similarly. Therefore $E_1(A)\neq E_1(B)$ and G is not cyclic.

This completes the proof of (3).

§6. Proofs of Proposition 1 and Theorems 1, 2, 3, 4 and 5.

PROOF OF PROPOSITION 1. Let m be an integer and put $A=\begin{pmatrix} m+2 & -1 \\ 1 & 0 \end{pmatrix}$. From the proof of Lemma 5, we have $\pi_1(M_{A(0)})=(x,\ y,\ t\,|\,t^{-1}xt=x^{m+2}y^{-1},\ t^{-1}yt=yxy^{-1},\ t=1)=Z_{|m|}$. Let $M_{A(0)}=V_1\cup V_2$ be the Heegaard splitting given in §4. Then by using the method in §4 and the above presentation, we can draw a Heegaard diagram of the Heegaard splitting of (V_1, V_2) and can see that $M_{A(0)}$ is homeomorphic to $L(|m|, 1)$. Furthermore by drawing pictures, we have Figure 1, and we can see that the core of V is the knot K_1 if $m\geqq 0$ and is the knot K_2 if $m\leqq 0$, where V is a solid torus and $M_{A(0)}=M_A\cup V$. This is a routine matter and we omit the detail.

If $m\neq 0$, then $\begin{pmatrix} m+2 & -1 \\ 1 & 0 \end{pmatrix}$ is not conjugate to $\begin{pmatrix} -m+2 & -1 \\ 1 & 0 \end{pmatrix}$ in $GL_2(Z)$. By Lemma 3 the complement of K_1 in $L(m, 1)$ is not homeomorphic to the comlement of K_2 in $L(-m, 1)$. Hence the knots K_1 and K_2 are not equivalent.

Now we prove Theorems 1, 2, 3, 4 and 5. Through the following proofs we assume that $M_{A(a)}=M_A\cup V$ and C is the core of V.

PROOF OF THEOREM 1. Put $A=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and suppose that $M_{A(a)}$ is homeomorphic to $S^2\times S^1$. Since $H_1(S^2\times S^1)\cong Z$, we have $p+s-2=0$ (see §3). Then by Lemma 4 we may put $A=\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ $(n\geqq 0)$. If $A=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, then $H_1(M_{A(a)})=\left(x,\ y\,\middle|\,\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)=Z+Z_n$. Hence $n=1$ and $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, after all we may assume $A=\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. Then by Lemma 5, $a=0$ and C is the knot K_1 $(=K_2)$ of Proposition 1.

PROOF OF THEOREM 2. Put $A=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

The case when $M_{A(a)}$ is homeomorphic to S^3 has been proved in [3] and [6].

The case when $M_{A(a)}$ is homeomorphic to $L(2, 1)$. From $|p+s-2|=2$, $p+s=4$ or 0. Then by Lemma 4, we may put $A=\begin{pmatrix}4 & q\\ r & 0\end{pmatrix}$, $\begin{pmatrix}3 & q\\ r & 1\end{pmatrix}$, $\begin{pmatrix}2 & q\\ r & 2\end{pmatrix}$ or $\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$. In any case, $|qr|$ is 1 or a prime number. By two formulas in the proof of Lemma 4, we may assume $A=\begin{pmatrix}4 & -1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Then by Lemma 5, $a=0$ and C is the knot K_1 or K_2 of Proposition 1.

The case when $M_{A(a)}$ is homeomorphic to $L(3, 1)$. From $|p+s-2|=3$, $p+s=5$ or -1 and we may put $A=\begin{pmatrix}5 & q\\ r & 0\end{pmatrix}$, $\begin{pmatrix}4 & q\\ r & 1\end{pmatrix}$, $\begin{pmatrix}3 & q\\ r & 2\end{pmatrix}$ or $\begin{pmatrix}-1 & q\\ r & 0\end{pmatrix}$. By the same reason as the above, we may assume $A=\begin{pmatrix}5 & -1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$. Then by Lemma 5, $a=0$ and C is the knot K_1 or K_2 of Proposition 1.

PROOF OF THEOREM 3. Put $A=\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ and suppose that $M_{A(a)}$ is homeomorphic to $L(4, 1)$. From $|p+s-2|=4$, $p+s=6$ or -2.

Case (1): $p+s=6$.

