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A note on Martin boundary of angular regions for Schrödinger equations

By Toshimasa TADA

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We denote by Ω the punctured unit disk $0<|z|<1$ and consider the Martin compactification Ω_P^* ([**4**, p. 166]) of Ω with respect to a Schrödinger equation

$$(1)\qquad (-\Delta+P(z))u(z)=0 \qquad \left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ z=x+yi\right)$$

with its potential P on Ω. The potential P on Ω is assumed to be nonnegative and locally Hölder continuous on $0<|z|\leqq 1$. We also consider the Martin compactification A_P^* of an angular region A with radius 1 and vertex at the origin $z=0$ with respect to (1). Let $\bar{\Omega}$ and $\bar{A}$ be the Euclidean closures of Ω and A, respectively. One might ask the following

QUESTION 1. *Does $A_P^*=\bar{A}$ for all angular regions A imply $\Omega_P^*=\bar{\Omega}$?*

Here the equality $\Omega_P^*=\bar{\Omega}$ ($A_P^*=\bar{A}$, resp.) means that the identity mapping of Ω (A, resp.) can be extended to a homeomorphism of Ω_P^* (A_P^*, resp.) onto $\bar{\Omega}$ ($\bar{A}$, resp.).

For a point p in the Euclidean boundary $\partial\Omega$ (∂A, resp.) of Ω (A, resp.), we denote by $\Omega_P^*(p)$ ($A_P^*(p)$, resp.) the set of all Martin boundary point ζ^* of Ω (A, resp.) for which there exists a sequence $\{\zeta_n\}_1^\infty$ in Ω (A, resp.) converging to p with respect to the Euclidean topology and at the same time converging to ζ^* with respect to the Martin topology. We call $\Omega_P^*(p)$ ($A_P^*(p)$, resp.) the Martin boundary of Ω (A, resp.) over p. We also denote by $\Omega_{P,1}^*(p)$ ($A_{P,1}^*(p)$, resp.) the set of Martin minimal boundary points over p, i.e. the subset of $\Omega_P^*(p)$ ($A_P^*(p)$, resp.) consisting of minimal points. In terms of $\Omega_{P,1}^*(0)$ and $A_{P,1}^*(0)$, Question 1 can be reformulated as

QUESTION 2. *Does $A_{P,1}^*(0)=\{$one point$\}$ for all angular regions A imply $\Omega_{P,1}^*(0)=\{$one point$\}$?*

Since P is locally Hölder continuous apart from the origin, we have $\Omega_P^*-\Omega_P^*(0)=\bar{\Omega}-\{0\}$ and $A_P^*-A_P^*(0)=\bar{A}-\{0\}$ (cf. [**1**]). By an argument similar to that

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in no. 2.2 we can see that the Martin kernel with pole in $A_P^*(0)$ vanishes on $\partial A-\{0\}$, and hence it is represented as the integral of the minimal Martin kernel with pole in $A_{P,1}^*(0)$. Therefore $A_{P,1}^*(0)=\{$one point$\}$ if and only if $A_P^*(0)=\{$one point$\}$. Similarly $\Omega_{P,1}^*(0)=\{$one point$\}$ if and only if $\Omega_P^*(0)=\{$one point$\}$. Our main purpose of this note is to construct a potential P on Ω for which the answer to the above question is in the negative:

THEOREM. *There exists a potential P on Ω such that $A_{P,1}^*(0)=\{$one point$\}$ for all angular regions A with radius 1 and vertex at the origin and yet $\Omega_{P,1}^*(0)=\{$two points$\}$.*

§ 1. Construction of the potential in the theorem.