In this case, by Lemma 4 we may put $A=\begin{pmatrix}6 & q\\ r & 0\end{pmatrix}$, $\begin{pmatrix}5 & q\\ r & 1\end{pmatrix}$, $\begin{pmatrix}4 & q\\ r & 2\end{pmatrix}$ or $\begin{pmatrix}3 & q\\ r & 3\end{pmatrix}$. If $A=\begin{pmatrix}4 & q\\ r & 2\end{pmatrix}$, then $|qr|=7$ and A is conjugate to $\begin{pmatrix}6 & -1\\ 1 & 0\end{pmatrix}$. If $A=\begin{pmatrix}3 & q\\ r & 3\end{pmatrix}$, then $|qr|=8$ and A is cojugate to $\begin{pmatrix}6 & -1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$. Hence we may assume $A=\begin{pmatrix}6 & -1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$. If $A=\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$, then $H_1(M_{A(a)})=\left(x, y\,\middle|\,\begin{pmatrix}4 & 2\\ 2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\right)\cong Z_2+Z_2\neq Z_4$. This is a contradiction. Then by Lemma 5, $a=0$ and C is the knot K_1 of Proposition 1.

Case (2): $p+s=-2$.

In this case we may put $A=\begin{pmatrix}-2 & -1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}-1 & n\\ 0 & -1\end{pmatrix}$ $(n\geqq 0)$. Suppose $A=\begin{pmatrix}-1 & n\\ 0 & -1\end{pmatrix}$ and that ϕ be the self-homeomorphism of T with $[\phi]=A$. Since $A=X^2Y^{-1}X^2Y^{-n-1}$, $\phi_*(x)=g_*^{-n-1}\circ f_*^2\circ g_*^{-1}\circ f_*^2(x)=y^{n+1}x^{-1}y^{-1}$ and $\phi_*(y)=yxy^{-1}x^{-1}y^{-1}$. Then we have $\pi_1(M_{A(a)})=(x, y, t, u\,|\,t^{-1}xt=y^{n+1}x^{-1}y^{-1}, t^{-1}yt=yxy^{-1}x^{-1}y^{-1}, u^a t=1, u=xyx^{-1}y^{-1})$. Then by Lemma 1 and Corollary B, $M_{A(a)}$ is not homeomorphic to $L(4, 1)$ for any $a>0$, for ϕ is of type I. Thus we have $a\leqq 0$. Put $tu^{-1}=\tilde{t}$. Then we have $\pi_1(M_{A(a)})=(x, y, \tilde{t}, u\,|\,\tilde{t}^{-1}x\tilde{t}=\tilde{\phi}_*(x), \tilde{t}^{-1}y\tilde{t}=\tilde{\phi}_*(y), u^{a+1}t=1, u=xyx^{-1}y^{-1})$, where $\tilde{\phi}_*(x)=uy^{n+1}x^{-1}y^{-1}u^{-1}=xyx^{-1}y^{n-1}x^{-1}$ and $\tilde{\phi}_*(y)=uyxy^{-1}x^{-1}y^{-1}u^{-1}=xy^{-1}x^{-1}$. Then $\tilde{\phi}$ is of type II. By Lemma 1 and Corollary B, $M_{A(a)}$ is not homeomorphic to $L(4, 1)$ for any $a+1<0$. Therefore we have $-1\leqq a\leqq 0$. If $a=0$, then $\pi_1(M_{A(0)})=(x, y\,|\,y^{n+1}x^{-1}y^{-1}x^{-1}=1, xy^{-1}x^{-1}y^{-1}=1)=(x, y\,|\,y^{n+2}x^{-2}=1, x^{-1}yxy=1)$. Then $\pi_1(M_{A(0)})/x^2=1=(x, y\,|\,y^{n+2}=x^2=(xy)^2=1)$. Since $n+2\geqq 2$ it is not cyclic by [12]. If $a=-1$, then $\pi_1(M_{A(-1)})=(x, y\,|\,yx^{-1}y^{n-1}x^{-1}=1, xy^{-1}x^{-1}y^{-1}=1)=(x, y\,|\,y^{n-2}x^{-2}=1, x^{-1}yxy=1)$. Then $\pi_1(M_{A(a)})/x^2=1=(x, y\,|\,y^{n-2}=x^2=(xy)^2=1)$. Since $n-2\geqq -2$ it is cyclic if and only if $n=1$ or 3 by [12]. If $n=1$ or 3, then $\pi_1(M_{A(a)})\cong Z_4$. Hence we may assume $A=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}-1 & 3\\ 0 & -1\end{pmatrix}$. Since $\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$ is conjugate to $\begin{pmatrix}-2 & -1\\ 1 & 0\end{pmatrix}$, after all we have $A=\begin{pmatrix}-2 & -1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}-1 & 3\\ 0 & -1\end{pmatrix}$. If $A=\begin{pmatrix}-2 & -1\\ 1 & 0\end{pmatrix}$, then by Lemma 5 C is the knot K_2 of Proposition 1. If $A=\begin{pmatrix}-1 & 3\\ 0 & -1\end{pmatrix}$, then by drawing pictures we can see that C is the knot K_3 illustrated in Figure 2. Note that $\begin{pmatrix}-1 & 3\\ 0 & -1\end{pmatrix}$ is not conjugate to $\begin{pmatrix}-2 & -1\\ 1 & 0\end{pmatrix}$ in $GL_2(Z)$.