1.1. We take four positive numbers a, b, c, d with $3/4<d<c<b<a<1$ and consider the following closed subsets of Ω which are of spiral shaped and converge to the origin windingly around it:

$$S_1 = \{re^{i\theta} : 2^{-\theta/2\pi}b \leqq r \leqq 2^{-\theta/2\pi}a,\ 0\leqq\theta<\infty\},$$
$$S_2 = \{re^{i\theta} : 2^{-\theta/2\pi}d \leqq r \leqq 2^{-\theta/2\pi}c,\ 0\leqq\theta<\infty\}.$$

There exists a conformal mapping from the simply connected region

$$U = \{0<|z|\leqq\infty\}-(S_1\cup S_2)$$

onto the exterior $\{1<|z|\leqq\infty\}$ of the unit circle. By the Carathéodory theorem every boundary element of U over the origin corresponds to a point in the unit circle. Here the boundary elements of U over the origin consist of two elements defined by two fundamental sequences $\{\alpha_n\}_1^\infty$ and $\{\beta_n\}_1^\infty$ of cross cuts

$$\alpha_n = [2^{-n}c,\ 2^{-n}b] \qquad \text{and} \qquad \beta_n = [2^{-n-1}a,\ 2^{-n}d]\,.$$

Therefore there exist exactly two Martin minimal boundary points of U over the origin.

The subregion

$$V = \Omega-(S_1\cup S_2)$$

of Ω is essential for the construction of the potential P on Ω. Since V is a subregion of U and $U-V$ is compact, the set $V_1^*(0)$ of Martin minimal boundary points of V over the origin also consists of two points.

1.2. Let $\{\delta_n\}_1^\infty$ be a sequence in $(0, \pi)$ with $\lim_n\delta_n=0$. We set

$$S_{1n} = \{re^{i\theta} : 2^{-\theta/2\pi}b\leqq r\leqq 2^{-\theta/2\pi}a,\ 2(n-1)\pi\leqq\theta\leqq 2n\pi-\delta_n\},$$
$$S_{2n} = \{re^{i\theta} : 2^{-\theta/2\pi}d\leqq r\leqq 2^{-\theta/2\pi}c,\ 2(n-1)\pi\leqq\theta\leqq 2n\pi-\delta_n\}$$

and consider the subregion

$$W=\Omega-\bigcup_{n=1}^{\infty}(S_{1n}\cup S_{2n})$$

of Ω. By the reasoning similar to that in [5, Example 1 on pp. 7-10] we can show that the cardinal number of the set $W_1^*(0)$ of Martin minimal boundary points of W over the origin is equal to that of $V_1^*(0)$ if we choose $\{\delta_n\}$ convergent to zero enough rapidly. The sequence $\{S_{jn}\}_{j=1,2;n\geqq 1}$ of closed Jordan regions S_{jn} satisfies that $S_{jn}\cap S_{km}\neq\emptyset$ $((j,n)\neq(k,m))$ and there exist only a finite number of S_{jn} such that $S_{jn}\cap\{\varepsilon\leqq|z|<1\}=\emptyset$ for any $\varepsilon>0$. Such a sequence of closed Jordan regions S_{jn} is referred to as a $\mathcal{Y}$*-sequence* in Ω.

Consider a potential P on Ω with its support contained in the closed subset

$$S=\bigcup_{n=1}^{\infty}(S_{1n}\cup S_{2n})$$

of Ω. We denote by $\mathrm{PP}(\Omega;\partial\Omega-\{0\})$ $(\mathrm{HP}(W;\partial W-\{0\})$, resp.) the set of nonnegative solutions u of (1) on Ω (nonnegative harmonic functions u on W, resp.) with vanishing boundary values on $\partial\Omega-\{0\}$ $(\partial W-\{0\}$, resp.). We also denote by H_u^W for each u in $\mathrm{PP}(\Omega;\partial\Omega-\{0\})$ the least nonnegative harmonic function on W with boundary values u on $\partial W-\{0\}$. If the mapping T_P from $\mathrm{PP}(\Omega;\partial\Omega-\{0\})$ to $\mathrm{HP}(W;\partial W-\{0\})$ defined by $T_Pu=u-H_u^W$ happens to be bijective, then the potential P is said to be *canonically associated* with the $\mathcal{Y}$-sequence $\{S_{jn}\}$. If a potential P on Ω is canonically associated with the $\mathcal{Y}$-sequence $\{S_{jn}\}$, then the cardinal number of $\Omega_{P,1}^*(0)$ is equal to that of $W_1^*(0)$. In view of [5, Theorem on p. 3] there exists a potential on Ω canonically associated with the $\mathcal{Y}$-sequence $\{S_{jn}\}$. From now on our potential P is supposed to be chosen on Ω so as to be canonically associated with the $\mathcal{Y}$-sequence $\{S_{jn}\}$, and therefore supp. $P\subset S$ and $\Omega_{P,1}^*(0)=\{\text{two points}\}$.