PROOF OF THEOREM 4. Put $A=\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ and suppose that $M_{A(a)}$ is homeomorphic to $L(5, 1)$ or $L(5, 2)$. Since $|p+s-2|=5$, $p+s=-3$ or 7.

Case (1): $p+s=-3$.

In this case, by Lemma 4 we may put $A=\begin{pmatrix}-3 & q\\ r & 0\end{pmatrix}$ or $\begin{pmatrix}-2 & q\\ r & -1\end{pmatrix}$. In both cases $|qr|=1$, and we may assume $A=\begin{pmatrix}-3 & -1\\ 1 & 0\end{pmatrix}$. Then by Lemma 5, $\pi_1(M_{A(a)})$ is cyclic if and only if $a=0$ or -1. If $a=0$, then C is the knot K_2 of Proposition 1. If $a=-1$, then by drawing pictures we can see that C is the knot K_2 of Proposition 1 again. Hence we can see that K_2 has two meridians.

Case (2): $p+s=7$.

In this case we may put $A=\begin{pmatrix}7 & q\\ r & 0\end{pmatrix}$, $\begin{pmatrix}6 & q\\ r & 1\end{pmatrix}$, $\begin{pmatrix}5 & q\\ r & 2\end{pmatrix}$ or $\begin{pmatrix}4 & q\\ r & 3\end{pmatrix}$. Then by using two formulas in the proof of Lemma 4 we may assume $A=\begin{pmatrix}7 & -1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}5 & 3\\ 3 & 2\end{pmatrix}$. If $A=\begin{pmatrix}7 & -1\\ 1 & 0\end{pmatrix}$, then by Lemma 5 C is the knot K_1 of Proposition 1.

Suppose $A=\begin{pmatrix}5 & 3\\ 3 & 2\end{pmatrix}$. Since $A=YXYX$, $\phi_*(x)=f_*\circ g_*\circ f_*\circ g_*(x)=xyxyx^2yx$ and $\phi_*(y)=yx^2yx$. Then we have $\pi_1(M_{A(a)})=(x, y, t, u \mid t^{-1}xt=xyxyx^2yx, t^{-1}yt=yx^2yx, u^at=1, u=xyx^{-1}y^{-1})$. Since ϕ satisfies the hypothesis of Lemma 2, we have $a=-1$, 0 or 1 by Corollary B. If $a=0$, then $\pi_1(M_{A(0)})=(x, y \mid yxyx^2yx=1, x^2yx=1)=(x, y \mid yxy=1, x^2yx=1)\cong Z_5$. In this case, by drawing pictures we can see that $M_{A(0)}$ is homeomorphic to $L(5, 2)$ and C is the knot K illustrated in Figure 3.