§ 2. The set $A_{P,1}^*(0)$.

2.1. In order to complete the proof of the theorem we will show that $A_{P,1}^*(0)=\{\text{one point}\}$ for the potential P on Ω constructed in § 1 and for all angular regions A with radius 1 and vertex at the origin:

$$A=\{re^{i\theta}:0<r<1,\ \sigma<\theta<\tau\}$$

with numbers σ,τ satisfying $0\leqq\sigma<\tau\leqq\sigma+2\pi<4\pi$. We set

$$A_n=\left\{re^{i\theta}:\frac{1}{2}2^{-\theta/2\pi}<r<\frac{3}{4}2^{-\theta/2\pi},\ \sigma<\theta-2(n-1)\pi<\tau\right\}$$

$(n=1,2,\cdots)$. Let u and v be positive solutions of (1) on A_n with vanishing boundary values on $\partial A\cap\partial A_n$. Since the support of P is contained in $S=\bigcup_1^{\infty}(S_{1n}\cup S_{2n})$ and $S\cap A_n=\emptyset$, the solutions u and v are harmonic on A_n. Then the boundary Harnack inequality

$$\frac{u(z)}{u(z_n)} \leqq c_n \frac{v(z)}{v(z_n)} \qquad (z \in \gamma_n) \tag{2}$$

is valid on the curve

$$\gamma_n = \left\{ re^{i\theta} : r = \frac{5}{8} 2^{-\theta/2\pi},\ \sigma < \theta - 2(n-1)\pi < \tau \right\}$$

for a positive constant c_n being independent of u and v, where z_n is the point in γ_n with its argument $(\sigma+\tau)/2$ ([**3**, Theorem 2.2] and its revisions by [**6**, Theorem 1 on p. 148] and also [**1**, Théorème 5.1 on p. 188], among others). The conformal equivalence of A_n, γ_n, z_n and A_1, γ_1, z_1 implies $c_n = c_1$ $(n=2, 3, \cdots)$. We say that the *boundary Harnack principle* is valid at the origin for the class of positive solutions of (1) on A with vanishing boundary values on $\partial A - \{0\}$ if the constant c_n in (2) can be chosen independent of n, which we have just established.

2.2. Although it is rather standard to derive that the set of Martin minimal boundary points over Euclidean boundary point p consists of one point from the boundary Harnack principle at p (see [**1**, pp. 193-195], cf. also [**2**], among others), we briefly include its proof in nos. 2.2-2.3 for the convenience sake.

We denote by $g_P(\cdot, \zeta)$ the Green's function on A with its pole at ζ with respect to (1) and by $k_P(\cdot, \zeta) = g_P(\cdot, \zeta)/g_P(z_1, \zeta)$ the Martin kernel on A, where z_1 is the point in γ_1 with its argument $(\sigma+\tau)/2$. Let ζ^* be an arbitrary point in $A^*_{P,1}(0)$. We remark that $A^*_{P,1}(0)$ contains at least one point by the definition. There exists a sequence $\{\zeta_m\}_1^\infty$ in A converging to the origin such that $\{k_P(\cdot, \zeta_m)\}$ converges to $k_P(\cdot, \zeta^*)$ uniformly on every compact subset of A. Consider the solution ω_n of (1) on the subregion

$$B_n = \left\{ re^{i\theta} : \frac{1}{2} 2^{-\theta/2\pi} < r < 1,\ \sigma < \theta - 2(n-1)\pi < \tau \right\}$$

of A with boundary values zero on $\partial B_n \cap \partial A$ and 1 on $\partial B_n \cap A$ $(n=1, 2, \cdots)$. Recall that $c_n = c_1$ $(n=1, 2, \cdots)$. Applying (2) with $c_n = c_1$ to $u = k_P(\cdot, \zeta_m)$ and $v = \omega_n$, we have