If $a=-1$, then $\pi_1(M_{A(-1)})=(x, y \mid yxy^{-1}xyx^{-1}y^{-1}=xyxyx^2yx, yxy^{-1}x^{-1}\cdot yxyx^{-1}y^{-1}=yx^2yx)=(x, y, a \mid ay^{-1}xya^{-1}=xa^2xa, y^{-1}x^{-1}aya^{-1}=xa, yx=a)=(y, a \mid ay^{-2}aya^{-1}=y^{-1}a^3y^{-1}a^2, y^{-1}a^{-1}yaya^{-1}=y^{-1}a^2)=(y, a \mid ay^{-2}ay=y^{-1}a^3y^{-1}a^3, yay=a^4)=(y, a \mid ay^{-3}a^4=y^{-1}a^3y^{-1}a^3, yay=a^4)=(y, a \mid ay^{-4}=y^{-1}a^3y^{-1}a^{-1}y^{-1}, yay=a^4)=(y, a \mid y^{-4}=a^{-1}y^{-1}a^{-1}, ayay=a^5)=(y, a \mid a^5=y^5=(ay)^2)$ is not cyclic by [**12**].

If $a=1$, then $\pi_1(M_{A(1)})=(x, y \mid xyx^{-1}y^{-1}xyxy^{-1}x^{-1}=xyxyx^2yx, xyx^{-1}yxy^{-1}x^{-1}=yx^2yx)=(x, y \mid xyx=yx^2yx^2yx^2y, xyx^{-1}yx=yx^2yx^2y)=(x, y, a \mid xyx=a^3y, xyx^{-1}yx=a^2y, yx^2=a)=(x, y \mid xax=a^4, xax^{-3}ax=a^3)=(x, a \mid a^4=xax, x^4=axa)=(x, a \mid a^5=x^5=(ax)^2)$ is not cyclic by [**12**], This completes the proof.

PROOF OF THEOREM 5. Put $A=\begin{pmatrix} p & q \\ r & s\end{pmatrix}$ and suppose that $M_{A(a)}$ is homeomorphic to a lens space whose fundamental group is a cyclic group of the order 19. Since $|p+s-2|=19$, $p+s=21$ or -17.

Case (1): $p+s=21$.

In this case we may put $A=\begin{pmatrix} 21 & q \\ r & 0\end{pmatrix}$, $\begin{pmatrix} 20 & q \\ r & 1\end{pmatrix}$, $\begin{pmatrix} 19 & q \\ r & 2\end{pmatrix}$, $\begin{pmatrix} 18 & q \\ r & 3\end{pmatrix}$, $\begin{pmatrix} 17 & q \\ r & 4\end{pmatrix}$, $\begin{pmatrix} 16 & q \\ r & 5\end{pmatrix}$, $\begin{pmatrix} 15 & q \\ r & 6\end{pmatrix}$, $\begin{pmatrix} 14 & q \\ r & 7\end{pmatrix}$, $\begin{pmatrix} 13 & q \\ r & 8\end{pmatrix}$, $\begin{pmatrix} 12 & q \\ r & 9\end{pmatrix}$ or $\begin{pmatrix} 11 & q \\ r & 10\end{pmatrix}$. In any case, $|qr|$ is 1 or a prime number. By two formulas in the proof of Lemma 4, we may assume $A=\begin{pmatrix} 21 & -1 \\ 1 & 0\end{pmatrix}$. Then by Lemma 5, $a=0$ and C is the knot K_1 of Proposition 1.

Case (2): $p+s=-17$.

In this case we may put $A=\begin{pmatrix} -17 & q \\ r & 0\end{pmatrix}$, $\begin{pmatrix} -16 & q \\ r & -1\end{pmatrix}$, $\begin{pmatrix} -15 & q \\ r & -2\end{pmatrix}$, $\begin{pmatrix} -14 & q \\ r & -3\end{pmatrix}$, $\begin{pmatrix} -13 & q \\ r & -4\end{pmatrix}$, $\begin{pmatrix} -12 & q \\ r & -5\end{pmatrix}$, $\begin{pmatrix} -11 & q \\ r & -6\end{pmatrix}$, $\begin{pmatrix} -10 & q \\ r & -7\end{pmatrix}$ or $\begin{pmatrix} -9 & q \\ r & -8\end{pmatrix}$. Then by two formulas in the proof of Lemma 4, we may assume $A=\begin{pmatrix} -17 & -1 \\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix} -16 & 3 \\ 5 & -1\end{pmatrix}$. If $A=\begin{pmatrix} -17 & -1 \\ 1 & 0\end{pmatrix}$, then by Lemma 5 $a=0$ and C is the knot K_2 of Proposition 1.