$$\frac{k_P(z, \zeta_m)}{k_P(z_n, \zeta_m)} \leqq c_1 \frac{\omega_n(z)}{\omega_n(z_n)} \qquad (z \in \gamma_n)$$

if $\zeta_m \notin B_n$. By the maximum principle the above inequality is valid for z in the subregion

$$D_n = \left\{ re^{i\theta} : \frac{5}{8} 2^{-\theta/2\pi} < r < 1,\ \sigma < \theta - 2(n-1)\pi < \tau \right\}$$

of A. The usual Harnack inequality for positive solutions of (1) yields $k_P(z_n, \zeta_m) \leqq c'_n k_P(z_1, \zeta_m) = c'_n$ for a positive constant c'_n and m with $\zeta_m \notin B_n$. Then $k_P(\cdot, \zeta^*)$

is dominated by $(c_1 c'_n/\omega_n(z_n))\omega_n$ on D_n. Therefore $k_P(\cdot, \zeta^*)$ has vanishing boundary values on $\partial D_n \cap \partial A$ $(n=1, 2, \cdots)$ and hence on $\partial A - \{0\}$.

2.3. Let u and v be positive solutions of (1) on A with vanishing boundary values on $\partial A - \{0\}$. We also assume that $u(z_1)=v(z_1)=1$. By (2) with $c_n = c_1$ and the maximum principle we have

$$c_1^{-1}\frac{v(z)}{v(z_n)} \leqq \frac{u(z)}{u(z_n)} \leqq c_1 \frac{v(z)}{v(z_n)} \quad (z \in D_n\,;\ n=1, 2, \cdots).$$

If we set $z=z_1$ in the above inequalities, then we have $c_1^{-1} \leqq u(z_n)/v(z_n) \leqq c_1$. Hence $c_1^{-2} v \leqq u \leqq c_1^2 v$ is valid on A. Set

$$\lambda_0 = \sup\{\lambda > 0 : \lambda v \leqq c_1^2 u\}.$$

The nonnegative solution $w = c_1^2 u - \lambda_0 v$ of (1) on A has vanishing boundary values on $\partial A - \{0\}$. If w is positive, then $u \leqq c_1^2 w/w(z_1)$ is valid on A and we have the contradiction

$$\lambda_0 v \leqq \left(c_1^2 - \frac{w(z_1)}{c_1^2}\right) u\,.$$

Therefore $w \equiv 0$ so that $v \equiv (c_1^2/\lambda_0)u$. This means that $A^*_{P,1}(0)$ contains at most one point.

The proof of the theorem is herewith complete.

2.4. We remark that the theorem is valid even if we replace the condition $\Omega^*_{P,1}(0) = \{\text{two points}\}$ with the condition $\Omega^*_{P,1}(0) = \{n \text{ points}\}$ $(n=3, 4, \cdots)$. For the purpose we consider disjoint closed subsets $T_1, \cdots, T_n$ of Ω which are of spiral shaped and converge to the origin windingly around it. In §1 we associated the potential P on Ω with the closed subsets S_1 and S_2 of Ω. Similarly we associate a potential Q on Ω with $T_1, \cdots, T_n$. Then Q satisfies that $A^*_{Q,1}(0) = \{\text{one point}\}$ for all angular regions A and $\Omega^*_{Q,1}(0) = \{n \text{ points}\}$. Moreover we can construct a potential Q on Ω such that $A^*_{Q,1}(0) = \{\text{one point}\}$ for all angular regions A and the cardinal number of $\Omega^*_{Q,1}(0)$ is that of the countable infinite set (the continuum, resp.). The constructions for the above two cases go along the same line as that for the case $\Omega^*_{P,1}(0) = \{n \text{ points}\}$ but this time by imitating [**5**, Example 2 on pp. 10-12] or [**5**, Example 3 on pp. 12-14] instead of [**5**, Example 1 on pp. 7-10] but the detail will be left to the reader.

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Toshimasa TADA
Department of Mathematics
Daido Institute of Technology
Daido, Minami, Nagoya 457
Japan