Suppose $A=\begin{pmatrix} -16 & 3 \\ 5 & -1\end{pmatrix}$. Since $A=X^{-1}YXYX^{-6}$, $\phi_*(x)=f_*^{-6}\circ g_*\circ f_*\circ g_*\circ f_*^{-1}(x)$ $=xyx^{-6}yx^{-5}yx^{-6}$ and $\phi_*(y)=x^6y^{-1}x^{-1}$. Then we have $\pi_1(M_{A(a)})=(x,\ y,\ t,\ u\,|\,t^{-1}xt$ $=xyx^{-6}yx^{-5}yx^{-6},\ t^{-1}yt=x^6y^{-1}x^{-1},\ u^at=1,\ u=xyx^{-1}y^{-1})$. Then by Lemma 1 and Corollary B, $M_{A(a)}$ is not homeomorphic to a lens space for any $a<0$, because ϕ is of type II. Thus we have $a\geqq 0$. Put $tu=\tilde{t}$. Then $\pi_1(M_{A(a)})=$ $(x,\ y,\ \tilde{t},\ u\,|\,\tilde{t}^{-1}x\tilde{t}=\tilde{\phi}_*(x),\ \tilde{t}^{-1}y\tilde{t}=\phi_*(y),\ u^{a-1}\tilde{t}=1,\ u=xyx^{-1}y^{-1})$, where $\tilde{\phi}_*(x)=$ $u^{-1}xyx^{-6}yx^{-5}yx^{-6}u=yx^{-5}yx^{-5}yx^{-5}yx^{-1}y^{-1}$ and $\phi_*(y)=u^{-1}x^6y^{-1}x^{-1}u=yxy^{-1}x^4y^{-1}$. Then $\tilde{\phi}$ is of type I. By Lemma 1 and Corollary B, $M_{A(a)}$ is not homeomorphic to a lens space for any $a-1>0$. Thus we have $0\leqq a\leqq 1$. If $a=0$, then $\pi_1(M_{A(0)})=(x,\ y\,|\,yx^{-6}yx^{-5}yx^{-6}=1,\ yxyx^{-6}=1)=(x,\ y\,|\,(x^6y^{-1})^3x^{-1}=1,$ $x^6y^{-1}=yx)=(x,\ y\,|\,yxyxy=1,\ x^6=yxy)=(x,\ y\,|\,yx^7=1,\ x^6=yxy)=Z_{19}$. By drawing pictures we can see that $M_{A(0)}$ is homeomorphic to $L(19, 3)$ and C is the knot K illustrated in Figure 4.

If $a=1$, then $\pi_1(M_{A(1)})=(x,\ y\ |\ xyx^{-1}y^{-1}xyxy^{-1}x^{-1}=xyx^{-6}yx^{-5}yx^{-6},$ $xyx^{-1}yxy^{-1}x^{-1}=x^6y^{-1}x^{-1})=(x,\ y\ |\ xyxy^{-1}=yx^{-5}yx^{-5}yx^{-5},\ yx^{-5}=xy^{-1}x^{-1})=$ $(x,\ y,\ a\ |\ xyxy^{-1}=a^3,\ a=xy^{-1}x^{-1},\ yx^{-5}=a)=(x,\ a\ |\ xaxa^{-1}=a^3,\ a=x^{-4}a^{-1}x^{-1})$ $=(x,\ a\,|\,a^4=xax,\ x^{-4}=axa)=(x,\ a\,|\,a^5=x^{-3}=(ax)^2)$ is not cyclic by [**12**].

By the way, by [**2**] there are five different lens spaces whose fundamental group is a cyclic group of the order 19, those are $L(19, 1)$, $L(19, 2)$, $L(19, 3)$, $L(19, 4)$ and $L(19, 7)$. Hence by the above arguments we can see that $L(19, 2)$, $L(19, 4)$ and $L(19, 7)$ do not contain GOF-knots. This completes the proof.

Now let L be a lens space, then from the arguments in this paper we can easily see that L contains only finitely many GOF-knots. In fact, it seems that L does not contain a great many GOF-knots. Thus we will set up the following question.

QUESTION. *Are the numbers of* GOF-*knots in all lens spaces bounded?*

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